Nuclear Astrophysics

Prof. Anil Kumar Gourishetty

Department of Physics

Indian Institute of Technology, Roorkee

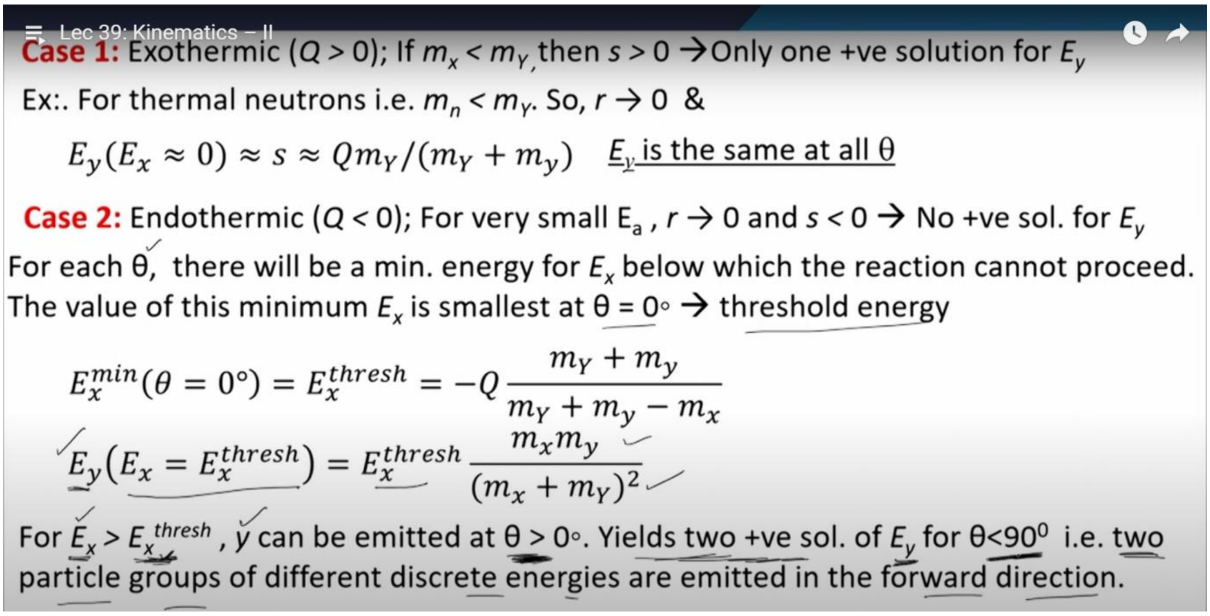
Lecture - 39

Kinematics - II

So, welcome back to the discussion on kinematics of nuclear reaction. In the previous class I have started discussing the relation between energies of the projectile and ejectiles and the angle with which the ejectiles are moving with respect to the incident beam direction. So, let us see more features of this kinematics of a nuclear reaction in the laboratory frame of reference. So, now the discussion is on lab frame of reference.

So, if you remember in the last class, I have ended the discussion with this expression of ejectile energy when the reaction is endothermic. It is endothermic and we have seen that because there is no positive solution for ejectile energy. And for every value of theta that is angle with which ejectile is moving forward with respect to the beam direction there will be a minimum energy for projectile beam below which it is not possible for the reaction to happen.

And the value of this minimum energy of the projectile is smallest when theta = 0 degrees which we termed as threshold energy. And at this threshold energy what is the value of the energy of the ejectile? Nothing but the threshold energy multiplied by the ratio of these two quantities. So, at threshold energy the value of Ey at theta = 0 given here. Now if we increase the projectile energy beyond the threshold energy of the projectile beam.



