Nuclear Astrophysics
Prof. Anil Kumar Gourishetty
Department of Physics
Indian Institute of Technology- Roorkee
Module – 4
Lecture – 17
Astrophysical S-Factor and Non-Resonant Charged Particle Induced Reactions

Welcome students to the ongoing discussion on the non-resonant charged particle induced reactions. So, let me take a quick review of the previous class. In the last lecture I have provided enough background to understand the concept of charged particle induced non-resonant reactions.

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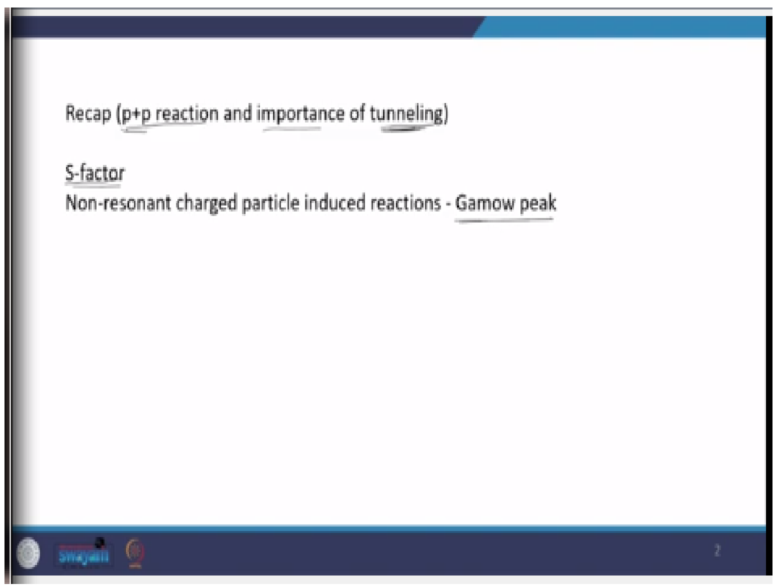

So, in the last lecture I have considered a p + p reaction, which is one of the most fundamental nuclear reactions responsible for the nucleosynthesis and also the evolution of uinverse we can say, and the importance of tunneling and what kind of arguments I had given? I hope you have followed it, let me quickly present before you again. The Coulomb barrier for p + p is around 550 keV and classically the fusion can happen only when the temperature corresponding to this 550 keV is available, it is like 10^9 K.

And the argument went like this, if the star has a temperature of 10^9 K and before that if no nuclear reaction occurs, the moment 10^9 K temperature is reached then all protons will react with each other immediately and one can expect catastrophic explosion which is not happening practically. So, that clearly indicates that before reaching the temperature required classically for the nuclear reaction to happen, the fusion is indeed happening, what could be the reason at that particular temperature?

So, people have given an argument with the Maxwell-Boltzmann distribution at the high tail end at a particular temperature. So consider the particles at a high energy range, high energy tail end and if high energy is available then it is okay, fusion can happen, it sounds bit good but one has to also see whether the number of particles at high energy tail end is sufficient to account for the energy production and we have seen it was 10^{-275}.

The ratio of probability for the particles to have 550 keV and to have 0.86 keV corresponding to this less temperature like 0.01 GK. We have seen 10^{-275}, so that is not sufficient, it is inadequate to account for the energy production in stars. Then it was a major obstacle even though Eddington had suggested that at very high density available in the sun and other stars the fusion could be responsible for the energy production.

That argument could not be supported by the mechanism of reaction but when Gamow and others independently proposed that, below Coulomb barrier also there is a finite chance for the charged particles to come out of the nucleus or to enter the nucleus by so called quantum tunneling process, after that people could confirm that it is because of the quantum tunneling process fusion reaction is taking place.

To support this I have presented few things like, let me quickly repeat those formulas and then I will cover astrophysical S-factor and the concept of Gamow peak. So, in today's lecture I am going to give a final touch to this p + p reaction and I am going to discuss S-factor concept and Gamow peak concept.

