Physics through Computational Thinking
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Non-dimensionalization and visual thinking

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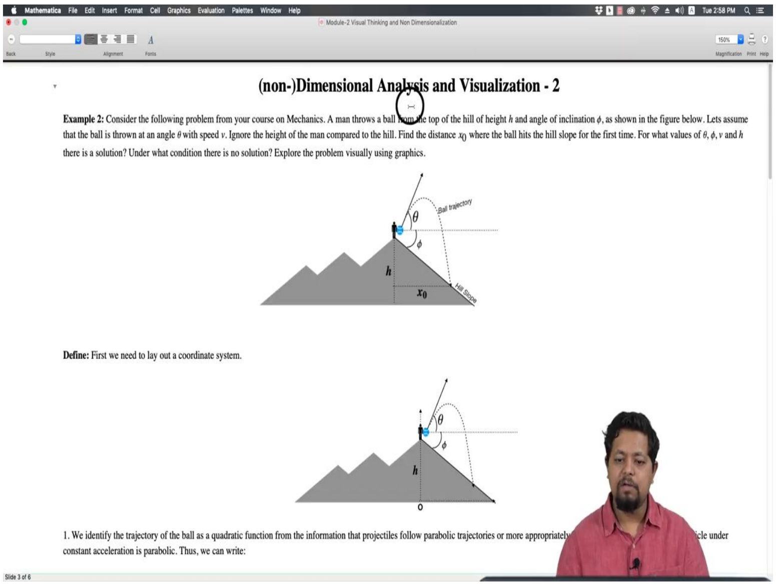

Coming to the second example of Non-Dimensionalization and Visualization, let us consider this problem. You may have seen this problem before from a course in mechanics. But if not, does not matter, it is a simple enough problem to solve. And we will take up this example as our next problem.

So, let us consider a hill and on top of the hill, let us say there is a man standing on top of the hill. The man holds a ball and he throws a ball. At some angle θ, the hill makes the angle ϕ with the horizontal. So, hill has a slope, the slope makes an angle ϕ with the horizontal, man throws a ball. Ball makes a parabolic trajectory and it falls down and hits somewhere on the hill slope downwards.

The problem is to find out what is the distance, what is the horizontal distance from the man's position that is right underneath the hilltop all the way to the point where the ball hit the slope.

So, in order to solve this problem, first we need to do is, define the problem that is layout the coordinate system, write down the basic physics components then we will translate it, we will take the equations that we have obtained from physics considerations, we will translate them into non dimensional form so that we can take then they become mathematical equations without any dimensions in them and we will analyze those mathematical equations, visualize them, and we will find, we will learn something about the solution from our visual results. Okay, excellent.

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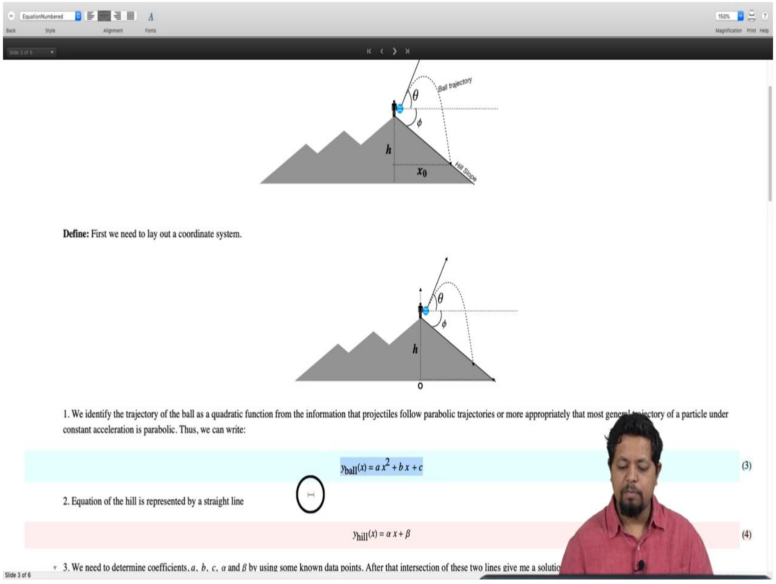

So, laying down the coordinate system, we need to first lay down the origin. So, my second picture over here shows the origin, let us take the origin right underneath the man at the bottom of the hill. This dotted line shows the level of the ground. This is the line of horizontal, hill makes the angle ϕ with the horizontal, we throw the ball at the angle θ, we will ignore the height of the man compared to the height of the hill.

Let us assume the hill is much much taller than the person. As the man throws the ball, it may so happen that the ball lands up somewhere away from the hill and then there is no solution. So, we have to find out the condition for which we will get a solution, that means we have to find out the relationship between the angle θ, ϕ, h so that the ball hits somewhere down the hill.

So, to do that, first we know, we start with some of the known facts that the trajectory of the ball is going to be parabolic because it is the constant acceleration along the negative y-direction.

The acceleration due to gravity is constant so there is a constant acceleration problem. For constant acceleration problems, the trajectory is always a parabola. So, we will take that trajectory as a parabola. And we will establish that y_{ball} that is the y coordinate of the ball is given by $a\,x^2 + b\,x + c$, where a, b, c are some constants. So, this is the equation of the trajectory of the ball, y_{ball} as a function of x is $a\,x^2 + b\,x + c$.

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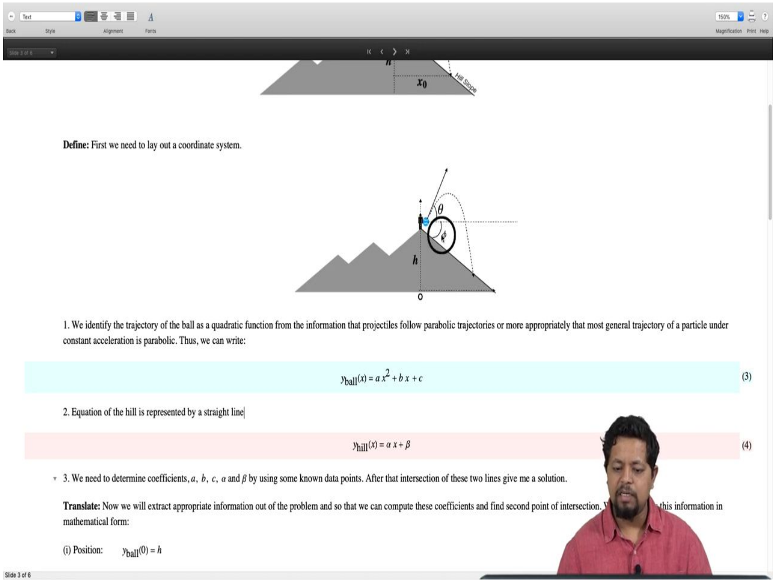

Now, this trajectory intersects somewhere with the hill, so we will also look at the equation of the slope of the hill, the equation of the slope of the hill is represented by a straight line and that we will take y_{hill}, that is the height of the hill as a function of the x-coordinate on this side of the hill as $\alpha\,x + \beta$, where α is the slope and β is some constant that we need to figure out. So, these are the 2 equations, this is the equation of trajectory and this is the equation of the slope.

Now, I have to find out: for what values of x, these 2 curves will intersect with each other. Of course, we already know 1 solution, that is when the ball is right on top of the hill, which corresponds to $y = h$ and $x = 0$. So, when $y = h$ and $x = 0$, the ball is placed at the top of the hill and there is a point of intersection for the slope of the hill and the trajectory of the ball.

The next point of intersection is what we need to find out which is this one and this may sometimes exist or may not exist depending on the various parameters of the problem. That is the height of the ball, the velocity of the ball, the angle θ and the angle ϕ.