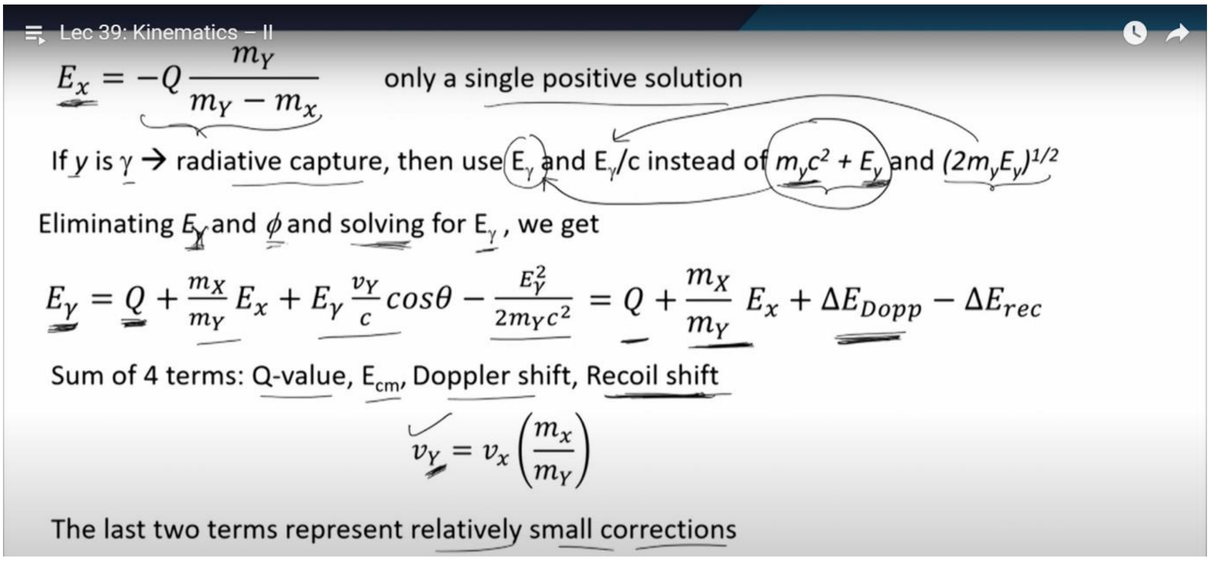
Then the ejectile the product particle. I am not using the word product nucleus. I am confining to the measurement of product particles. It could be protons, it could be sometimes alpha, it could be neutrons, it could be gamma rays. So, they can be emitted at theta = 0. If theta is greater than 0 degrees, earlier it was only theta = 0. Now theta greater than 0. Comes into picture when the projectile energy is greater than the threshold energy of the projectile.

So, you need to understand the relation between threshold energy of the projectile beam and energy of the projectile beyond the threshold energy and the value of theta the value with which the ejectile is moving forward with respect to the incident beam direction. Now in this particular case the expressions of r and s yield two positive solutions for Ey. Now the theta value is greater than 0 but I am confining to less than 90 degrees.

Now you can see instead of one, energy of the ejectiles there will be two groups of different energies. What does it mean? At theta = 0 the minimum energy with which projectile energy can initiate a nuclear reaction is threshold energy. And if you go beyond threshold energy values then you will see up to 90 degrees there will be two sets of energies for the particles. So, when you plan the experiment and keep the detectors with respect to the incident beam direction.

This relation between the energies of the ejectiles and the theta values will be very useful to you. Of course, it is valid when the projectile mass is much less than the mass of the product nucleus. Let us see what more is there in store for us. So, there will be two particle groups of different discrete energies which are emitted in the forward direction because I am confining theta to be less than 90 degrees.

Now if the projectile energy exceeds the value of this, Q into m y by m y - m x then there will be single positive solution. So, you have to understand different scenarios in this case. Now up to this I have considered a reaction in which particles are having rest masses. I have discussed earlier that most of the reactions in nuclear astrophysics deal with radiative capture type. So, in this case ejectile is a gamma which has no rest mass.



So, you need to replace few quantities if the nuclear reaction is a radiative capture type. What are the quantities to be replaced? So, if the ejectile is gamma ray then we call this reaction as a radiative capture. Now use energy of the gamma in the place of total energy of the ejectile that is rest mass of the ejectile and the kinetic energy of the ejectile. This sum whatever earlier wherever we have used this rest mass plus kinetic energy of the ejectile.

This we have to replace with this energy of the gamma and square root of 2mE for the ejectile has to be replaced with energy divided by the velocity of light. That is how we represent the momentum. Now do all this mathematics again by replacing these two quantities. That is total energy of the gamma and total momentum of the gamma which has no rest mass. Now in the similar way, we can eliminate the Ey and phi then solving for E gamma we get this expression.

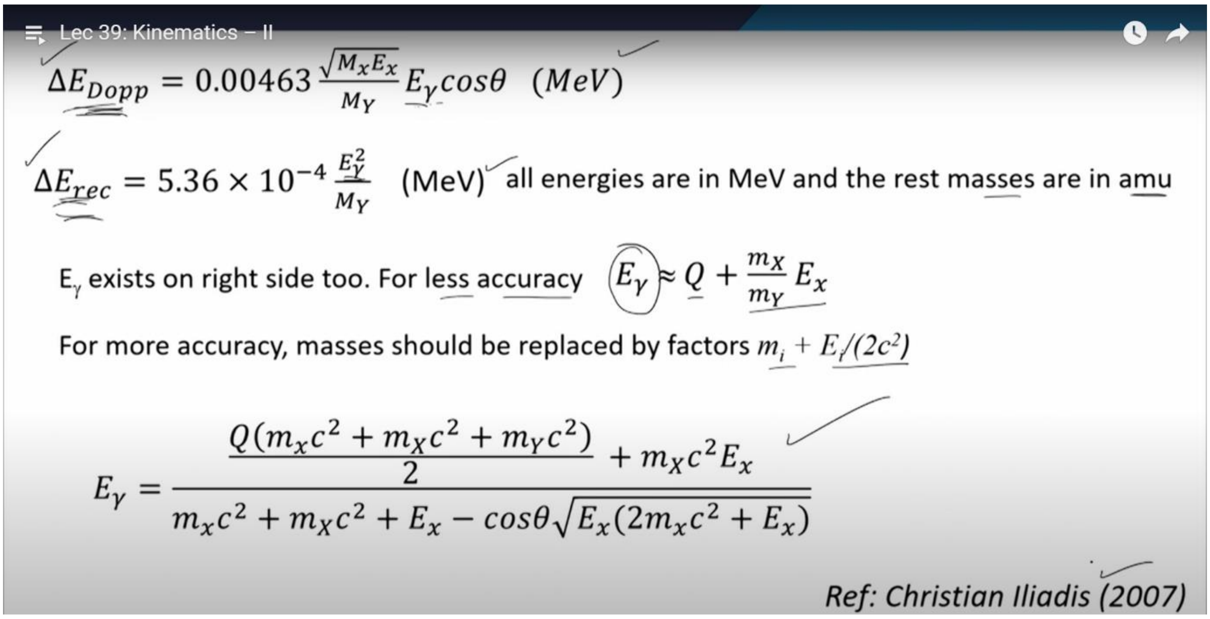
The energy of the gamma in radiative capture reaction can be found using this relation. To study the nuclear reaction of interest when we deal with the astrophysics many times you come across radiative capture reactions. I have said many times earlier the measurement of gamma energy is very important. So, this energy of the gamma mathematically is sum of four terms.