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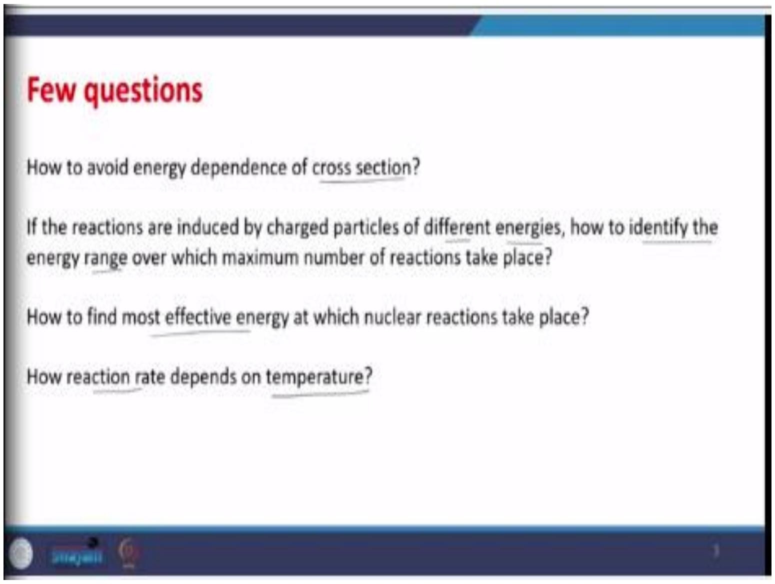

So, same questions which I have posed in the previous class, those are listed in the above slide no 3.

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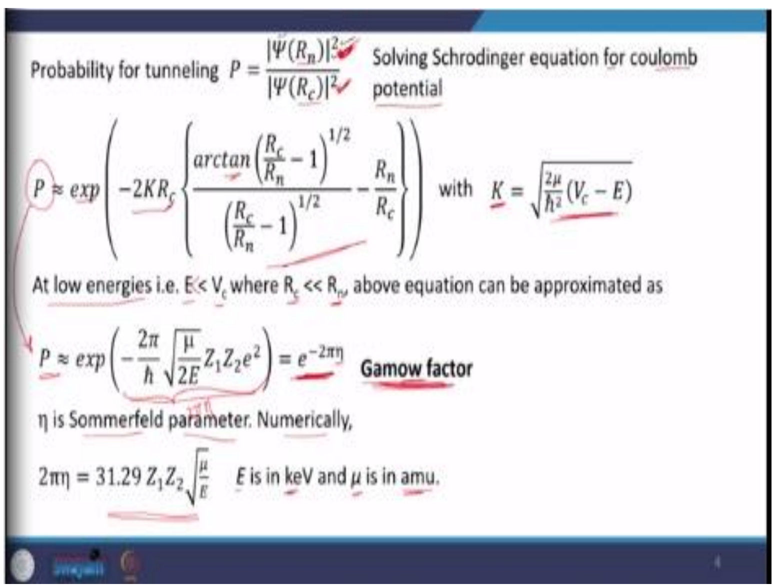

Now going back to this p + p reaction. Quantum mechanically you know square of the wave function at nuclear radius is nothing but probability for finding the particle at nuclear radius and is equal to $\frac{|\psi(R_n)|^2}{|\psi(R_C)|^2}$. $|\psi(R_n)|^2$ tells us the probability of finding the particle at a nuclear radius and $|\psi(R_C)|^2$ tells the probability of finding the particle at classical radius. In the previous lecture I have denoted the locations of nuclear radius and classical turning point. Now solving the

Schrodinger equation for the Coulomb potential, you can arrive to the equation $P \approx exp\left(-2KR_c\left\{\frac{arctan\left(\frac{R_c}{R_n}-1\right)^{1/2}}{\left(\frac{R_c}{R_n}-1\right)^{1/2}}-\frac{R_n}{R_c}\right\}\right)$ with $K = \sqrt{\frac{2\mu}{\hbar^2}(V_c - E)}$.

