Physics through Computational Thinking
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Lecture 07
Hyperbolic Trigonometric Function

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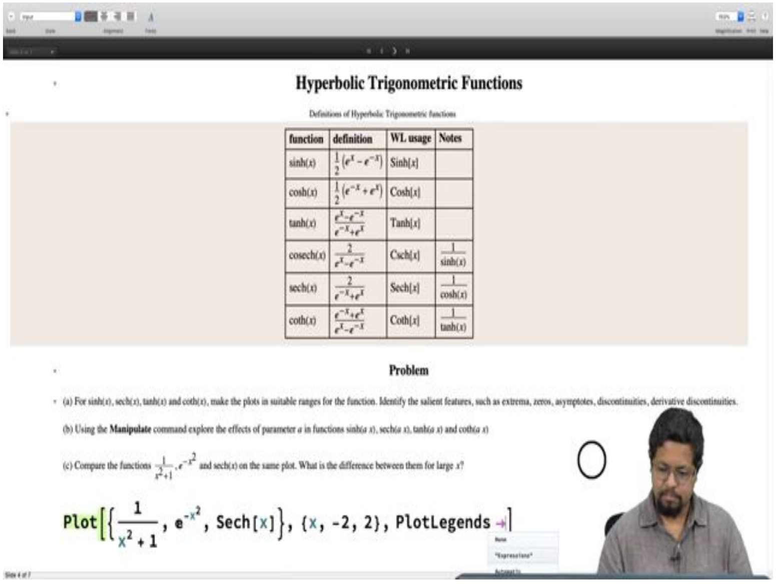