What are those four terms? Number one Q value of the nuclear reaction. Then the energy of the projectile in centre of mass system. Then Doppler effect because recoil nucleus is moving with some velocity. What is it velocity? I will write down. vy is the velocity with which recoil nucleus is moving. And because recoil nucleus is moving with this velocity which can be written as velocity to the projectile multiplied with the mass of the projectile divided by mass of the product nucleus.

So, because the recoil nucleus is moving with a velocity vy there will be Doppler shift in the gamma ray. Not only that because of the shift in the energy of the recoil nucleus one has to consider the recoil shift as well. So, the energy of the gamma in radiative capture is the sum of these four terms. Last two terms represent relatively small corrections but of course sometimes they may play important role.

So, we need to have numerical expressions for last two terms that is Doppler and recoil effects. What are those relations? Let me present here directly.

The Doppler effect corresponds to the energy given in here in MeV. Recoil shift energy corresponding to it is given by this relation in MeV. Here all the energies are in MeV and the rest masses are in amu. So, these two relations correspond to the last two terms in the energy of the gamma ray expression. But remember in here if you see the third term and fourth term Doppler Effect and recoil shift on the hand side you can see E gamma is there.

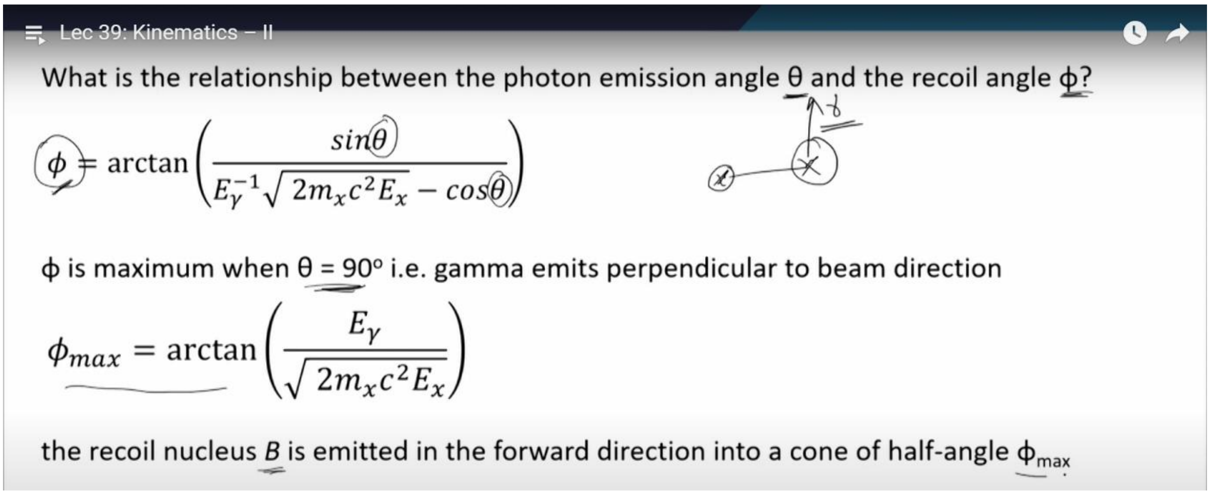


On the left-hand side also E gamma is there, is not it? On one hand we are writing expression for the energy of the gamma ray on another hand in third and fourth term which corresponds to the Doppler Effect and recoil shift E gamma term is coming into picture. This also has to be handled. So, if we are okay with some kind of less accuracy then we can write down the expression for E gamma as simply the sum of first two terms in the expression.