Up to this I have discussed in the last class. Now let me introduce one of the most important parameter in nuclear astrophysics. At low energies that is less than Coulomb barrier where the classical turning point is much less than the nuclear turning point, this equation can be approximated by writing like this, $P \approx exp\left(-\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{2E}}Z_1Z_2e^2\right) = e^{-2\pi\eta}$. This $e^{-2\pi\eta}$ is called as Gamow factor. What is the importance of this Gamow factor? It considers the effect of quantum tunneling process at low energies of charged particles. So, the Gamow factor tells us the exponential dependence of the probability, where η is Sommerfeld parameter. And numerically, $2\pi\eta = 31.29\, Z_1Z_2\sqrt{\frac{\mu}{E}}$, where E is in keV and μ, that is reduced mass in amu, so this value is important while calculating the Gamow factor.

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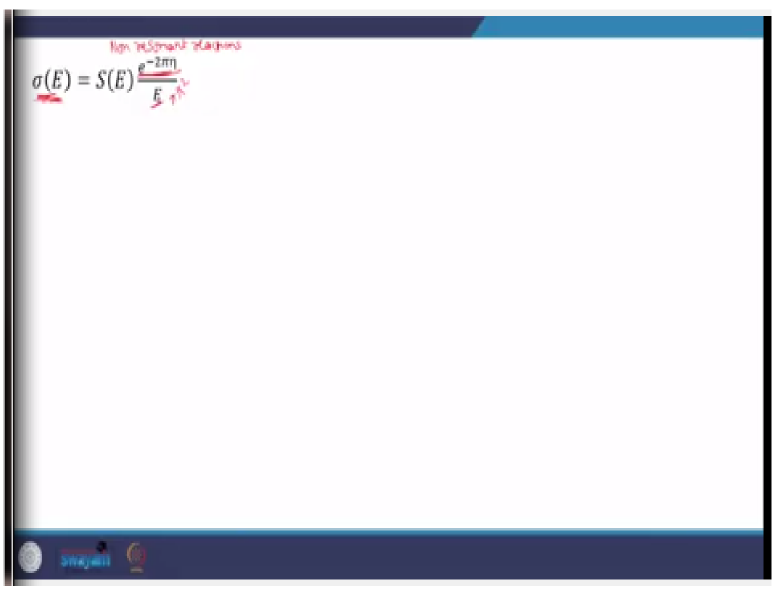

Now after introducing the Gamow factor let me introduce the cross-section formula. This cross section which depends on the energy for charged particle induced non-resonant reactions; I am talking about non-resonant reactions induced by charged particles. See, cross section anyway it depends on $e^{-2\pi\eta}$, I have discussed already. And earlier if you remember replacing classical area $\pi r^2$with $\pi\lambda^2$, do you remember?

When I have discussed inverse reactions and forward reactions, I have provided a general formula for the cross section of a nuclear reaction, what are those four terms? $\pi\lambda\!\!\!^{-2}$which consists of de Broglie wavelength, replacing the classical radius with the de Broglie wavelength. Number 2, statistical factor, number 3 identicalness in terms of Kronecker symbol, number 4 matrix elements which represents the nature of interactions in the entrance channel and in the exit channel, is not it?

So, then at that time I have discussed the representation of cross section in terms of ƛ that is de Broglie wavelength, is not it? So, $ƛ^2$ is nothing but in terms of energy you can always write down as 1/E. So, 1/E dependence you know on cross section because as energy decreases below the Coulomb barrier cross section decreases.

And Gamow factor already I have introduced is not it? It is other way around as energy increases the cross section is happening in other way and this is Gamow factor which takes into account the effect of quantum tunneling.