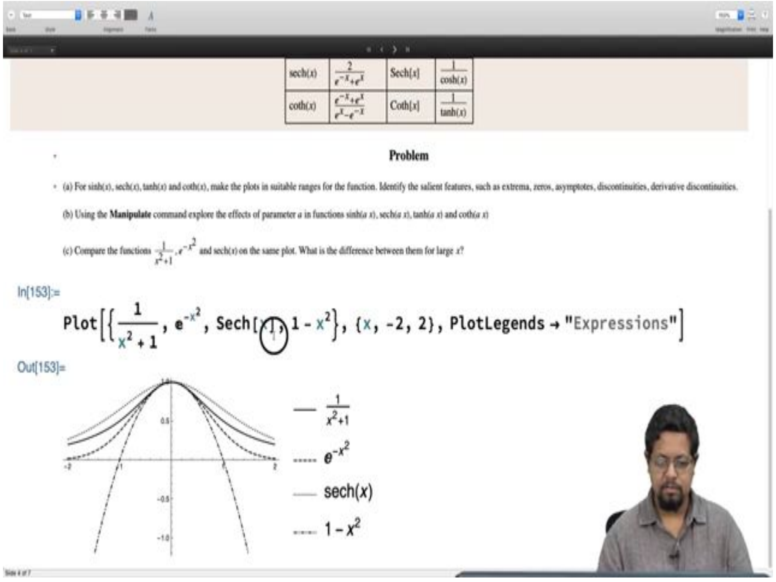

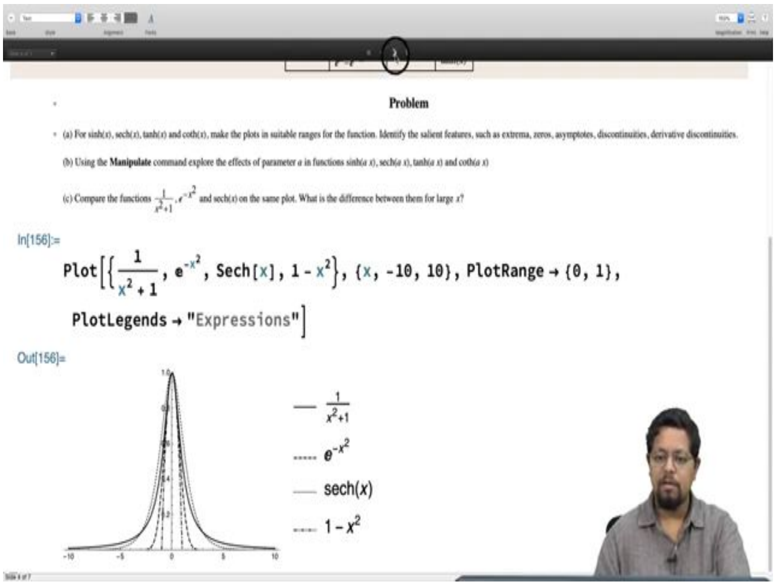

Let us look at another class of functions called hyperbolic trigonometric functions. Hyperbolic trigonometric functions are defined as the variants of trigonometric functions sine, cosine, tan: cosecant, secant, cotangent etc the sinh functions defined as ½ e^x-e^{-x}. In Mathematica or in Wolfram language, it is expressed as this expression sinh(x) and similarly the other, the rest of the hyperbolic, hyper geometric, sorry hyperbolic trigonometric functions.

Cosecant hyperbolic is inverse of sine hyperbolic, secant hyperbolic is inverse of cos hyperbolic, cotangent hyperbolic is inverse of tan hyperbolic. So, if you are not familiar with this function, this table gives you the definition and gives their usage in Mathematica or wolfram language.

So, let us go ahead and look at the following problem, I want to you to, for sine hyperbolic, secant hyperbolic, tan hyperbolic and cotangent hyperbolic I want you to make the plots of these functions in some suitable range and identify the salient features such as extremas, zeros, diversions and asymptotes, discontinuities and whether they are differentiable or not. And where they are differentiable and where they are not differentiable.

So, go ahead and play around with these four functions plot them, and identify these salient properties, then I also want you to use Manipulate command to explore the effects of parameter ‘a’ in the functions, sinh(ax), sech(ax), tanh(ax) and coh(ax). Finally I want you to compare the functions, Lorentzian with a = 1, $1/(x^2+1)$, Gaussian e^{-x^2} and sech on the same plot.

So the important question here is, what is the difference between them for large x. so, I want you to try these problems by pausing the video and work it out as home work. I am going to quickly demonstrate you the last part of this problem by doing a plot.

So, I will plot the Gaussian, the Lorentzian and the last function I want to plot is sech. Interesting enough all these three functions behave the same way. So, here is the plot of all three functions. It is interesting to note that all the three functions have very similar behavior near x = 0. In fact that behavior is the same as $1-x^2$.

I can fix the plot range as just 0 to 1, it is interesting enough that all the functions all the three functions behave the same way at x = 0 but they fall off differently as x goes to plus infinity or x goes to minus infinity. Let us increase the range in which we are plotting to see that behavior. It is quite interesting to note that the Gaussian which is represented by the dashed line over here falls off the fastest.

The Lorentzian appears to fall off faster than secant hyperbolic but eventually secant hyperbolic takes over and falls off faster than Lorentzian, we can increase the range further and verify that sech represented by the dotted line is falling off faster than the Lorentzian. The reason is simply because sech goes like e^{-x^2} for large x.

While, Lorentzian goes like $1/x^2$ for large x and exponential falls off faster than $1/x^2$. Gaussian goes like e^{-x^2}, it falls off faster than e^{-x} and that is why the Gaussian function dies off the fastest among all of these curves.