That is Q value and the energy of the centre of mass system. If you want better accuracy then what to be done? Then replace the masses with the relativistic effects. This gives us energy of the gamma this kind of relation. So, one can derive after seeing these relations and all this discussion is available in the textbook which is one of the important references for this course "Nuclear Physics of Stars" that is Iliadis.

Continuing the discussion on the kinematics of nuclear reaction. The next question is, what is the relationship between angles with which gamma is emitting with respect to the incident beam. The angle with which product nucleus is moving with respect to the incident beam which is characterized by the angle phi. What is the relation between this theta and phi? How to get them?

So, earlier expressions can be solved to obtain the expression for recoil angle. This expression that when theta becomes 90 this phi value is maximum. What does it mean? In the reaction if this is the ejectile and this is the target nucleus the gamma is emitted perpendicular to the beam direction. This is the case when theta = 90 that is when gamma emits perpendicular to the beam direction phi but recoil angle is maximum.



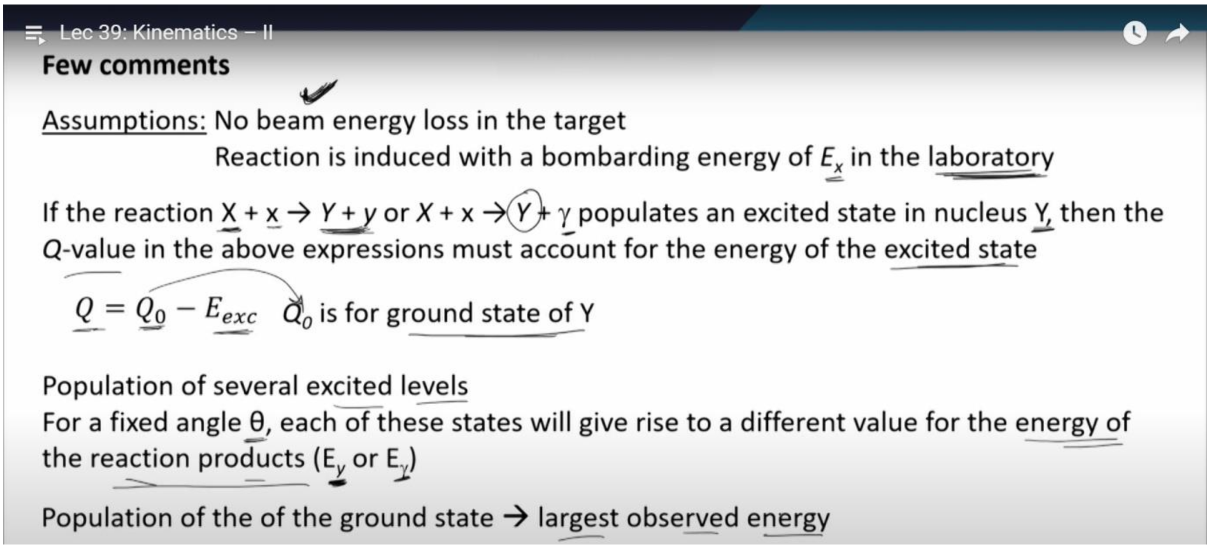
And that maximum recoil angle is expressed by this relation substituting theta = 90 in the above expression. Now when this is maximum the recoil angle is maximum that means the recoil nucleus B it is emitted in the forward direction only but into a cone of half angle phi max.

So, let me provide a few comments on these relations. See we have assumed that there is no beam energy loss in the target this is very important. Otherwise, the projectile energy if it undergoes loss within the target materials at different stages of its transport journey within the target material then the projectile's energy is changing then accordingly for one particular case when we fixed the projectile energy then the whole scenario will change.

So, the important assumptions which we have made here is beam energy loss is ignored and then the reaction is induced with a bombarding energy in the laboratory frame of reference. So, next I will discuss the relation between laboratory and the centre of mass frame of references when we deal with the energies and the angles. So, with these assumptions what are the comments I would like to make if at all the projectile after reacting with the target nucleus capital X.

It may give particles or nuclei with rest masses are particle with no rest mass like gamma. So, it is a general case I am discussing. If the capital Y gets populated that is the reaction populates an exited state in nucleus Y, then the Q value in the above expressions they have to consider for the energy of the exited state also. Until now we have considered it is reaching to the ground state. We have considered the situation where the product nucleus capital Y is in the ground state.

But it is quite common that the product nucleus goes into the excited state, it has some energy. So, when you deal with the Q value it has to consider the effect of energy of the exited state as well. How to take it all? It is very simple. The Q value of the situation corresponding to the ground state of the product nucleus - the excitation energy. So, this Q_0 is nothing but the Q value corresponding to the ground state of the product nucleus.