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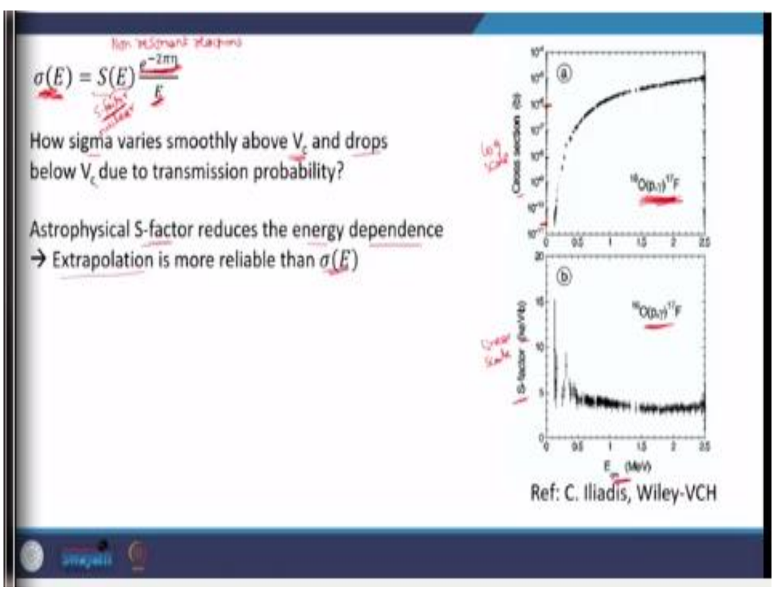

And the astrophysical S-factor, this term S-factor considers all nuclear effects. So, for non-resonant reactions induced by charged particles the cross section consists of three terms, and written as, $\sigma(E) = S(E)\frac{e^{-2\pi\eta}}{E}$. So, this is our understanding of non-resonant nuclear reactions where cross section is a smooth variation of energy. Later I will separately discuss resonant reactions, where cross section will have sudden change at particular values of energy. Right now, the assumption is that cross section is smooth function of energy, is not it? So, these three terms we are going to use extensively and we are going to rework on the reaction rate equations. Now how cross section varies smoothly above the Coulomb barrier and it drops below Coulomb barrier due to transmission probability.

And astrophysical S-factor, this reduces the energy dependence because by multiplying E with $\sigma(E)$ you can always write down the expression for astrophysical S-factor. Now here let me give you one good example how to understand the relation between cross section and the S-factor? So, above Coulomb barrier the cross section is varying smoothly, for the reaction p + ^{16}O, Coulomb barrier is about 2.2 MeV.

And the moment you go down to the Coulomb barrier the cross section is decreasing because the classical level probability is coming down because the energy is below the Coulomb barrier and it is because of the tunneling process. And the tunneling probability decreases with decrease in energy. See how cross section is varying from 10^{-11} to 10^{-6}, so much variation.

Now if you want to find out the cross section at low values of energy then one need to extrapolate this cross section. And this extrapolation of cross section induces a lot of error if there is a huge change in the y axis value, for small change in the x axis value. In most of the cases the Gamow peak energy values are not available in earth's laboratory I mean except very few facilities in the world.

So, people always prefer to go for extrapolation kind of work. And if you depend on cross section dependent energy then extrapolation induces lot of uncertainty. That is where the significance of astrophysical S-factor comes into picture, how? See, the second diagram if S-factor is plotted whose unit is keV barn versus center of mass energy, you see the variation is not very high and remember the first diagram is in logarithmic scale and this second diagram is in linear scale, is not it?

So, even at linear scale the change is not very high for this particular energy, proton induced radiative capture reaction with ^{16}O. So, the S-factor basically reduces the energy dependence and this is where extrapolation is more reliable when compared to cross section, I hope it is clear to you.

