Electromagnetism
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Lecture – 21
Introduction to Dirac delta function

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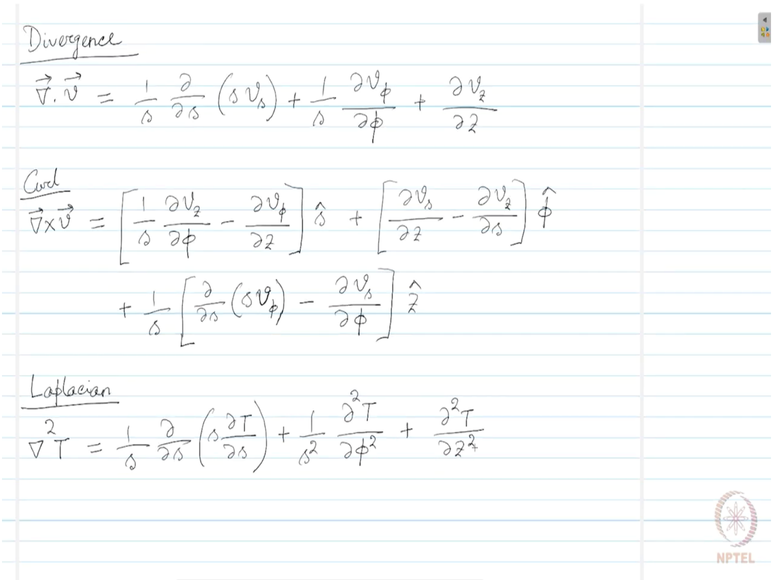

Now, let us consider an interesting example.

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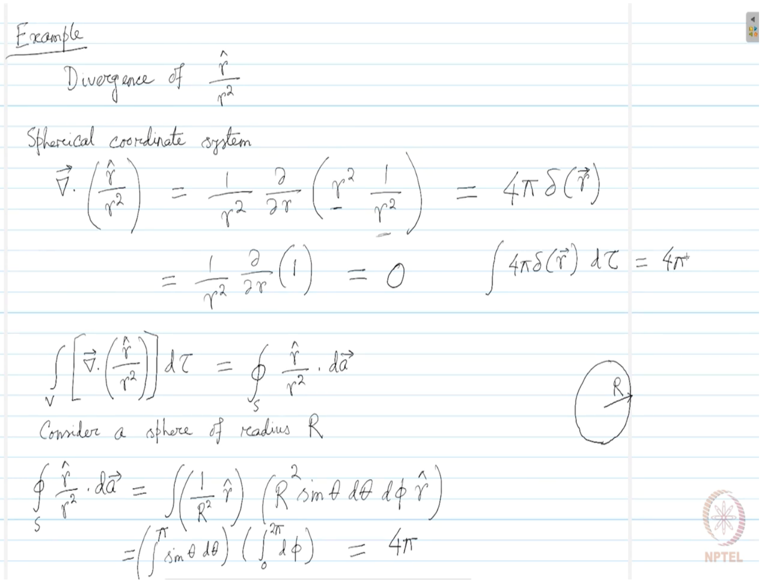

Let us consider in spherical coordinate system the divergence of the vector r cap over r squared. Actually it can be calculated in any coordinate system because we are involving position vector and it is valid in every coordinate system, but it is easy it is the easiest in spherical coordinate system that is why we will move to spherical coordinate system to calculate the divergence of r cap over r squared. How do we calculate this? In spherical coordinate system what is the expression for divergence?

Because we have only the r component here if we move into spherical coordinate system that is how the problem gets simplified. So, we will have only the r component in the expression for divergence. So, we will write it down as divergence of r cap over r squared as 1 over r squared del del r r square times 1 over r squared, this would be the divergence. So, r squared here and r squared here these two cancel each other; if that happens we are left with 1 over r squared del del r of 1 which is a constant so, it gives us 0.