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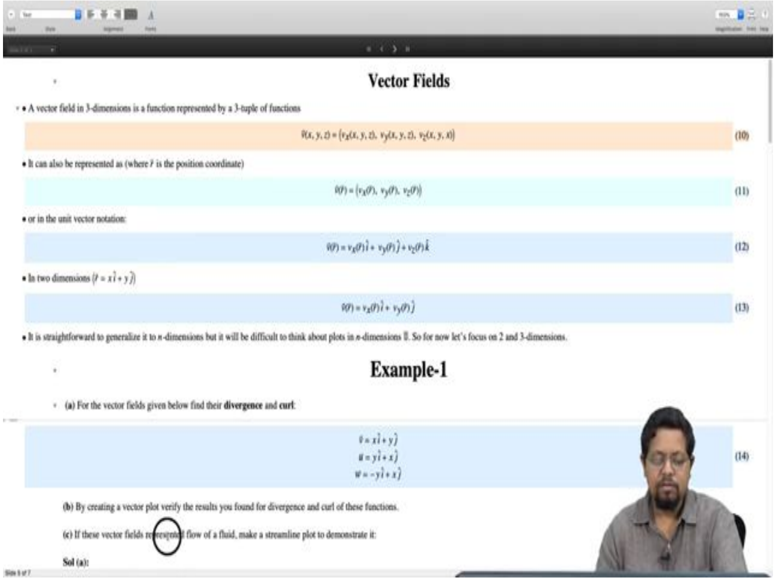

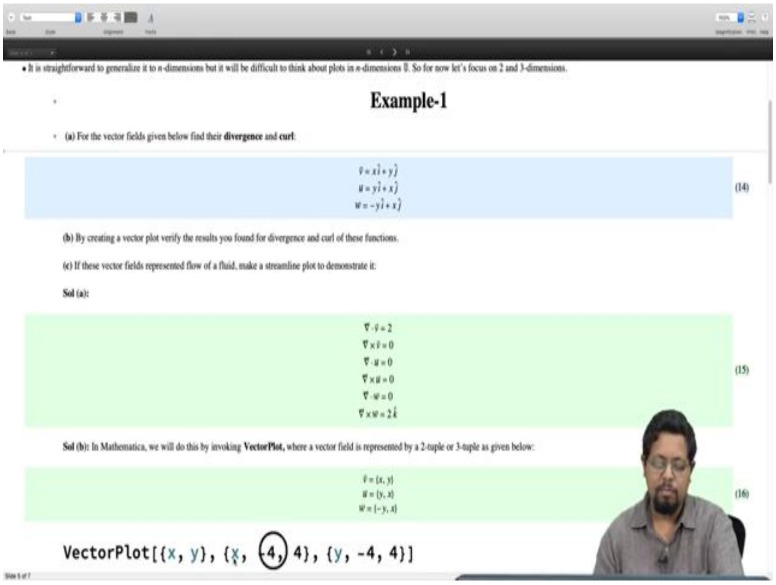

Let us go to the next slide, let us talk about the vector fields, and let us understand how to plot vector fields and how to understand these plots. Vector fields are defined as a n-tuple, in 3 dimensions there are 3-tuples they have 3 components each of the components is a function of the coordinates x, y and z. So, if v is a vector field is a function of x, y and z it is a vector function of x, y and z that is it has got 3 coordinates v_x, v_y and v_z at each of the three components v_x, v_y and v_z are functions of the coordinates x, y and z.

So, vector field has 3 components, v_x, v_y and v_z and each of them is a function of 3 coordinates x, y and z, we can also represent the vector field with a vector in terms of coordinates R like in this expression. You can also represent vector field in vector notation as $v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ where v_x, v_y and v_z all three of them are functions of the coordinate R.

In 2 dimensions this simply reduces down to $v_x\hat{i} + v_y\hat{j}$ and you can also go ahead and generalize the vector fields, in n dimensions but plotting something in n dimensions will be extremely difficult so we will just explore 2 dimensions and 3 dimensions in this lecture.

Okay, let us go ahead with the first example, in this first example I want to consider 3 vector fields v, u and w. v is $x\hat{i} + y\hat{j}$, u is $y\hat{i} + x\hat{j}$ and w is $-y\hat{i} + x\hat{j}$. I want you to calculate divergence and curl for these three vector fields. So if you want at this point you can pause the video and calculate the divergence and curl for these three simple vector fields.