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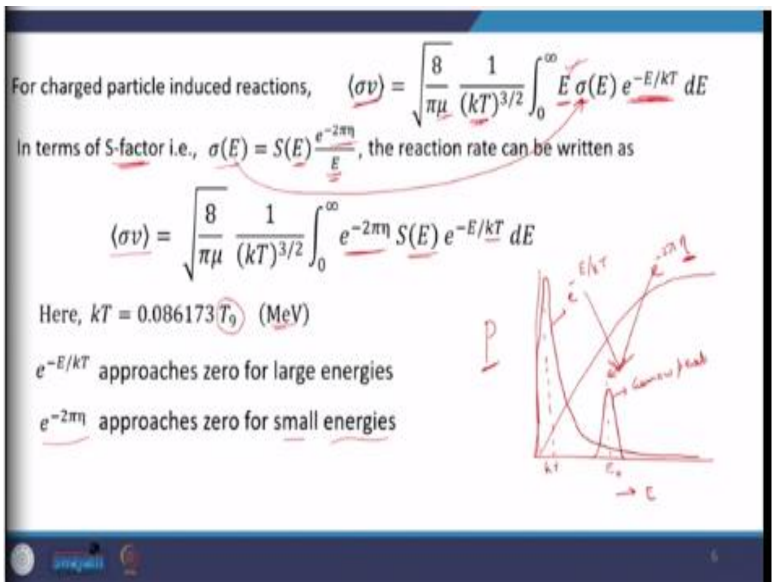

And for charged particle induced reactions this reaction rate expression is well known to you, $\langle\sigma v\rangle = \sqrt{\frac{8}{\pi\mu}}\ \frac{1}{(kT)^{3/2}} \int_0^\infty E\ \sigma(E)\ e^{-E/kT}\ dE$, where μ is the reduced mass, this kT is a well known parameter, energy E and σ is cross section, why are you using this $e^{-E/kT}$ in this expression because of the Maxwell -Boltzmann distribution. Now I am going to discuss one of the most fascinating and interesting aspect of nuclear astrophysics.

When charged particles are playing role in terms of energy production from the stars and nuclear synthesis, so please pay attention to this mathematical treatment. So, this discussion I am starting with the well-known expression for nuclear reaction rate which considers energy dependent cross section and the Maxwell -Boltzmann distribution of the particles because they are non-degenerate though they are fermions, we are using Maxwell- Boltzmann distribution.

Please recollect the previous discussion that is Gamow factor and S-factor, how that expression is useful in this reaction rate formula? Let me go one after the other. Now let me express this reaction rate formula in terms of S-factor, substituting this expression of cross section in the expression of reaction rate, that is, $\sigma(E) = S(E)\frac{e^{-2\pi\eta}}{E}$.

Because reaction rate cannot be calculated without the information of cross section and this cross section has 3 terms, remember. And each and every term the significance we need to be clear about it. Now I am going to write the reaction rate by using this terminology of S-factor.
$\langle\sigma v\rangle = \sqrt{\frac{8}{\pi\mu}}\ \frac{1}{(kT)^{3/2}}\int_0^\infty e^{-2\pi\eta}\ S(E)\ e^{-E/kT}\ dE$

So, this is the reaction rate for a non-resonant reaction induced by charged particles. Now here the value of kT already I have given in one of the previous lectures in terms of GK it is 0.086 T_9 in MeV. Now here please analyze $e^{-E/kT}$ approaches 0 for large energies, whereas, $e^{-2\pi\eta}$ approaches 0 for small energies.

In the integrand, please observe there are 2 exponential terms, what are those 2 exponential terms? $e^{-2\pi\eta}$ which came from the tunneling phenomenon and $e^{-E/kT}$ which came because of the distribution of the energies of particles at a specific temperature T. Remember this η contains the energy term, please look into the formula of our Sommerfeld parameter. When these 2 exponential terms are there within the integrand can you guess what the product of these 2 exponentials gives us?

What kind of information the product of these 2 exponential terms gives us? That is very interesting and in terms of some diagram let me try to explain. Now, here if you see the tunneling probability finally all types of probability I am expressing here. The tunneling probability increases with increase in the energy and finally after Coulomb barrier there is no tunneling.