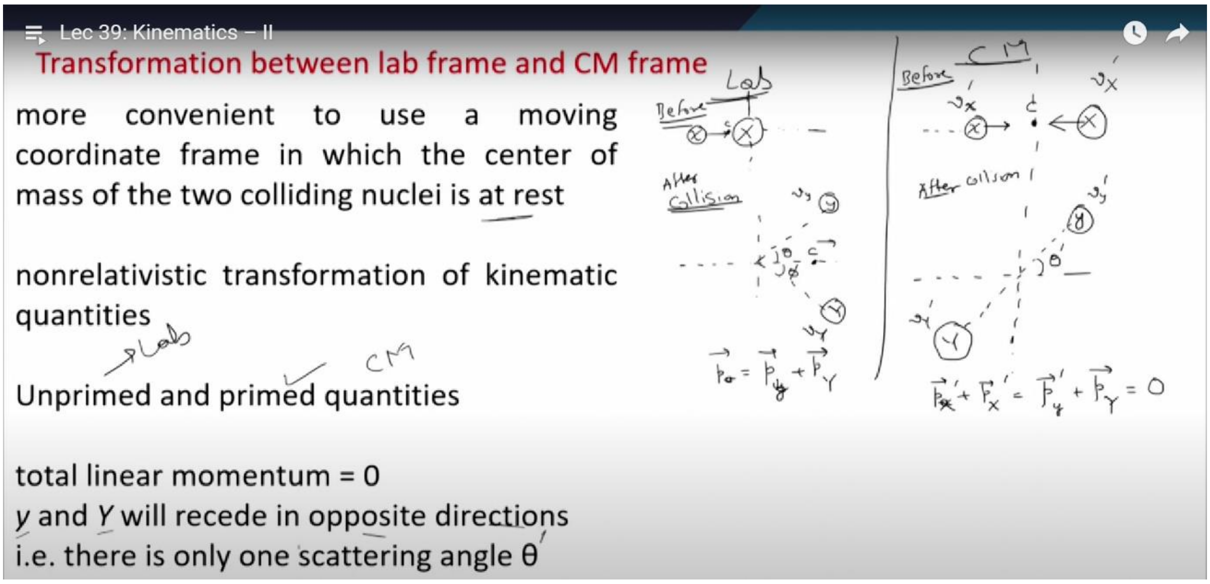
Now it is very much possible that the population can be at different excited levels. So, for a fixed angle of emission of the ejectile each of these states which were populated by the incident particle. Each of these excited states will give rise to a different value of the energy of the reaction products. Remember for a fixed value of theta that means ejectile is emitting with respect to a fixed value of theta with respect to the incident beam direction.

Different levels of this compound nucleus formed in this case capital Y they lead to the different value for the energies of the reaction products. It could be ejectile and the population of the ground state that is the largest absolute energy.

Now consider a case when the situation has to be discussed in centre of mass frame of reference. What is the importance of CM frame of reference? From now onwards I am going to use the word CM frame. From theoretical point of view the motion of centre of mass has no consequence in a nuclear reaction. So, in general we prefer to represent the values like a cross section, reaction rate or s factor all these things in centre of mass frame of reference.

So, the cross section when we measure in the nuclear reaction, we deal with the quantities in CM frame of reference. Because earlier when I have drawn the collision between two entities, I have shown you the way with which centre of mass is moving. Now in CM frame of reference both projectile and the target they are moving in opposite direction. So, that centre of mass is at rest. So, for the sake of convenience let me draw both situations again.

So, in lab frame small x it is moving centre of mass frame is moving and we have taken the target nucleus and, in this direction, when we draw the things, it is before collision and after collision. It is the emission of ejectile with velocity y and angle is theta and this is the way the centre of mass was moving and the angle with which product nucleus is emitted is phi and this is basically capital Y moving with velocity y.



Here the momentum of the incident particle because in lab frame target nucleus is at rest, projectile is moving. So, when you apply the conservation of momentum the momentum of the projectile is equal to the momentum of ejectile + momentum of the product nucleus. Now in centre of mass frame of reference you can see that the projectile is moving in forward direction. Now I am using a symbol prime all primary quantities refers to the quantities in CM frame of reference. All unprimed quantities refers to the laboratory frame of reference. So, in CM frame I am using prime symbol. So, before collision in centre of mass frame of reference, how to draw the things? So, here the centre of mass is at rest and projectile is moving in the opposite direction. Projectile and target nucleus capital X they are moving in opposite direction. So, v X, v capital X prime and v small x prime and after collision how to draw the things?