Once you have calculated the divergence I want you to create a vector plot and verify the results what you found out about the divergence and curl. And finally I will demonstrate to you that these vector field represent flow of a fluid, I will make a streamline plot to demonstrate that to you.

So, let us go ahead and check the answer if we calculate the curl and the divergence for v, u and w this is what you going to get, v has a non-zero divergence, divergence of v is 2, but curl of v is 0. On the other hand u has both its divergence and curl vanishing and w has zero divergence but non-zero curl pointing in the $\hat{k}$ direction that is perpendicular to $\hat{i}$ and $\hat{j}$.

Let us see how these three fields appear different when I do a vector plot, that is if I make a vector plot of these three vector fields. What does it mean to have a vector field to have a divergence but no curl, what does it mean for a vector field to have curl but no divergence what it means for a vector field to have neither divergence nor curl. Let us explore it by plotting these three examples.

So, Mathematica in order to plot this will use this by invoking a function called vector plot. And to do so, we will need to represent the vector field as a 2-tuple or 3-tuple depending on we are working in 2 dimensions or 3 dimensions. So, in this case, my vector fields are xy, yx and -yx in the flower brackets, so that is how we represent them in Mathematica and we will make a vector plot by invoking the vector plot function.

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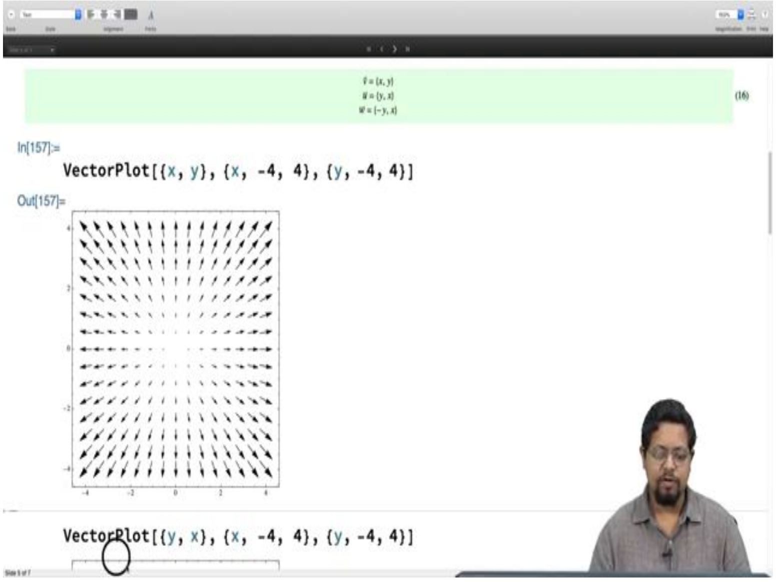

So let me demonstrate that to you, this is my vector field (x, y) I have given its first argument that is the range, the first argument is the vector field itself, which is 2-tuple (x, y) the second argument is the range of x and the third argument is range of y.

When I make a plot of this I get a plot like this where you see, that in the center I got nothing and then I got very tiny vectors pointing outwards and then the length of those vectors increase and keeps on increasing as I go radially outwards what does this mean, for a divergence and the curl? Clearly this vector field has a divergence because something is diverging out of a volume.

So, if I create a volume over here, think of an imaginary volume I see that something is coming out of that volume that means this vector field has a divergence. In fact locally anywhere you make a small volume or small circle you see whatever is going in more is coming out of that. So, less is going in but more is coming out that means this vector field has divergence.

In fact it has divergence everywhere and that divergence is constant in this case divergence of v is 2. This has no curl because there is no sense of rotation in this vector field. From this plot it is clear that this vector field is diverging out but there is no sense of rotation. Let us go ahead and follow another example for vector field u for that we can simply go and plot vector fields u by exchanging the coordinates here.