However, according to this Maxwell-Boltzmann distribution you have $e^{-E/kT}$, this is much sharper not so much broader. So, you can see very sharp initially and then you can see it is decreasing like this exponentially, is not it? Now the product of these 2 gives rise to a beautiful peak where the centroid is E_0. This is called as Gamow peak, that means the peak denotes an area or the region about which the most of the nuclear reactions are taking place; this is the answer for one of the questions.

The first question which I have asked in one of the previous slides was how to reduce or how to avoid the energy dependence of cross section; using S-factor. Second question was at what energy range most of the nuclear reactions take place? This range can be calculated easily by taking the product of 2 terms $e^{-2\pi\eta}$ and $e^{-E/kT}$. So, let us see the salient features of this peak which is called as Gamow peak.

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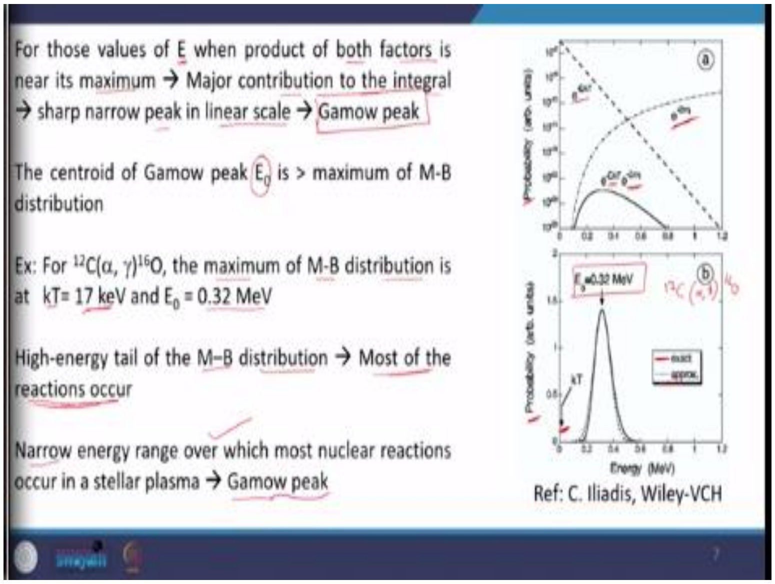

Now for those values of energy when the product of both factors is near its maximum that is where major contribution to the integral comes into picture. And it results into a sharp narrow peak in linear scale; I am talking about in linear scale which we are calling as Gamow peak. Now the centroid of the Gamow peak is E_0 and this is always greater than the maximum of Maxwell - Boltzmann distribution kT, please understand the relation between E_0 and kT.

Here let me give you one example where in logarithmic scale if I draw $e^{-E/kT}$, it will be some kind of straight line and $e^{-2\pi\eta}$ exponential increasing. And the product $e^{-E/kT}$ and $e^{-2\pi\eta}$ is nothing but some kind of peak; please remember this is in logarithmic scale on y axis. The same thing if I plot in linear scale for a particular reaction, it is about 0.32 MeV that is 320 keV.

You see the location of kT and you see the location of E_0 and here you can see the dotted line is approximate one and the solid line is exact one. So, this peak is not exactly Gaussian, it is very close to Gaussian, I will explain more about this in due course. The point which you have to understand here is the product of these 2 quantities decides the energy range over which maximum number of nuclear reactions are indeed taking place within the stars.

How does it matter? When you want to understand the energy production from a star because of a particular nuclear reaction at what energy range you will carry out the measurement? This Gamow peak will help us to decide the energy range. Now here I have taken an example of $^{12}C(\alpha,\gamma)^{16}O$. So, the maximum of Maxwell-Boltzmann distribution if you see the kT is 17 keV only.

Whereas it is 320 keV for the centroid of the Gamow peak. So, earlier when you have drawn the Maxwell-Boltzmann distribution the maximum when it comes at kT you might have thought the maximum reactions are taking place at that value of kT. No, for charged particle induced non-resonant reaction the maximum reactions are not taking place at kT but at some other energy which is higher than kT, that we are denoting using a symbol E_0.

