Electromagnetism
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Lecture - 63
Divergence and curl of a generic magnetic field

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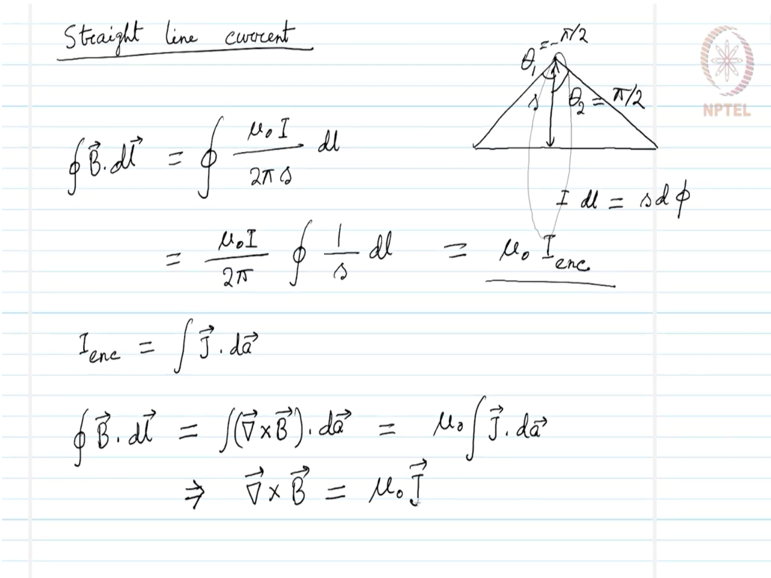

Now, if we do not consider this special case of a straight line current. If we consider a general situation, what do we arrive at? Do we arrive at some different value of curl of B, let us find that out.

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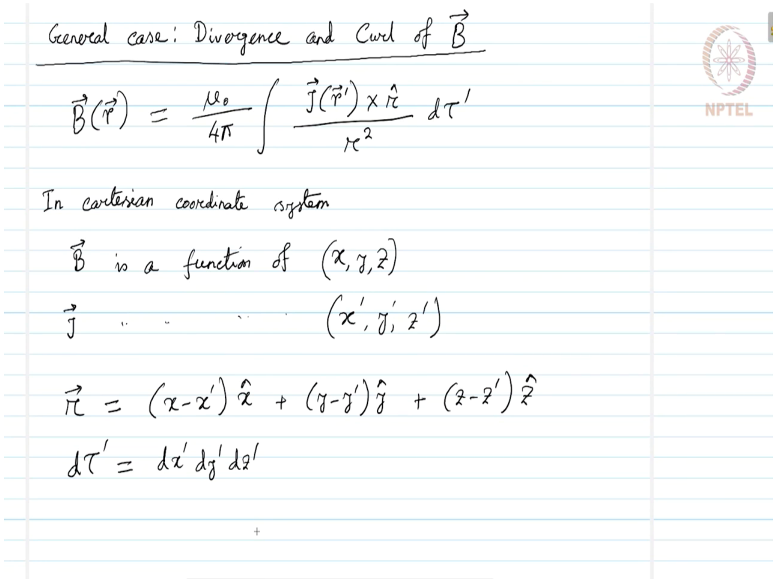

Let us also think about the divergence of the magnetic field that would also be very interesting to know. So, let us try to calculate the divergence and curl of B from a general consideration general expression of the magnetic field that we obtain from the Biot Savart law.

So, the Biot Savart law, gives us that the magnetic field. In the presence of a volume current can be expressed as mu naught over 4 pi integration over J that is the function of r prime. The source coordinate system cross r cap over r squared d tau prime. Now, if we consider Cartesian coordinate system we can explicitly write that B is the function of x y and z.

While, our current density volume current density J is a function of x prime y prime and z prime, the source coordinate system. And, we have this curly r vector that relates these two coordinate system and that is given as x minus x prime x cap, plus y minus y prime y cap plus z minus z prime z cap; I wrote this down explicitly. So, that we do t have any confusion while working out the divergence and curl in these systems.

How about d tau prime? That can be expressed as dx prime times dy prime times dz prime. So, our integral is over the primed coordinates and we will try to calculate the divergence and curl with respect to the unprimed coordinates. And how do we do that? Let us try doing that.