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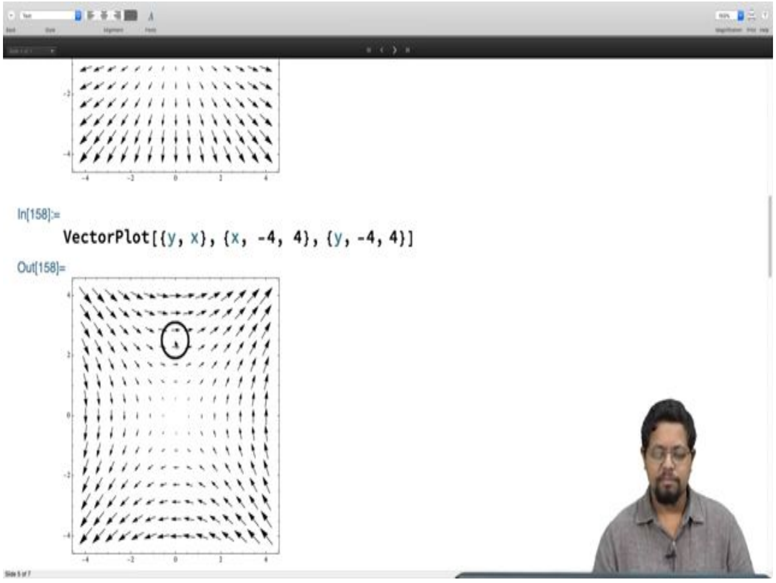

Vector field u is (y, x). So, when I plot that I got something like this. It's an interesting vector field you see that there are two two special lines along this diagonal line, flow is inwards and along this diagonal line, flow is outwards and in between the flow is from is this line towards that line. So, over here in the center we have what we called as the saddle point.

If you are exactly on this line the flow will take you to the saddle point but, if you are exactly on this line take away from the saddle point, for all the others you eventually end up going away from the saddle point and you will flow towards this line which is the line of stability. The line of instability, this is the line of stability and this diagonal is the line of instability.

For any other point the vector field is always taking you towards the direction of the line of instability. Therefore, this field neither has a divergence nor has a curl. To see that, imagine drawing a small volume anywhere, if I draw a small volume somewhere around here, you see that whatever goes in, equal amount comes out and as a consequence this has no divergence.

It also has no sense of rotation as for any small volume flow is coming in and going out but there is no sense of rotation, therefore, it also has a zero curl.

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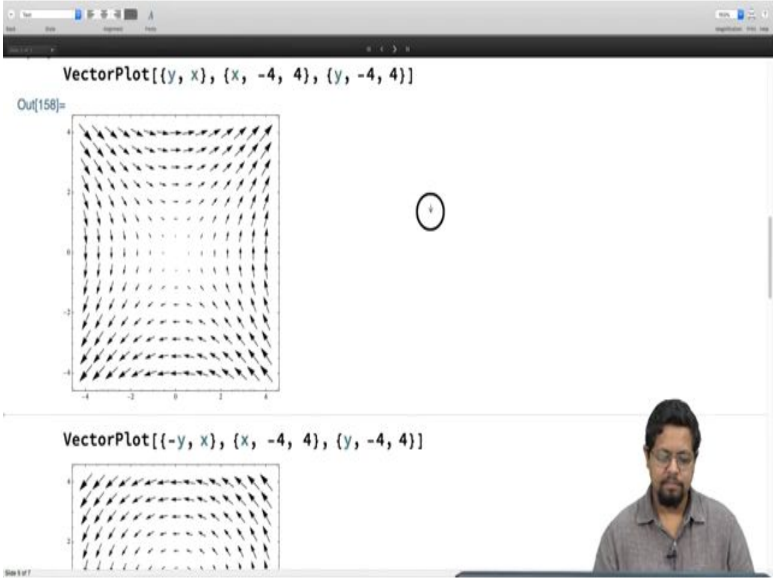