So, the reactions are not occurring at the peak area of the Maxwell-Boltzmann distribution, but high energy tail of the Maxwell-Boltzmann distribution the most of the reactions are occurring. Now this narrow energy range over which most nuclear reactions are taking place I am using the word Gamow peak.

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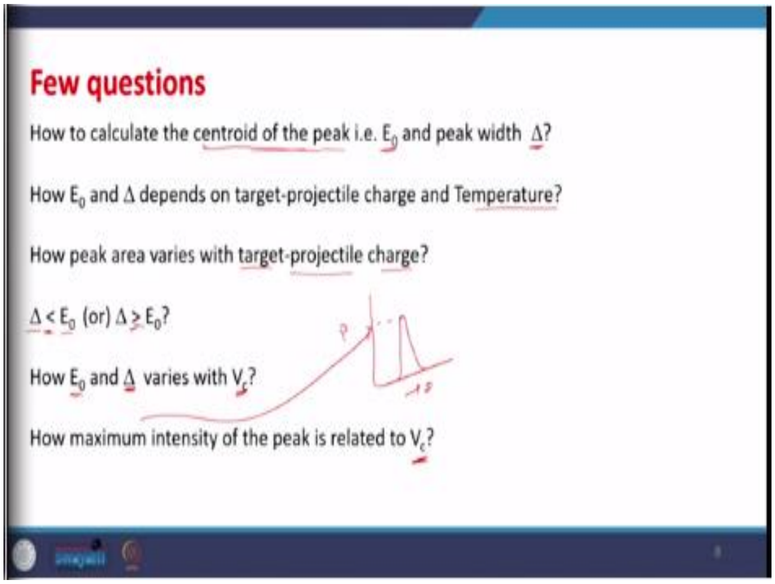

Once we have this Gamow peak in our mind, several questions should come to our mind. See the moment you say Gamow, the value at y axis corresponding to the centroid that is maximum of the integrand and the area of the peak and width of the peak. Because in today's lecture I am using all these parameters, initially it may look like confusing but it is very simple to understand, just you have to be little patient to relate various parameters.

So, most of the mathematics I am doing in today's lecture requires derivation from your side as part of homework. So, I am very happy to introduce a few questions related to Gamow peak, I hope this will help you in understanding the concept of non-resonant nuclear reactions. And how reaction rate is depending on various parameters. Finally, we have to come up with temperature dependence.

Because at that temperature which is frequently changing in the star, reaction rate is also changing and we are in process of obtaining analytical expression for reaction rate at different values of temperature. So, what is the list of questions have a look. How to calculate this centroid of the peak? Just qualitatively I have shown you the location of E_0, now how to calculate the width of the peak also and how this centroid and peak width depends on the charge of the target and projectile and also on temperature.

3rd question, the peak area, does it vary when the charge of the target and projectile changes? If yes, what is the analytical expression for that? We have to derive. Now whether the peak width which is also in terms of energy is greater than the value of the centroid or less than the centroid that we need to understand and with respect to the Coulomb barrier how the centroid of the peak and the peak width changes? This we need to understand, let us see. And also the maximum intensity of the peak is related to V_C.