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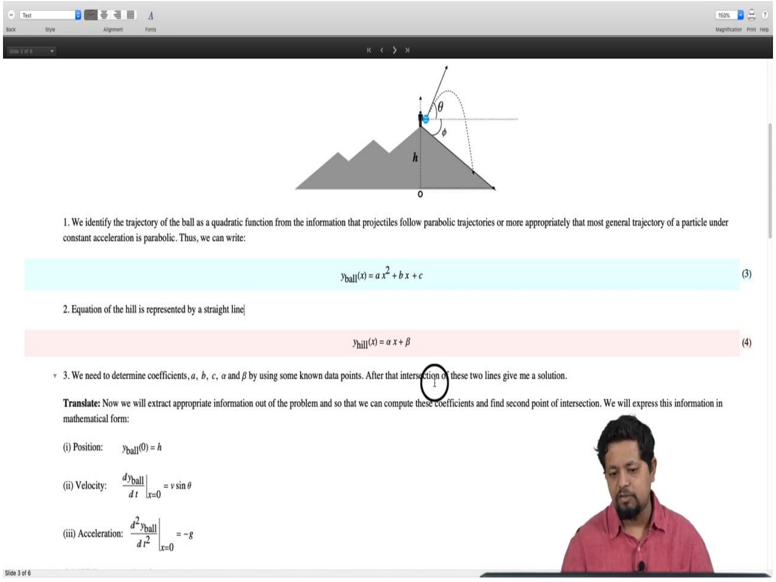

In order to find out these coefficients a, b, c, α and β, we will find out the physics constraints that we already know about this problem. And we will use those physics constraints to find out the coefficients a, b, c, α and β.

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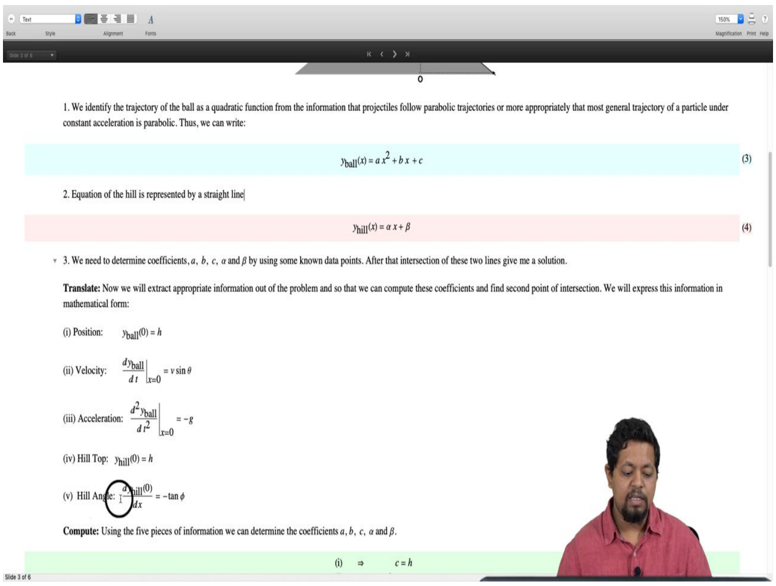

That brings us to the translate stage. We want to translate this into a mathematics problem. So, let us add a few more physics inputs. Here I have taken 5 conditions, because I have got 5 constants a, b, c, α and β. So, here I have taken 5 different conditions. First one is the position of the ball. I know that y_{ball} at $x = 0$ is h so that is one condition.

I also know that when I throw the ball at the top of the hill $d\,y_{\text{ball}}/d\,t$ at $x = 0$ is $v\,\sin\theta$, because I know the angle θ. I also know that $d\,y_{\text{ball}}/d\,t^2$, that is acceleration of the ball in the y-direction is $-g$, which is always $-g$ independent of what the position is and for the hill, I know the top of the hill is $y_{\text{hill}}(0) = h$, and the hill angle is $d\,y_{\text{hill}}/d\,x = -\tan\phi$.

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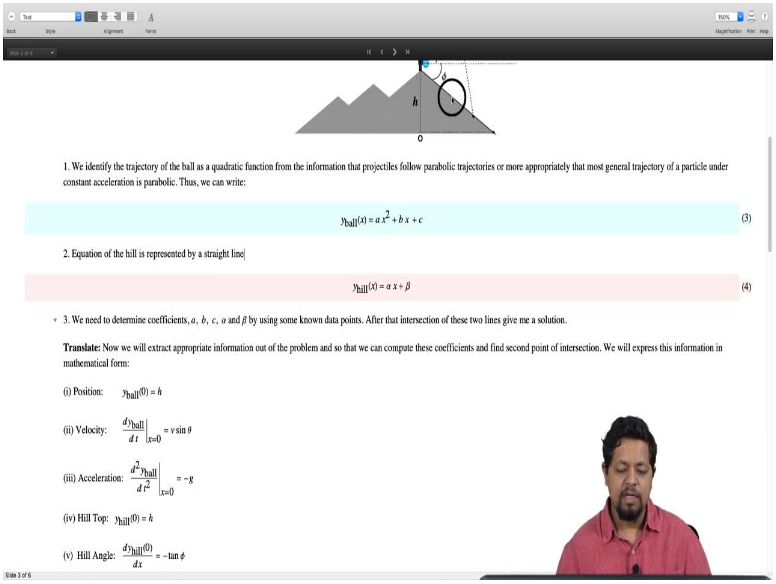

So at $x = 0$, we have top of the hill. So that is why $y_{\text{hill}}(0) = h$, and $d\,y/d\,x$, that is the slope of the hill is equal to $-\tan\phi$. Okay, so will take these 5 pieces of information and use them to determine a, b, c, α and β. Let us go ahead and solve this with the paper and pen. If you want to solve it yourself and take a pause, go ahead do that.

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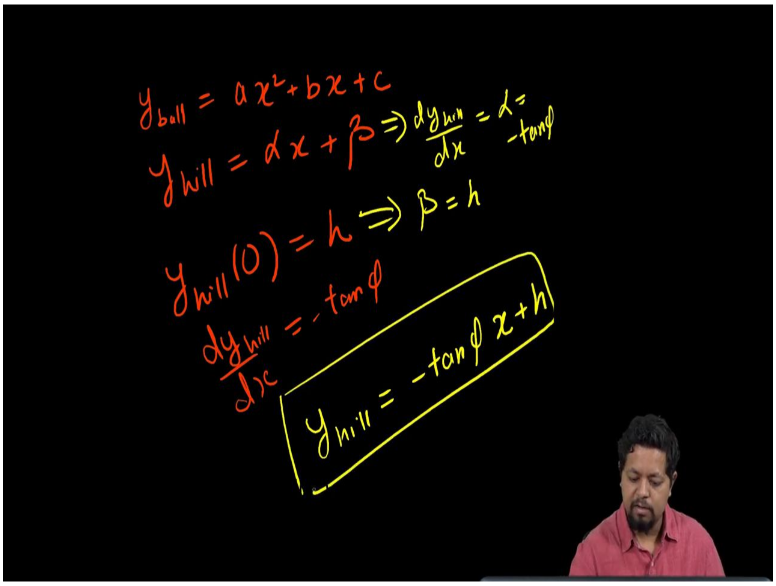

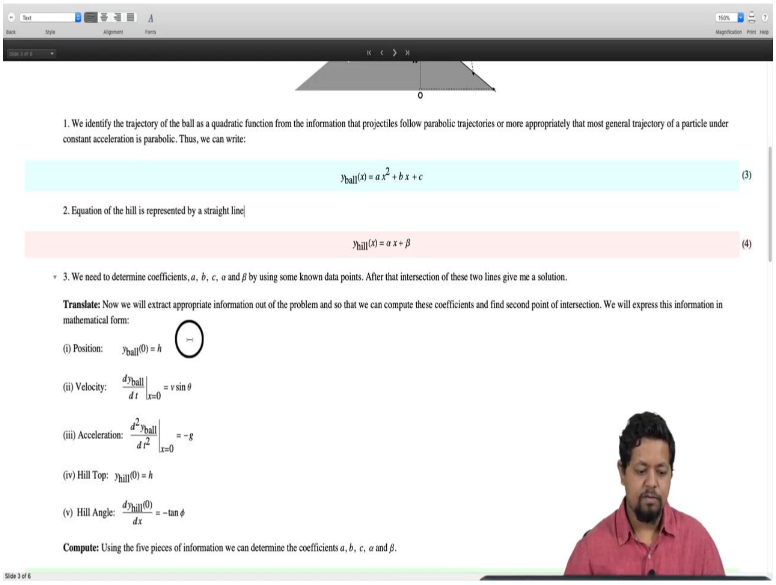

Alright. So, our equations were y_{ball} is equal to $a x^2 + b x + c$ and y_{hill} equal to $\alpha x + \beta$ using the condition that y_{hill} at $x = 0$ is h and $d\, y_{hill}/d\, x = -\tan\phi$, we can go ahead and fix α and β first using this we can calculate $d\, y_{hill}/d\, x$ and that is just α which must be equal to $-\tan\phi$. From this condition we get $y_{hill} = h$ that means, $\beta = h$.

