Electromagnetism
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Lecture - 40
Boundary conditions and the uniqueness theorems

Now, let us consider Boundary Conditions and Uniqueness Theorem.

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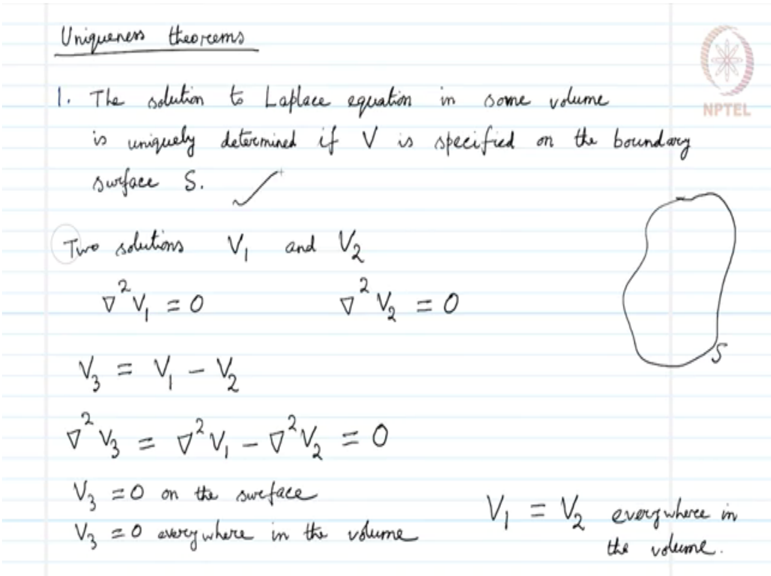

So, we have in the form of Laplace equation a second order differential equation at hand and in order to solve that second order differential equation we, need two boundary conditions at least. And the uniqueness theorems are about the uniqueness of the solutions to this Laplace equation that we will find.

We have two uniqueness theorems, the first theorem says that the solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S. So, let us try to understand what it means.

Suppose, we have two solutions to Laplace equation inside the volume, let us consider this kind of a volume arbitrary and it is bound by a surface S. Let us not call the volume V here because that way we will confuse with the potential. So, we will try to prove this by the method of contradiction. How does that work? We will first suppose that there were two solutions to Laplace equation inside this volume. The solutions were V 1 and V 2. So, the solutions were not unique, we will first assume that and by the method of contradiction, we would prove that our idea contradicts itself and we will show the uniqueness. Let us try doing that.

Then the Laplace equation for the first potential, first solution is del square V 1 equals 0, Laplace equation for the second potential is del square V 2 equals 0 and both of them assume the given value on the surface, on the surface we know the value of the potentials. So, they take the same value on the surface each point of the surface.

Then we can define a new quantity V 3 taking the difference of these two solutions. So, V 3 is V 1 minus V 2 and because this is the difference between these two potentials, now let us consider these two solutions. Let us consider taking the Laplacian of V 3 which will be nothing else, but del square V 1 minus del square V 2 and because del square V 1 and del square V 2 are individually 0, this quantity has to be 0. If this quantity is 0 and because V 1 and V 2 take the same value on the surface of this volume whatever we have considered, then V 3 is a solution to the Laplace equation. It will take the value 0 everywhere on this surface S that we have considered. V 3 equals 0 on the surface.

Now, if V 3 is equal 0 on the surface and this V 3 is a solution to the Laplace equation, then V 3 does not allow any maximum or minimum within that volume and if that is the fact, then V 3 will be 0 inside the inside everywhere in the volume. And V 3 will be 0 everywhere in the volume means V 1 equals V 2 everywhere in the volume. And that means what we have considered to begin with that V 1 and V 2 were two different solutions, we have already contradicted that. So, V 1 and V 2 could not be two different solutions to the Laplace equation, they are the same solution basically and that way we have prove the first uniqueness theorem. Let us consider an example of this.

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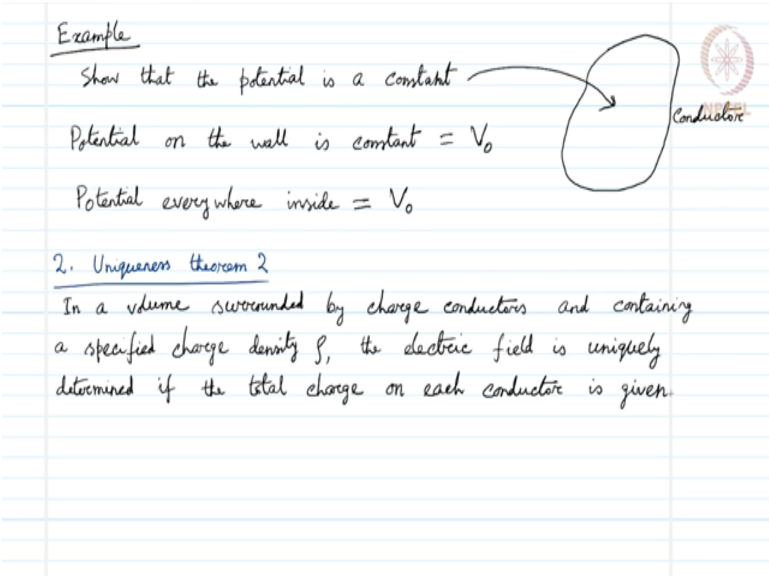

If we consider an enclosure completely surrounded by a conductor; this is the conductor we have drawn. Let us assume its 3D and it encloses a volume, so within that enclosure we are asked to show that the potential is a constant here. How do we do that? In case of spherical shell we have seen that the potential inside the shell is constant, and here we are supposed to find that out in case of an enclosure of arbitrary shape enclosed by a conductor. And just by applying Laplace equation nothing else. That means, there is no source charge, there is no charge within this inside the conductor. Let us try doing it.

The potential on the conducting wall is some constant. Why is it a constant? Because we have a conductor surface and if the potential is not constant there would be flow of charge until it becomes a constant. So, we can consider that on the surface of a conductor the potential will always be constant. Let us consider that is V naught.

The potential inside this conductor is a function that satisfies Laplace equation and that has a value V naught at the boundary because Laplace the solution to a Laplace equation does not allow any maximum or minimum therefore, the potential inside that volume that will also be V naught. Everywhere inside that is equals V naught. There is no other possibility.

Now, let us consider the second uniqueness theorem, the statement of it. It says in a volume surrounded by charged conductors and containing a specified charge density rho, the electric field is uniquely determined if the total charge on each conductor is given. We will not prove this. This is something we will just state and move on to something more interesting with the help of the first uniqueness theorem of Laplace equation.