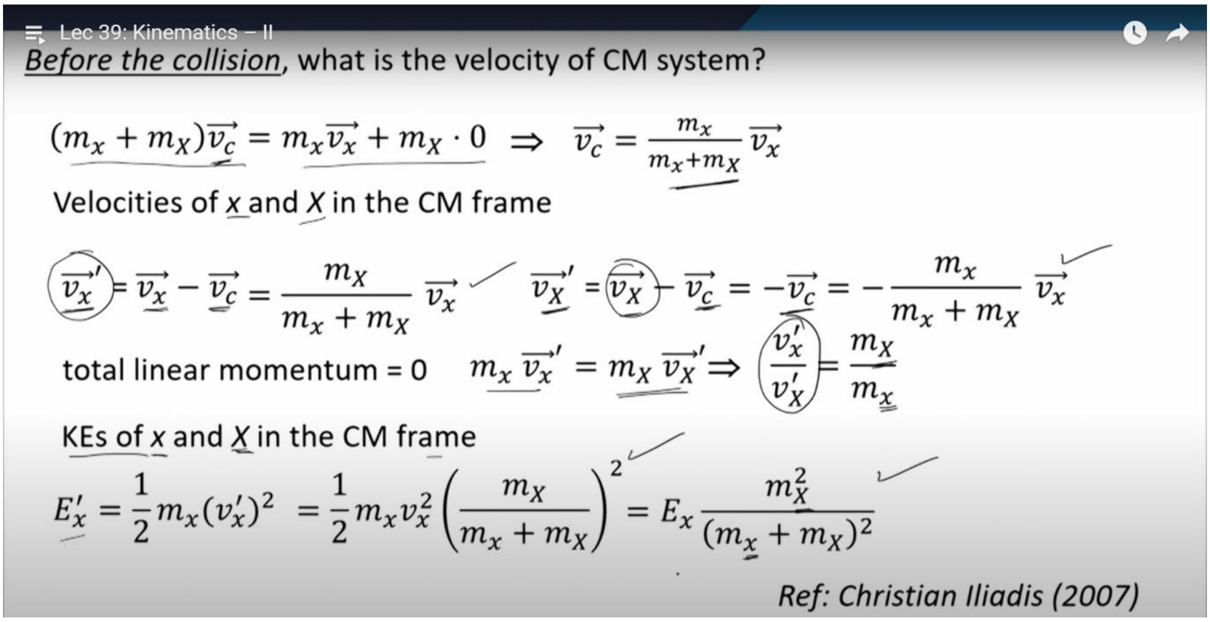
So, this is after collision and this is before collision. After collision you can see the product nucleus is moving and here the ejectile that is a small y it is also moving. Now only one angle is coming into picture that is theta prime because in the CM frame of reference products that are ejectile and the product nucleus they are emitting in opposite directions. Because we are considering CM frame of reference and there is no point in bringing the phi that is recoil angle into picture.

This v y prime small y and v capital Y prime. In this particular case momentum of projectile and momentum of capital X = momentum of ejectile + momentum of capital Y which is also vector it is 0. Total momentum before and after collision in CM frame is equal to 0. This has to be kept in mind when you try to solve the relations in kinematics of a nuclear reaction.

So, this CM frame of reference is more convenient to use a moving coordinate frame in which the centre of mass is at rest. Now the non-relativistic effect if we consider how the transformation of a kinematic quantities looks like. Primed quantities refers to CM frame and unprimed refers to lab frame of reference. In the CM frame total linear momentum is 0.

And these product entities that is ejectile and the product nucleus they will recede in opposite directions. So, there is only one scattering angle that is theta prime.

So, before the collision, what is the velocity of centre of mass system? It is basically given by this expression where the terms are self-explanatory. Now I am talking about the velocity to the centre of mass is given by this relation. And what is the what are the velocities of this projector and target nucleus in the centre of mass frame? Here it is given by the difference of projectile velocity and the centre of mass velocity is nothing but the velocity with which projectile is moving in centre of mass frame.



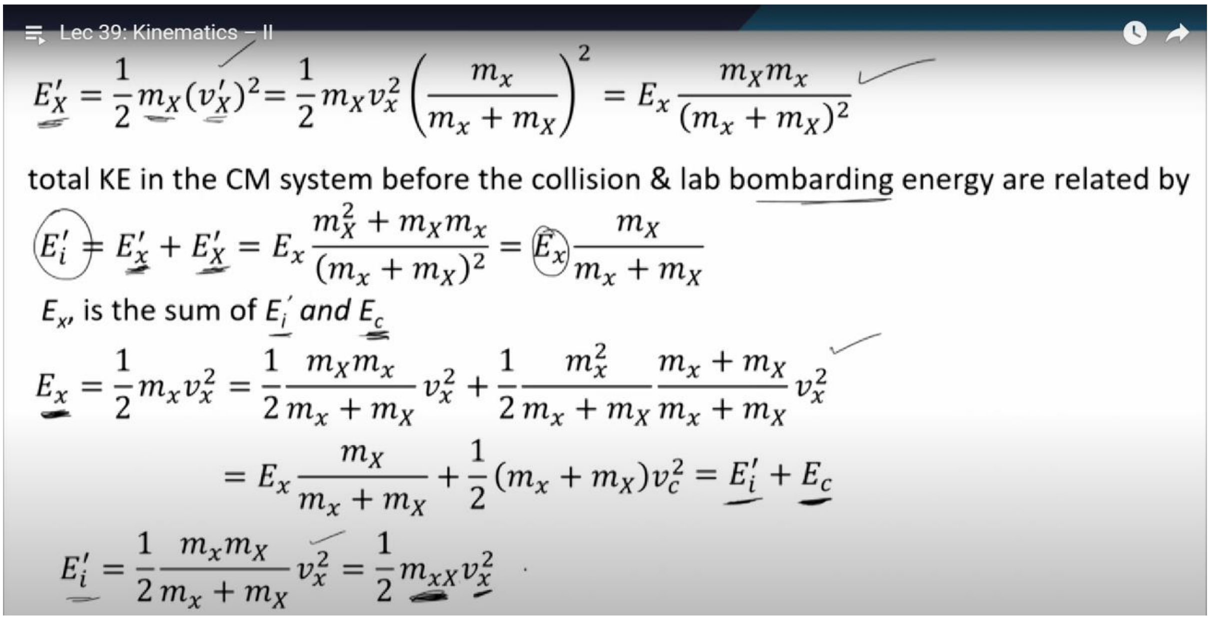
And in terms of masses, we can use this relation. What about velocity of target nucleus in centre of mass frame? Nothing but velocity of target nucleus - values to the centre of mass frame and anyway it is 0 and velocity of centre of mass frame is given here. So, in terms of masses we can write like this. So, the total linear momentum is 0 in the case of centre of mass system. So, the total linear momentum is 0 gives giving rise to this kind of relation.