So, therefore $y_{hill} = -\tan\phi\, x + h$. So that is the equation for y_{hill}. I want to use other pieces of information. The other 3 conditions to find out a, b, c. Okay, let us go ahead and solve for the second equation.

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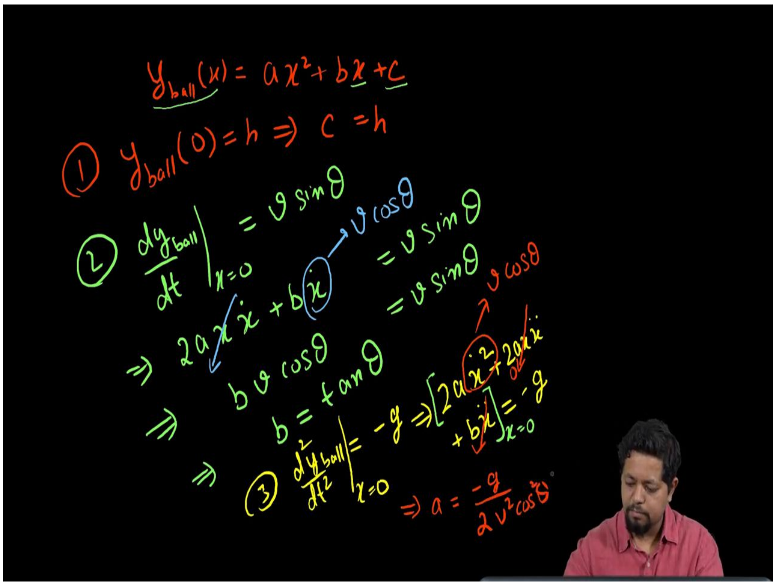

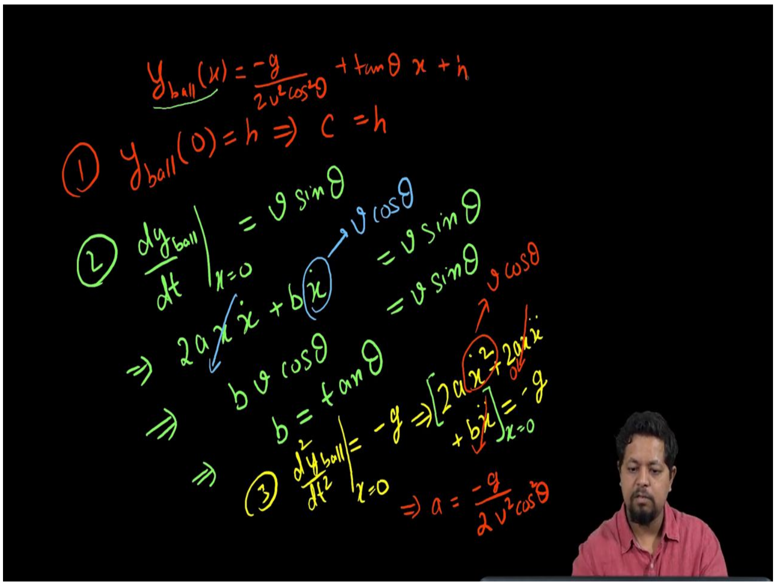

Equation of y_{ball} as a function of x is $a\,x^2 + b\,x + c$. My 3 conditions are y_{ball} at $x = 0$ is h, this implies that $c = h$. I take the second condition which says that $d\,y_{\text{ball}}/d\,t$ at $x = 0$ is $v\,\sin\theta$, we evaluate left hand side $d\,y/d\,t$, we get $2a\,x\dot{x} + b\dot{x} + c = v\sin\theta$.

If we substitute $x = 0$, and that gives rid of this term. And over here we substitute $\dot{x} = v\cos\theta$, which is the velocity of the ball or horizontal direction of the velocity of the, horizontal component of the velocity ball at $x = 0$.

Therefore, we get from here $b\,v\cos\theta + c$, value of $c = h$, so this is equal to $v\ \sin\theta$. I am sorry there is a mistake here, I should have ,when I take $d\,x/d\,t$, there is no c. So, we get rid of that, we get rid of that and so we just have $b\,v\cos\theta = v\sin\theta$ and that gives me $b = \tan\theta$.

At this point, you can verify that $b = \tan\theta$ is dimensionless, y has dimensions. y has dimensions of length, x has dimensions of length, so b should be dimensionless and that is what we find. Similarly, c has dimensions of length and we find $c = h$ which has dimensions of length so these are some of the cross checks we can make during the calculation.

Now, using the third condition it says $d\,y/d\,t^2 = -g$ at all x's but definitely at $x = 0$ so we can calculate $d\,y/d\,t$ of the ball, we already have calculate $d\,y/d\,t$ as $2a\,x\,\dot{x} + b\dot{x}$ taking another derivative of that we get $2a\,\dot{x}^2 + 2a\,x\,\ddot{x} + b\ddot{x} = -g$.

Evaluating the left hand side at $x = 0$ simplifies a few things. First of all, this term vanishes then also because $\ddot{x}$ is always 0 there is no acceleration in the $\ddot{x}$ this term is 0 and I am left with a equal to $-g$ divided by $2a\,\dot{x}^2$ evaluated at evaluated $x = 0$, $\dot{x} = v\cos\theta$ so I get $-g/(2\,v^2\cos^2\theta)$ okay, so putting it all together, we have got $-g/(2\,v^2\cos^2\theta)$ $+b\,x$ which is $\tan\theta\,x + h$. Okay, so let us go ahead and now look at those 2 equations.

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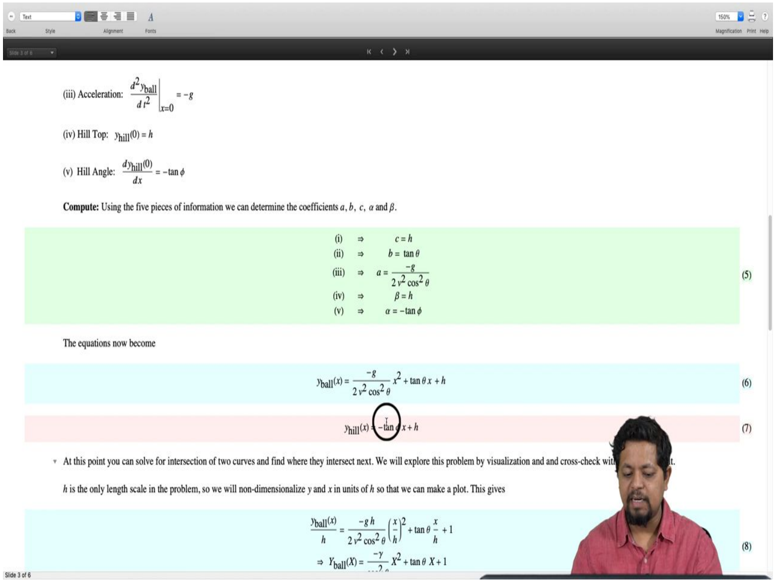

Putting these conditions, I found that y_{ball} is $-g/(2v^2\cos^2\theta)x^2 + \tan\theta\, x + h$. Similarly, for y_{hill} I have got $-\tan\phi\, x + h$. Now these 2 are equations which contains dimensions in them. The next thing I want to do is, I want to non dimensionalize these equations. In order to non dimensionalize these equations, I want to pull out the dimensions of all the quantities.

So, y is measured in dimensions of length, x is measured in dimensions of length and both of them, I can non dimensionalize by a natural dimension of length that is present in the problem, which is simply h, I can measure both x and y in units of h.