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$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

Product rule

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{(\vec{\nabla}\times\vec{J})}_{\to 0} - \vec{J}\cdot\underbrace{\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{\to 0}$$

$$\frac{\partial}{\partial x}J(x', y', z') = 0$$

$$\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec{\nabla}\times\vec{E} = \frac{q}{4\pi\epsilon_0}\underline{\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} = 0$$

First we try to calculate the divergence, the divergence of B, can be expressed as mu naught over 4 pi integration over the primed coordinate and divergence over the unprimed coordinate.

So, they will not interfere with each other, they will commute with each other we take the divergence inside the integral that is not a problem in this case. So, we take the divergence of

J cross r cap over r squared. And, integrate it with the volume element d tau prime. Now, we make use of a product rule that we have learnt in the mathematical background.

Divergence of J cross r cap over r squared, this is given as r cap over r squared dot the curl of J minus J dot the curl of r cap over r squared. Now, what is the curl of J? J is a function in the primed coordinate system and curl is an operator in the unprimed coordinate system. So, we will have terms like del del x of J that is a function that is not a function of x at all.

We will have this kind of terms and this kind of terms will go to 0 therefore, this quantity is 0. And, when we have that J does not depend on any unprimed coordinate we can set this one curl of J equal to 0. And, how about the curl of r cap over r squared how about this quantity? In case of electric field, we the expression of electric field was 1 over 4 pi epsilon naught.

Let me put it as q over 4 pi epsilon naught r cap over r squared for a point charge q at any position vector. And we know that the curl of the electric field equals q over 4 pi epsilon naught, curl of r cap over r squared and this quantity is 0 we k now this already. And that can only happen if this curl goes to 0; that means, this quantity is also 0.

So, both the terms in the curl of J cross r prime over r squared goes to 0. That means, the divergence of magnetic field must be 0; that means, the magnetic field is solenoidal that is always true. So, we have obtained one very important result, while in case of electric field. We had divergence of it that was the volume charge density over epsilon naught and here for magnetic field we have found the divergence to be 0.

In the previous example we have seen that curl of the magnetic field is non-zero and we want to explicitly work out the expression for curl of the magnetic field in a generic situation. Let us do that now.

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Curl

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\vec{J} \times \frac{\hat{r}}{r^2} \right) d\tau'$$

Product rule

$$\vec{\nabla} \times \left(\vec{J} \times \frac{\hat{r}}{r^2} \right) = \vec{J} \underbrace{\left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right)}_{4\pi\delta(\vec{r})} - \left(\vec{J} \cdot \vec{\nabla} \right) \frac{\hat{r}}{r^2}$$

Integrates to zero
verify.

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \, 4\pi\delta(\vec{r} - \vec{r}') \, d\tau'$$

$$= \mu_0 \vec{J}(\vec{r})$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}} \quad \text{Ampere's law}$$

Let us write down the expression for curl of the magnetic field, which will be from considering the Biot Savart law expression for magnetic field. Due to a volume current, mu naught over 4 pi integration curl of J cross r cap over r squared d tau prime.

Now, we make use of another product rule with another product rule. We can write that curl of J cross r cap over r squared this quantity can be written as J times the divergence of r cap, over r squared minus J dot divergence r cap over r squared. There were terms involving derivatives of J that we have already ignored because J is the function of the primed coordinates and its derivative in unprimed coordinates are not going to sustain.

There would have been 4 terms we have kept only 2 by ignoring the other terms that will certainly go to 0. Now, we have divergence over r cap divergence of r cap over r squared and we know that this quantity becomes 4 pi times Dirac delta. So, this gives us let me write it in a different colour, this brings us 4 pi Dirac delta curly r vector. And how about the second term the second term once.

So, in order to find the expression for the curl of B, we will have to integrate over the second term. And once we integrate the second term goes to 0; please verify this. So, what we are left with is curl of B equals mu naught over 4 pi integration J that is the function of r prime times 4 pi Dirac delta curly r vector.

So, let me write curly r vector in its full glory that is r vector minus r prime vector and this is integrated over d tau prime. If we do that then this integration is going to give us a result provided r prime equals r upon this integral and that is going to give us. So, 4 pi and 4 pi here and there will cancel that will give us mu naught J; that is a function of r vector and not r prime vector that is the result; we will obtain from this. And, this result that curl of B is mu naught J is known as the Amperes law.