So, this gives us an idea about the ratio of the velocities of projectiles and target nuclei. They are nothing but the ratio of masses of product nucleus and the projectile. Now let us see how the kinetic energy of projectile and the target nucleus looks like in the centre of mass frame. Because we have written the velocities of small x and capital X in CM frame. Once we have the velocities in hand by multiplying with half m and squaring the velocities one can get the values of kinetic energies.

So, how do they look like mathematically? So, E x prime is nothing but half mv square and if you expand this you will get this kind of expression and simplifying leads to this kind of expression which involves mass of the target nucleus and mass of the projectile and the energy of the projectile.

Similarly, energy of the target nucleus kinetic energy is nothing but half mv square where m is the mass of the product target nucleus and velocity with which target nucleus is moving in centre of mass frame that is why we are using prime symbol. So, one can get this kind of expression. Now what is the total kinetic energy before the collision in the centre of mass frame? And how it can be linked with the lab bombarding energy?

Very simple take the Ex prime when projector energy is the CM frame and target nucleus kinetic energy in the CM frame together you can get the total kinetic energy. And here the Ex if you see it is the sum of Ei prime total kinetic energy and the Ec the total kinetic energy of the centre of mass system. And this Ex if you expand you will get this kind of relation and similarly, you can see that this energy of the projectile has the sum of total kinetic energy in the CM system and energy associated with the centre of mass.

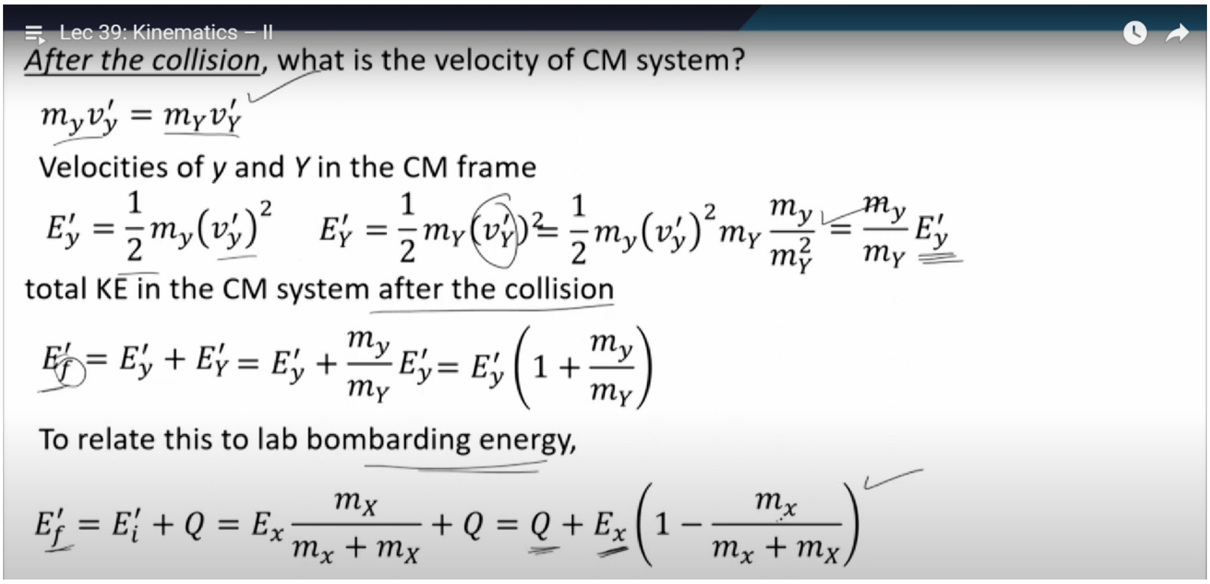


And Ei prime as can be seen from this expression it is the half mv square kind of thing where m is the reduced mass of the projector and target nucleus and velocity with which projectile is moving in the lab frame. So, that is why I have said how the relations can be related with each other.