Now, let us have a look at it from a different viewpoint. We have divergence of a vector. Now, if we perform a volume integral on the divergence of this vector over any arbitrary volume, then that will be according to the divergence theorem equal to the surface integral of the vector itself over the surface that binds the volume. In that situation we can write it as volume integral of the divergence of r cap over r squared, multiplied by the volume element d tau is the closed surface integral the surface that closes this volume encloses this volume r cap over r squared dot da.

Now, let us consider a volume and the surface in such a way let us choose a volume and a surface in such a way that our calculation is calculation becomes simple. We can actually choose any arbitrary volume and the surface binding that volume, that is all we need. So, if we consider a sphere of radius capital R, let us draw it here. We have this kind of a sphere the radius is capital R and we will perform these two integrals over that surface. Now, we have already calculated the divergence of this vector to be 0, so, we know that the volume integral is going to be 0.

Now, we have to calculate the surface integral, if we have to calculate the surface integral let us write it down this way, r cap over r squared dot da; this is given as integration. So, it is the spherical surface, so, we will put the expression of da for the spherical surface. It is after putting the value for r that is smaller that is capital R because we are restricted; we have restricted ourselves on the spherical surface r cap times capital R squared sin theta d theta d phi r cap. This is the integral that we are supposed to perform over appropriate ranges of r and so, r sorry over appropriate ranges of theta and phi, R is a constant that is capital R.

So, we will have the integration over theta ranging from 0 to pi sin theta d theta multiplied with the integral over phi from 0 to 2 pi d phi and that gives us 4 pi. What did we get from the expression for divergence in spherical coordinate system? We found that the divergence of this vector r cap over r squared was 0. So, the volume integral over any arbitrary volume of this divergence of the vector must be 0 and the divergence theorem tells us that a closed surface integral enclosing that particular volume of the vector itself should give us the same value as the volume integral over the divergence of the vector, but it is it gives us 4 pi the surface integral.

So, how do we reconcile these two things? Is the divergence theorem wrong? Must not be. Here in case of calculating divergence what we have done is we have cancelled r squared from the numerator with the r squared from the denominator, but this cancellation is not valid at r equals 0. So, what is valid at r equals 0? At r equals 0 we must have a different kind of function actually Dirac solved this problem by introducing a Dirac delta function.

So, according to Dirac this divergence must be 4 pi delta r and what does that mean? How do we define Dirac delta function and how it works to give this answer and that reconciles the surface integral let us see.

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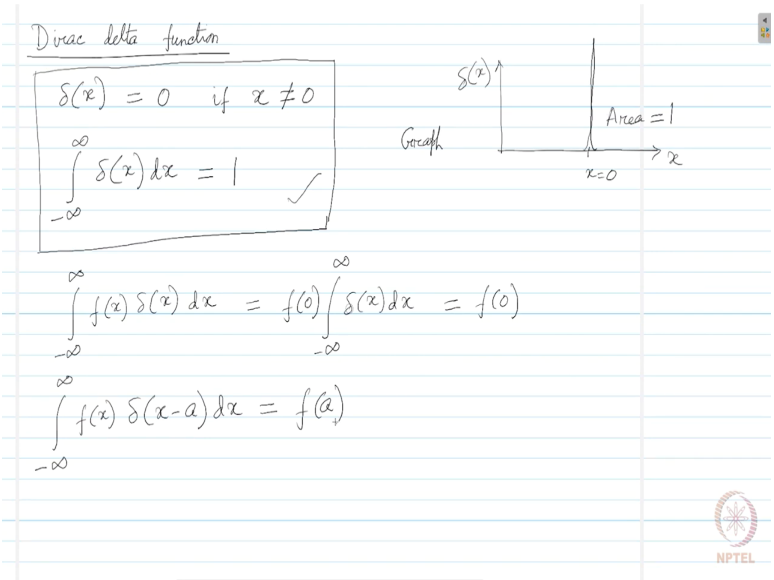

The Dirac delta function in one dimension is defined as delta x equals 0 if x naught equals 0. So, if the argument of the function is not 0, then the function returns the value 0 and if we integrate over minus infinity to plus infinity delta x dx then this integral gives as 1; that means, whenever we get a value of x in the argument of the Dirac delta that becomes 0, then this kind of an integral gives us the value 1. Only the integral gives us 1, that does not mean that the function itself becomes 1.