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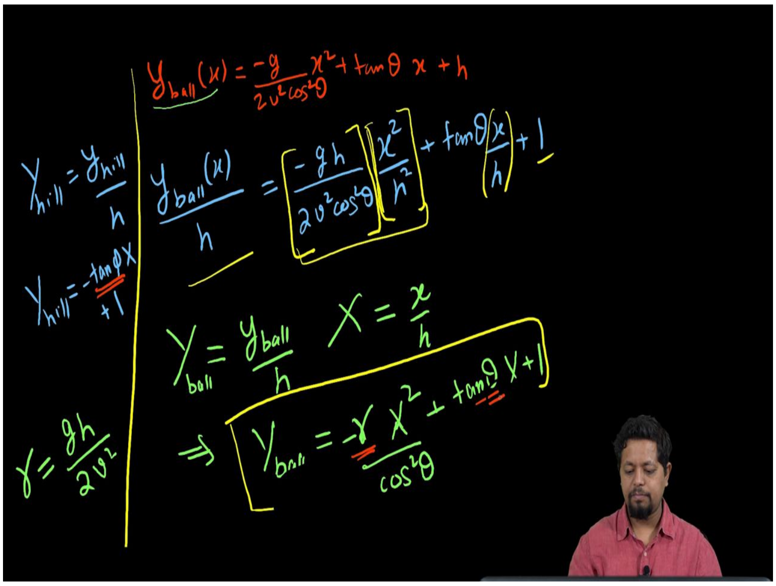

Which means, erasing all of that. I can go ahead and divide y_{ball} of x by h and when I do that there should be x^2 here, when I do that, I get $-g/(2v^2\,\cos^2\theta)x^2/h + \tan\theta\, x/h + 1$.

Now, it is interesting to note that, you know, this term is dimensionless. This term is dimensionless. And this term is dimensionless so this term should also be dimensionless. But I want to represent x also in terms of h, so what I will do is, because I have got x^2 here, I will make this h^2 and multiplying h over here.

Now, given that this term is dimensionless, x by h is dimensionless. I also conclude that this must be dimensionless and in fact dimensions of v^2 and $g\,h$ are same. So indeed, this is dimensionless. What I will do now is I will define capital Y or block Y as y_{ball}/h and capital X as x/h. In terms of this, my equation simply becomes y_{ball} equal to this constant which I am going to define as γ, so I get $-\gamma\,X^2 + \tan\theta$ which is already dimensionless times $X + 1$.

So, that is my non Non Dimensionalize equation for the trajectory of the ball. And similarly, I can Non Dimensionalize the equation for for hill. So, I will define Y_{hill} as y_{hill}/h, then I simply get $Y_{\text{hill}} = -\tan\phi\, X + 1$, sorry, I should make one correction. I did not define γ sorry, so I should define γ here and γ is a dimensionless constant.

So, I will define γ as $g\,h/2v^2$ then I will have a $\cos^2\theta$ here, theta is another different parameter. So, I want to keep it as a separate parameter do not want to absorb θ inside γ. γ is just a measurement of h/v, g is a universal constant, h is the height of the hill and v is the speed with which we throw the ball and $g\,h/v^2$ is dimensionless quantity.

So, v is a measure of the speed with which we should throw the ball or the inverse square of the velocity of the ball. So, I have got 3 parameters in this problem, the 3 parameters that I have in this problem are $-\tan\phi$, γ, and θ. So, ϕ, θ and γ are 3 of the constants that are given to me in the problem. And depending on what the values of these 3 constants are, my solutions can change.

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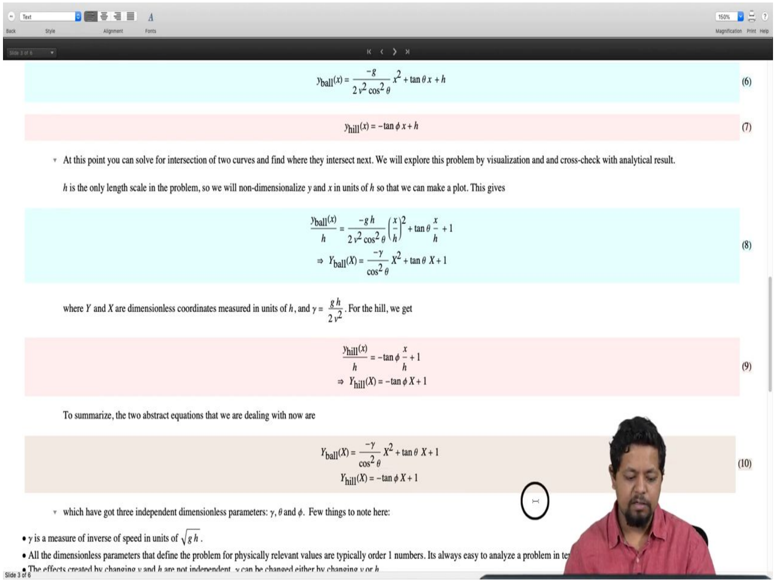

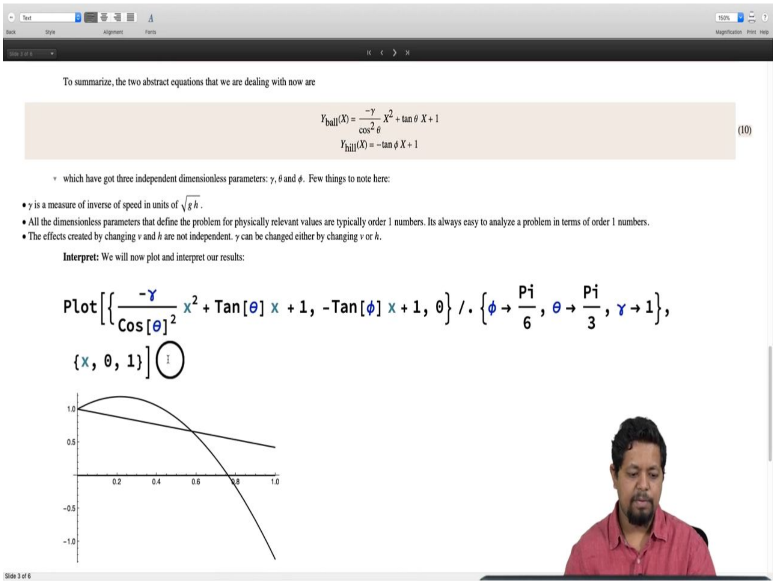

Let us go ahead and analyze that here is my final equation for Y_{ball} and here is my equation for Y_{hill} and now since these are 2 abstract equations without any dimensions, y is dimensionless, x is dimensionless, γ, θ and ϕ are parameters, I can go ahead and plot this by using the plot command like this.

So, I make a plot for this equation for Y_{ball} and for Y_{hill} and the ground level which is $y = 0$ that is the x-axis and I choose some values of the parameters ϕ and θ, I set

$\phi = \pi/6$ and $\theta = \pi/3$ and I set $\gamma = 1$, remember that always the dimensionless parameters of the problem once you Non Dimensionalize are always order 1 quantities.

So, you will always tend to pick some order 1 numbers for these quantities. That is why I have said $\gamma = 1$, too small values of γ or too large values of γ will end up giving you unphysical situations.

If I plot all of these 3 things with these values of θ, ϕ and γ, I get these 3 curves. And for this particular case, these 3 choices, I do get a solution.

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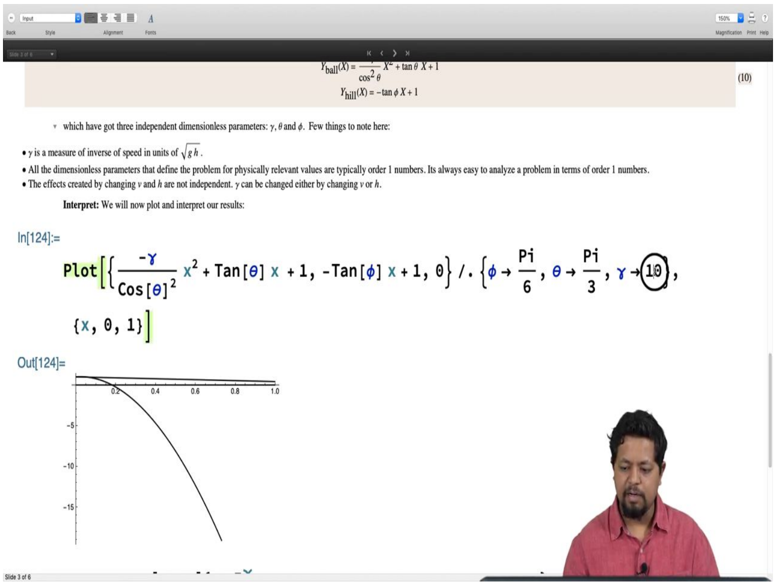