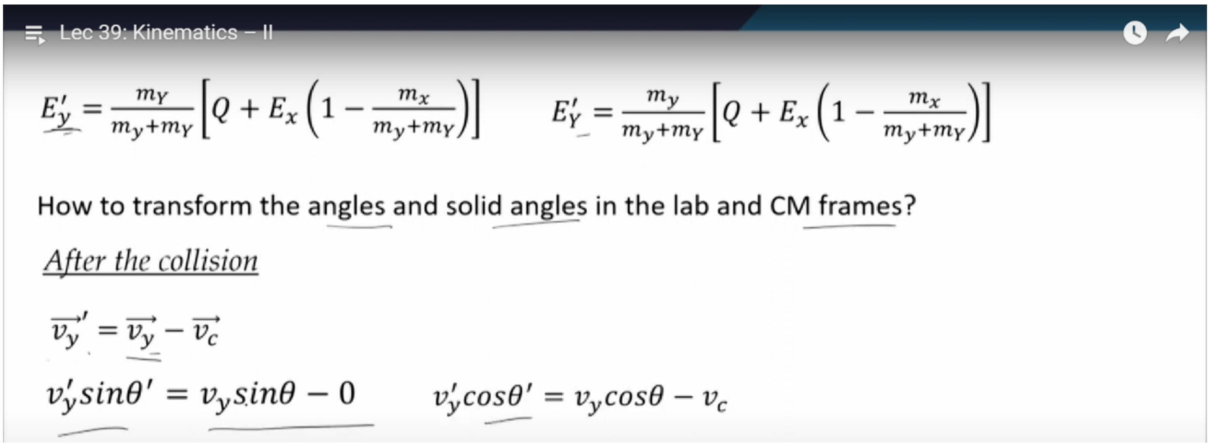
Now the situation after the collision, how it looks like? What is the velocity? Writing this expression of the momentum, we can write like this and velocities of a small y and capital Y in this centre of mass frame we can write in terms of half m v square. And here I can use for the velocity of the product nucleus in CM frame. From this relation expanding gives this kind of relation Ey prime.

Now total kinetic energy in the CM system after the collision can be given like instead of E y earlier that is initial before the collision. After the collision we are writing small f and doing the routine mathematics you will get this kind of relations. And how can we relate to this lab bombarding energy every time? We want to relate how the quantity will look like when we relate to the incident particle energy. So, Ef prime can be written in terms of Q value and energy of the projectile, mass of the projectile and the target nucleus. So, all these mathematics one can do easily.

And Ey prime ejectile you know kinetic energy in the CM frame and product nucleus kinetic energy in the CM frame can be obtained using simple algebra relations. And the final thing is how the angles and solid angles can be related in lab and CM frame. After the collision if you see the velocity of the ejectile is related by vy - vc which I have written earlier. And if you take the components, it is 0. Cos theta prime and sin theta prime you can write the expressions like this.



And using those relations one can come up with the tan theta which relates to theta prime and cos theta which relates in terms of theta prime only where here this gamma is not gamma ray, it is some constant, some symbol. So, please do not get confused. So, this can be represented by this relation which can be approximated by this relation. And if we consider the further radiative capture then this cos theta can be simplified as this relation where you can see the relation between theta and theta prime and where beta can be written like this.



So, to summarize today's lecture I have related various quantities between lab frame and CM frame. Please work out all these relations yourself and I have ended the lecture by relating the theta and theta prime when we come across lab frame and centre of mass frame. So, in the next lecture which is the final lecture of the nuclear astrophysics course, I will cover time of flight method and the essence of important indirect methods. Thank you so much.

$$tan\theta = \frac{v'_y sin\theta'}{v'_y cos\theta' + v_c} = \frac{sin\theta'}{cos\theta' + {}^{v_c}/_{v'_y}} = \frac{sin\theta'}{cos\theta' + \gamma}$$

$$\cos\theta = \frac{\gamma + cos\theta'}{\sqrt{1 + \gamma^2 + 2\gamma cos\theta'}}$$

$$\gamma \equiv \frac{v_c}{v'_y} = \sqrt{\frac{m_x m_y E_x}{m_Y(m_y+m_Y)Q + m_Y(m_Y+m_y-m_x)E_x}} \approx \sqrt{\frac{m_x m_y}{m_X m_Y}\frac{E_x}{(1+m_x/mX)Q + E_x}}$$

For radiative capture,

$$\cos\theta = \frac{cos\theta' + \beta}{1 + \beta cos\theta'} \quad where \quad \beta = \frac{\sqrt{E_x(E_x + 2m_x c^2)}}{m_X c^2 + m_x c^2 + E_x}$$