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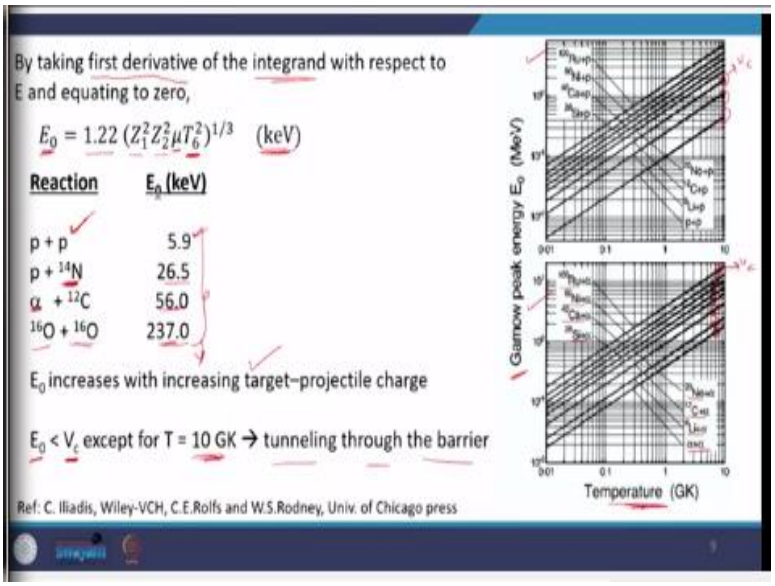

The first question let us take to calculate the centroid, the simple way is to take the first derivative of the integrand. This integrand involves 2 terms $e^{-2\pi\eta}$ and $e^{-E/kT}$. Take its first derivative, then equate to zero, that gives you the value of energy at which maximum probability is occurring. And equating to 0 after taking the first derivative you can derive this centroid of the Gamow peak as $E_0 = 1.22\ (Z_1^2 Z_2^2 \mu T_6^2)^{1/3}$ in keV.

For better understanding I will discuss some of the reactions as examples. For example, starting with the fundamental nuclear reaction p + p, E_0 is 5.9 keV. Here I must tell you to correlate with what I have said earlier, Coulomb barrier is 550 keV and earlier we have seen while discussing the effect of tunneling even at 1 keV the probability of tunneling is 10^{-10}.

It is small of course but there is a finite probability for fusion to take place at low energy 100 times less than the temperature corresponding to the Coulomb barrier, now the centroid energy is 5.9

keV. Now I increase the target charge, ^{14}N you see E^0 is increasing, now I am increasing the projectile charge from proton to α it is a 56 keV. And I am increasing the charges of both the target and projectile, it is 237 keV (refer to slide no 9).

So, the centroid of the Gamow peak increases with increasing the charge of target and projectile, this you have to understand and this number helps you to get convinced with this statement. Because this is one of the questions how centroid of the peak changes with the charges of target and projectile. Now remember E_0 is less than Coulomb barrier unless it reaches very high temperature.

So, except at high temperature it is because of the tunneling the fusion is taking place. This diagram (in the slide) should help you to understand the relation between Gamow peak centroid and temperature in GK. You see Rh + α, Ni + α, Ca + α, Si + α, ^{20}Ne + α, ^{12}C + α, ^{6}Li + α and $\alpha + \alpha$. For this case you can see that and for here also for protons and here for alpha particles you can see, can you see some circles here? I am drawing with red pen; this is a Coulomb barrier V c.

Somewhere here at near 10 GK, below this Coulomb barrier corresponding temperature the Gamow peak energy centroid is decreasing with decrease in the charges. So, for different reactions induced by protons and alpha, this picture should help you in understanding the relation between E_0 and the charges of target and projectile. So, in today's lecture I have covered the importance of astrophysical S-factor, it is the less dependence on energy of the system.

And this less dependence on the energy of the system helps us to come up with better estimate on the values of the cross section during extrapolation. If you do direct measurements there is no point in going for the extrapolation business. But most of the measurements have been carried out by doing the extrapolation, initially whatever energies are available you do the nuclear reactions and measure the values of cross section and S-factor then you do the extrapolation. To reduce the uncertainty in the cross-section value one has to go for the S-factor.

Then I have considered the concept of Gamow peak, the centroid of the peak is the most effective energy at which nuclear reactions are taking place within the stars when we discuss the charged particle induced reactions. And then I have shown you how the target charge and projectile charge decides the value of the centroid peak. And in the next lecture I will discuss more features of this Gamow peak and some more points in the next lecture. I hope you are following what I am trying to convey in the lectures, thank you so much for your attention.