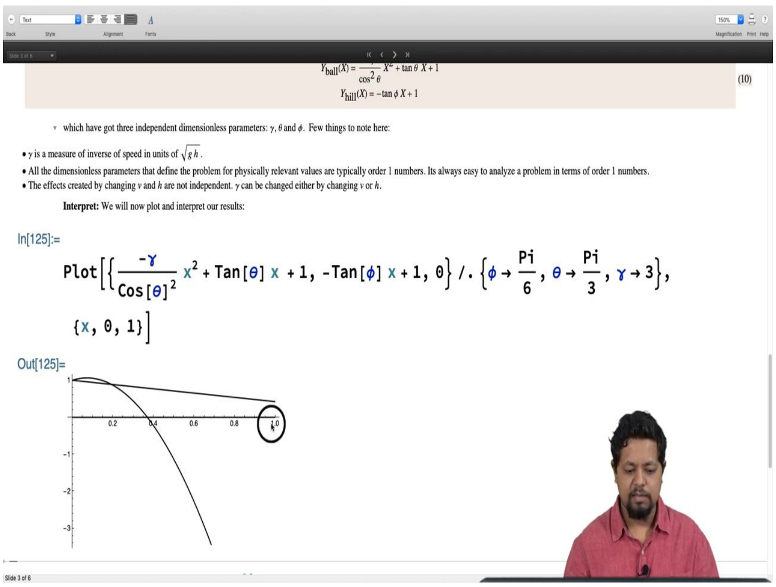

If I change γ, you see that I may end up getting no solution. And $\gamma = 10$ becomes very large. So, at some point, you know, $\gamma = 3$ corresponds to a situation like this.

Now, it may appear to you that the angle of the hill is changing, but really is the relative value of these parameters that is changing and it appears in the plot that you are changing the value of the hill but we have kept the value of the hill, angle of the hill as constant and we are changing gamma, which is the essentially changing the speed of the ball.

So, large value of γ means small value of the speed or a large value of the height of the hill. And that shows you how the solution changes as you increase γ.

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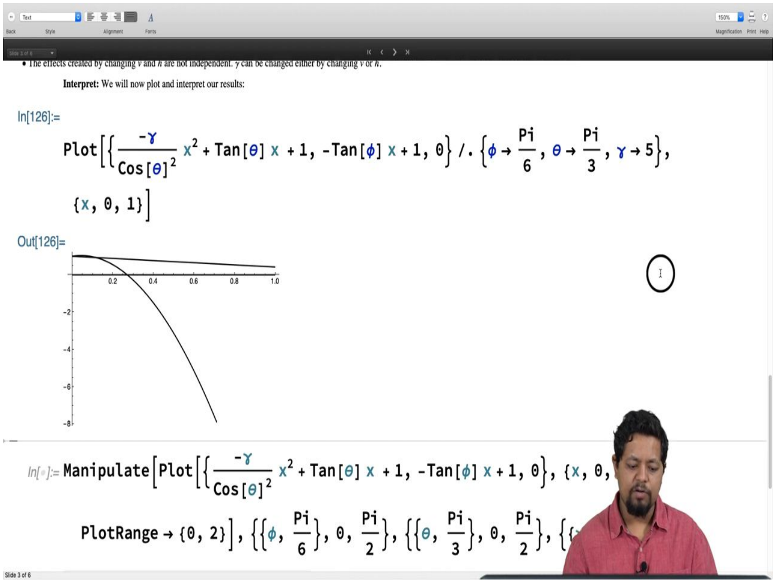

Now, we can actually go ahead and analyze this by manipulate command, by manipulating all 3 parameters and find out, when do we have a solution and when we do not have a solution.

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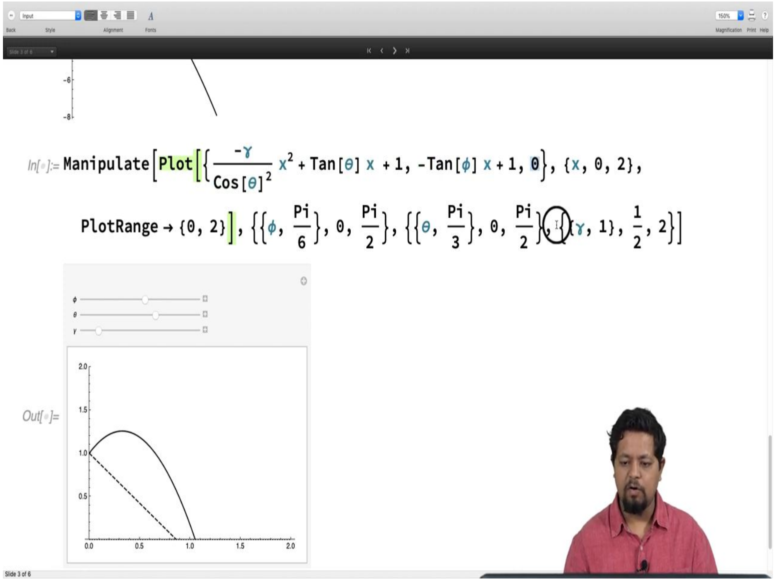

In order to manipulate this, watch the lecture on how to use the manipulate command. And go ahead and make a plot of these 3 functions, Y of the ball, Y of the hill, and the ground level when I make a plot, do a manipulate of all 3 of them.

I have set some ranges of the parameters for θ, ϕ and γ, over here. For example, γ, I have set it from ½ , and I have taken the canonical value as 1, I can go ahead and move the sliders now. And you can see what happens when I move the slider.

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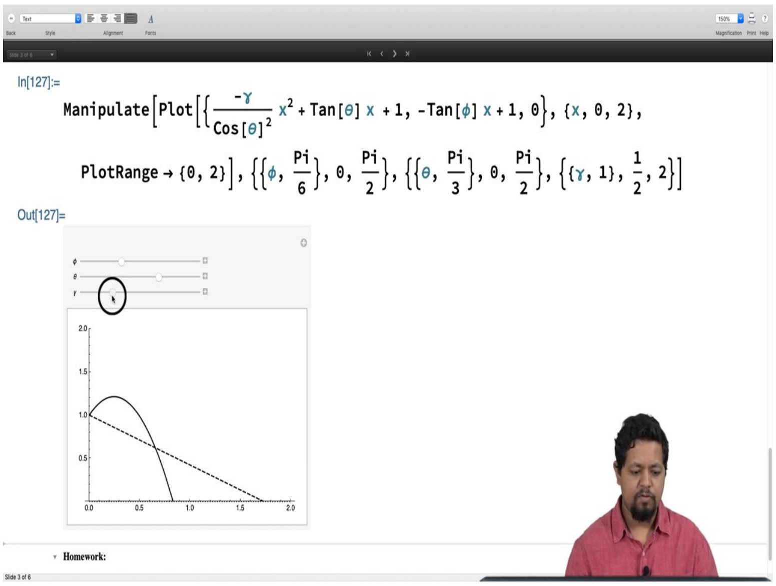