So, what should be the value of the function? If we want to represent it graphically it should look something like. So, if this is the x axis and if this is the function axis, x is let us say 0 here, then the function would have a singularity at x equals 0 the function would look something like this graphically. Although this is not a true representation the true representation is the definition that is here what we have drawn on the right hand side is just a naive way of understanding this nothing else.

So, we must have the area under this curve equal to 1 and if this is the definition, then we can write that integration over minus infinity to infinity a function f x delta x dx this will give us the value of the function at 0 multiplied with the integration over the Dirac delta function from minus infinity to infinity this that is. So, this integral becomes 1, we are left with the value of the function at 0 that is what we will have and this is the proper definition of the Dirac delta function.

We can also write that minus from minus infinity to infinity integral over f x delta x minus a, but the argument of the delta function will go to 0 when x becomes a dx, this quantity is f a. We can generalize the delta function this way. Now, if we define the delta function this way, then we can go back and see here that if we define if we equate the divergence of r squared oh sorry r cap over r squared to 4 pi Dirac delta, then when we perform the volume integral over 4 pi Dirac delta we will actually have 4 pi as the result because the Dirac delta integration over the Dirac delta in the range where and the argument of delta becomes 0 at least once will give us the value 1. So, we will be left with 4 pi.

So, using this Dirac delta function, Dirac reconciled the two approaches; one is calculating the divergence and then integral performing a volume integral and the other one is directly performing a surface integral over the volume sorry over the surface that encloses the volume in the first case. So, this has been reconciled.

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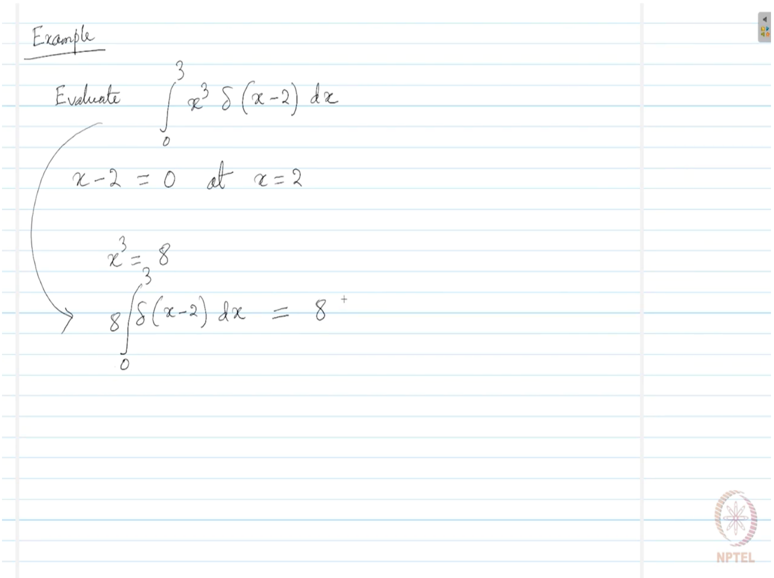

And, now let us consider an example of Dirac delta. We want to evaluate the integral from 0 to 3 x cubed times Dirac delta x minus 2 dx. So, where does the argument of Dirac delta go to 0? x minus 2 is the argument of Dirac delta that is 0 at x equals 2 and is x equals 2 within the range of the integration? Yes, the integration ranges from 0 to 3 so, x equals 2 is included. Therefore, we will have x cubed equals 8 and then the integral will become 8 delta x minus 2 dx over this range and this will become 1 at x equals 2. So, we will be left with 8 that will be the answer of this problem.