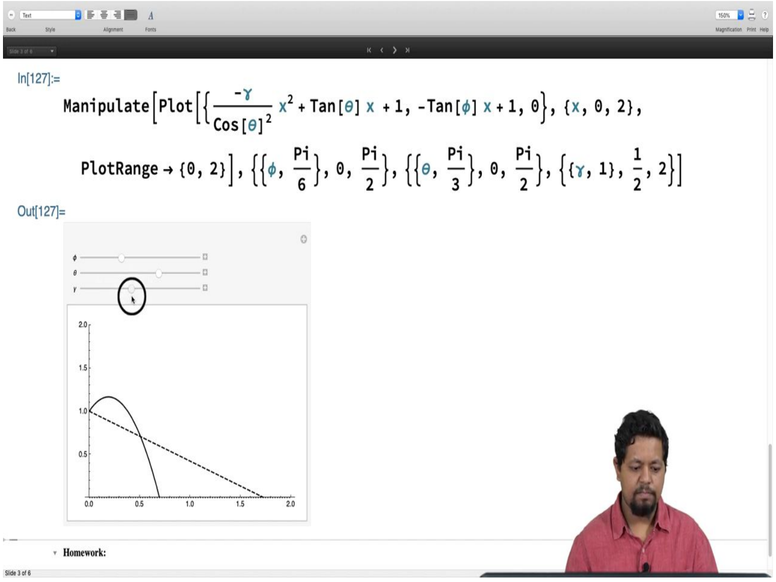

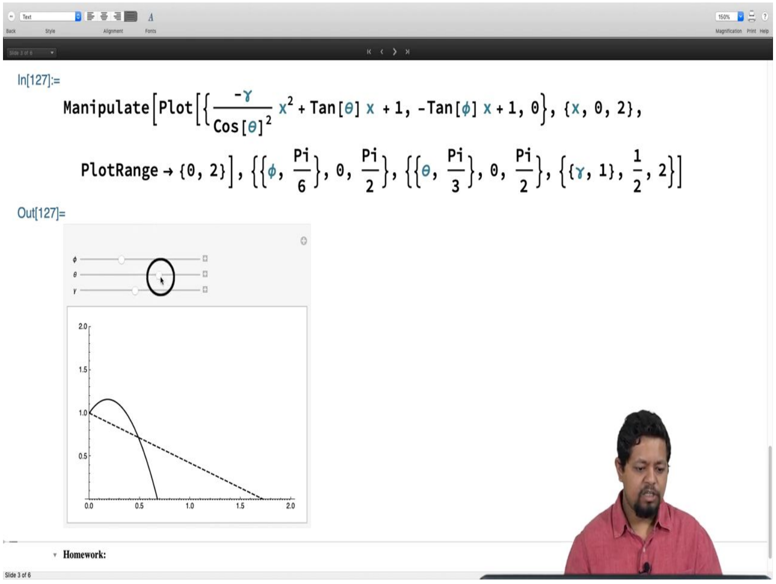

As I move γ, the velocity of the ball changes and as a consequence, the trajectory of the ball changes and you can see how the solution that is where the ball falls on the hill changes.

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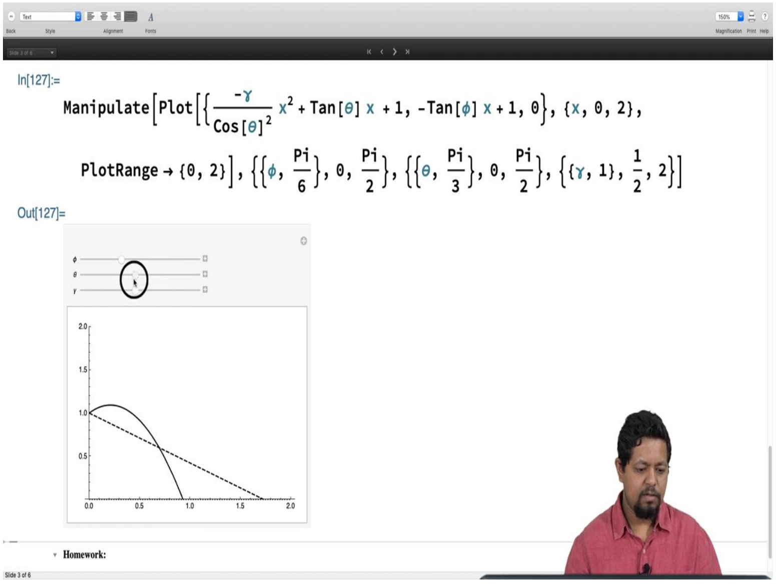

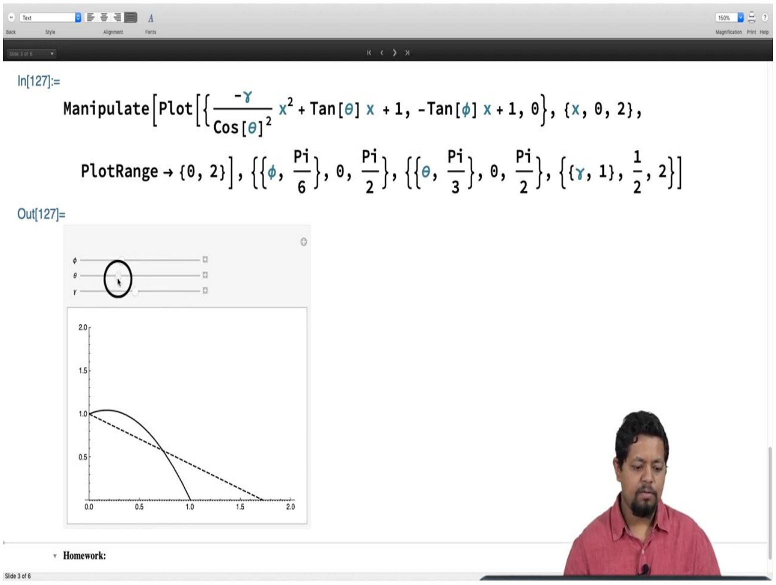

Let me go ahead and change θ now as I change θ, again, I am changing the angle at which I throw the ball and that also changes the trajectory.

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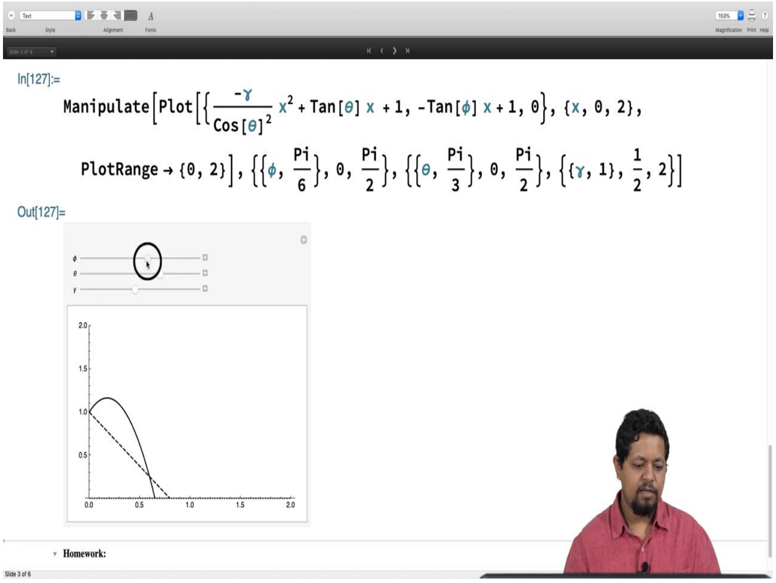

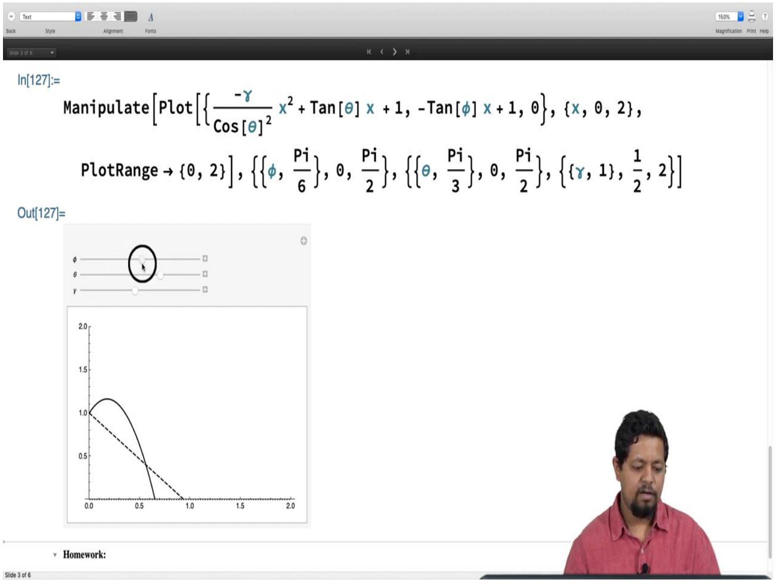

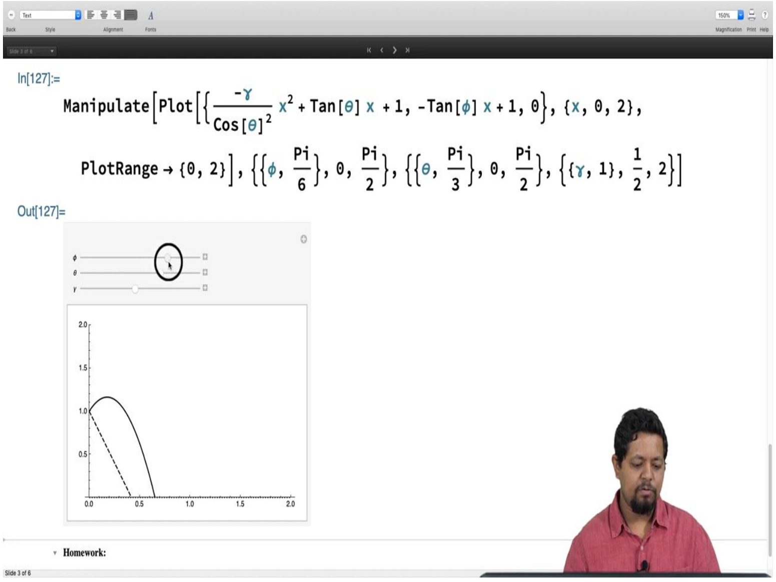

Let me go and change ϕ now, as I change ϕ, I change the angle of the hill. And as you see, the hill becomes more steep, we lose a solution.

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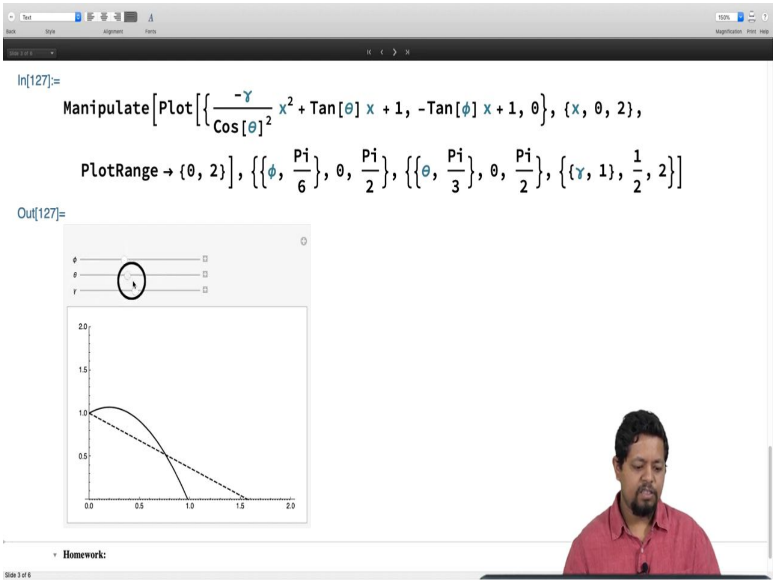

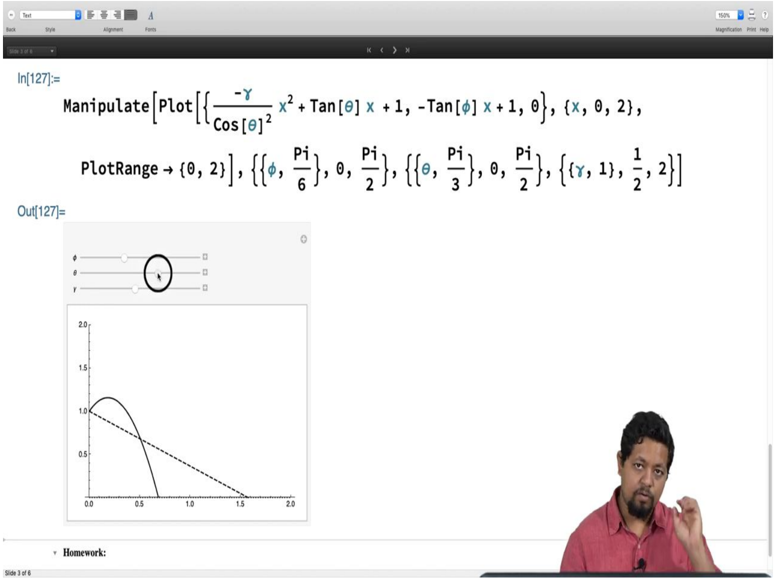

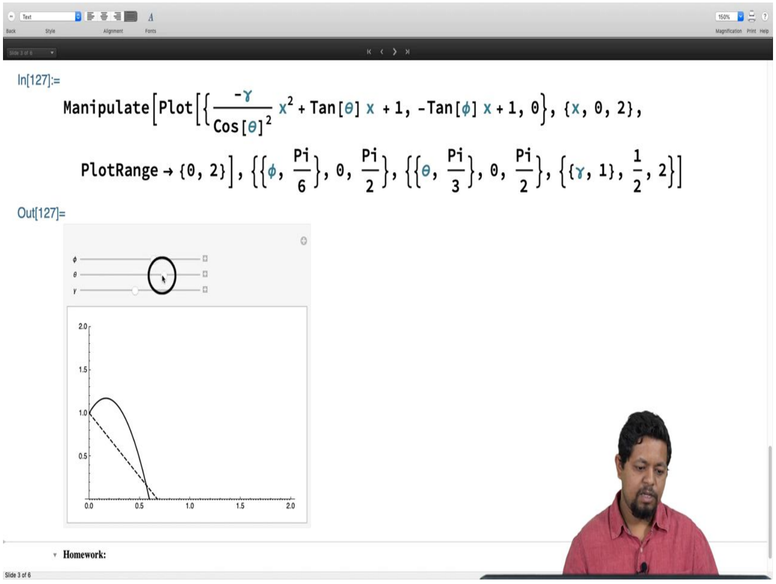

If I change θ, you see that does not really affect the solution. For as θ goes to 0, I still I that is I throw the ball horizontally I have a solution and as I increase θ my throw is more and more vertical. I still have a solution. Given that you know the hill is not too steep.

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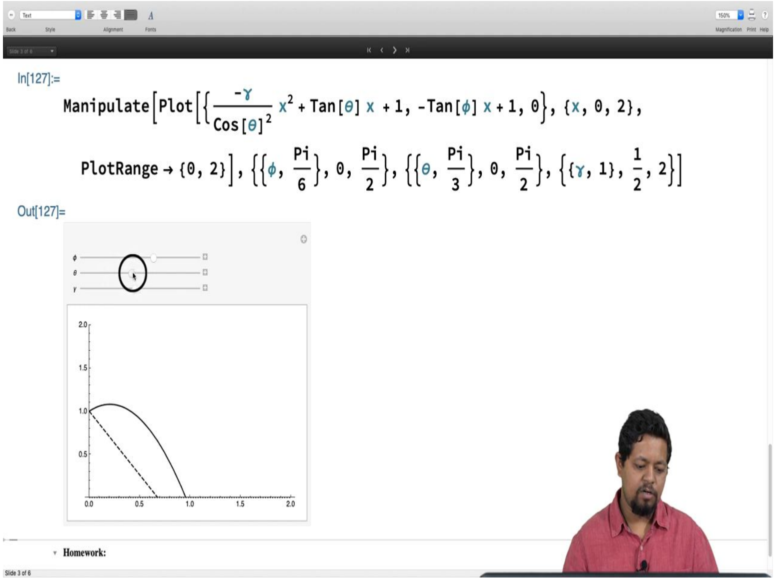

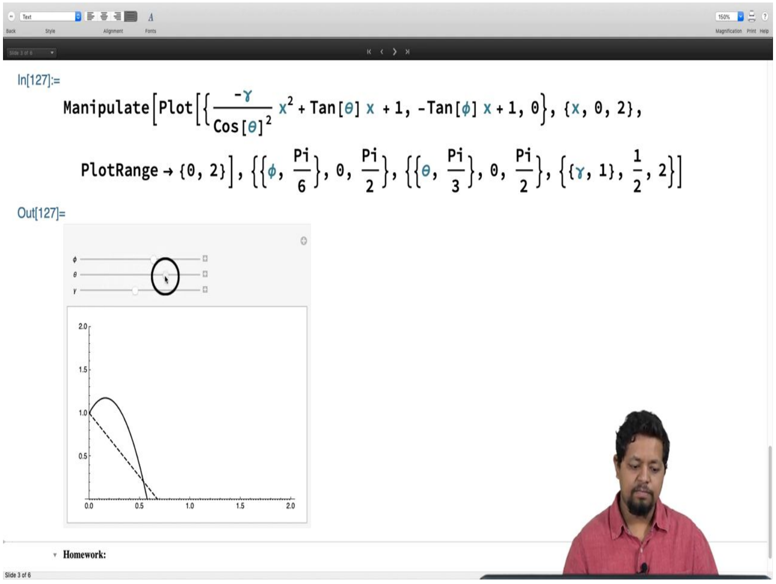

For a steep hill, a horizontal throw will probably not lead to a solution and you will have to probably end up throwing it vertically.

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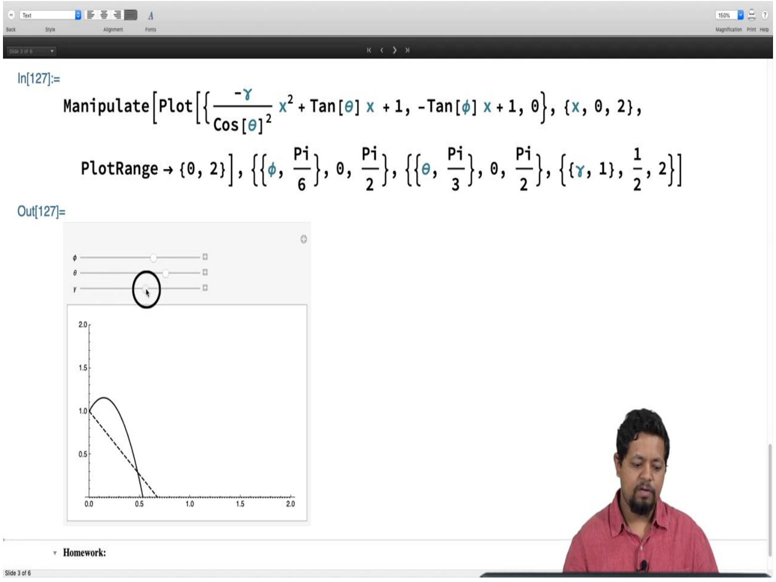

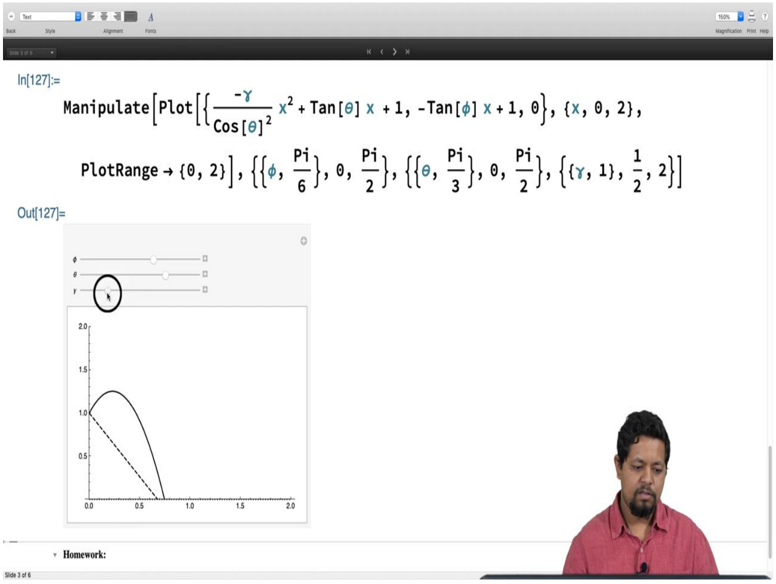

Similarly, γ affects the speed of the ball and changes the trajectory. So, if γ is very small, then also you end up having a solution and the ball will end up going very far away. So, this simple example shows how the interplay of various parameters in the problem can affect the solution.

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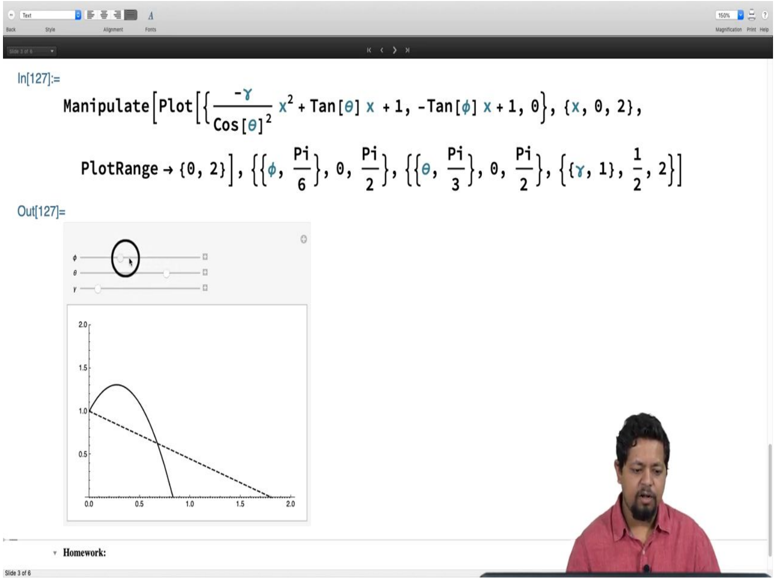
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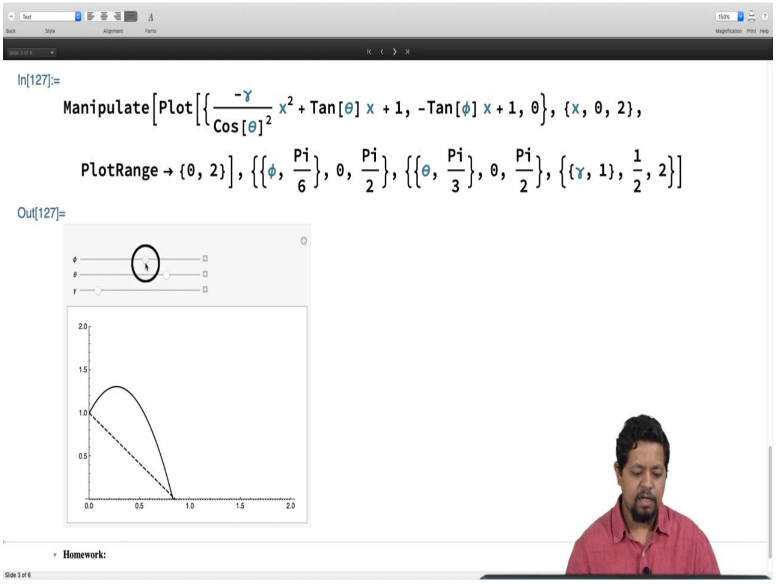
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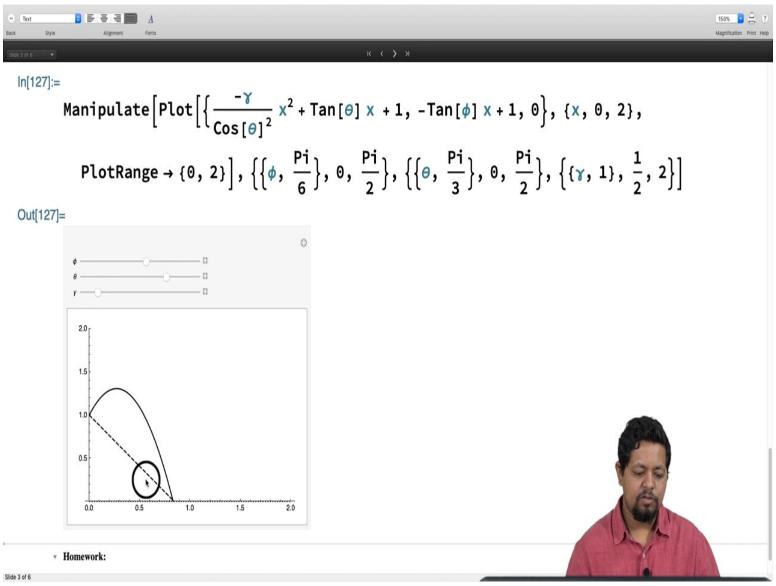
2.0
Homework:


In[127]:=
Manipulate[Plot[{-γ/Cos[θ]^2 x^2 + Tan[θ] x + 1, -Tan[ϕ] x + 1, 0}, {x, 0, 2},
PlotRange → {0, 2}], {{ϕ, Pi/6}, 0, Pi/2}, {{θ, Pi/3}, 0, Pi/2}, {{γ, 1}, 1/2, 2}]
Out[127]=
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
Homework:

Now, we can actually go ahead and find the condition when do we critically have a solution that is, when do we have a situation like this, that the ground, the hill and the ball all meet at the same point, a situation like this, this is a critical solution, you slightly deviate away from this critical condition and you lose a solution.