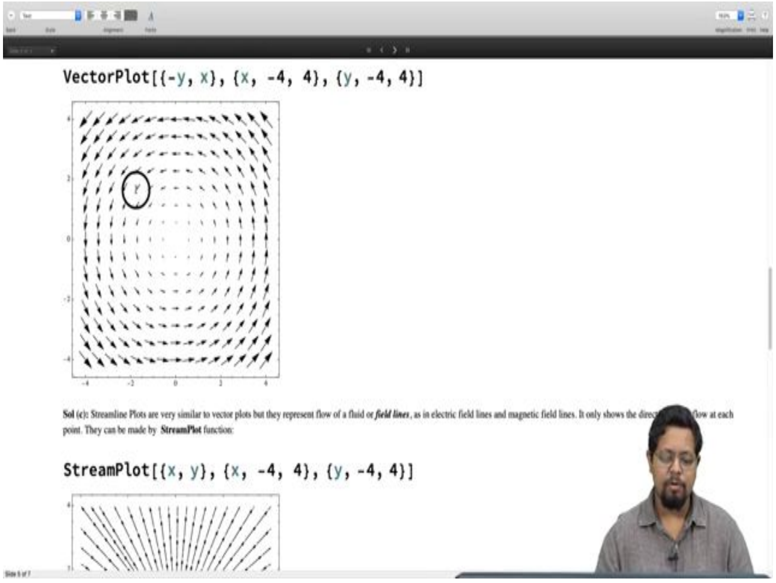

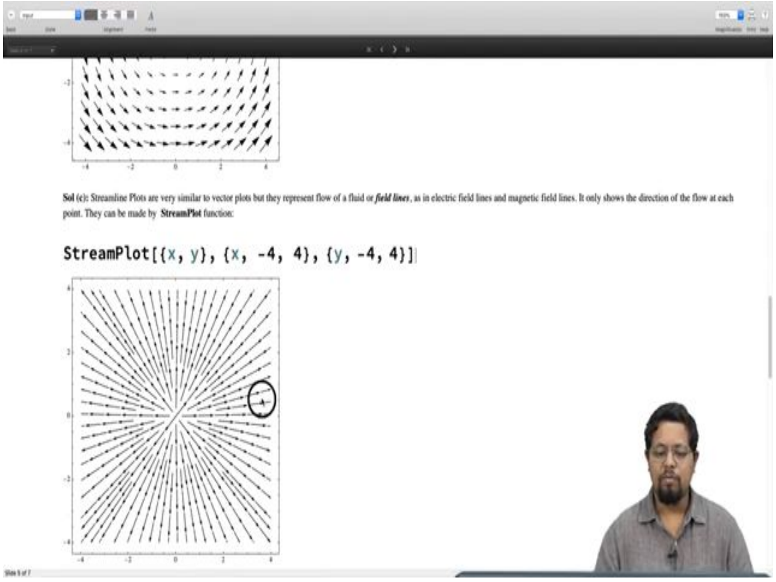

Let us go and check out the third example w equal to (-y, x). For this vector field there is a clear sense of rotation. When I make a plot of this I get something like this, there is a clear sense of rotation. If I make a small volume anywhere, it gives you a sense of rotation: the clockwise rotation. Overall also it gives you a sense of rotation in the counterclockwise direction. If we make a small volume we will see here the sense of rotation in the counterclockwise direction.

But, there is no net flow in or net flow out therefore, there is no divergence. So, it has got divergence equal to zero everywhere but, it has got a sense of curl or sense of rotation no matter where you are. Always the outside vector fields in any volume are bigger compared to the vector fields at towards the inside and that is why, there is a sense of rotation in this volume.

And as a consequence, this vector field has a non-zero curl while it has a zero divergence. We can also look at this by using a stream line plot or stream plot. I can plot a vector field as a vector plot and sometimes it is useful to also plot it as the stream plot. So rather than doing a vector plot over here if I replace this vector plot by stream plot as done over here, I will get a plot like this which is quite similar to the plot that I have got over here.

Except the length of the vector was getting bigger and bigger which was saying that the strength of the vector field is becoming bigger and bigger as I go outwards. But for the stream plot the length of the vector is ignored and straight lines are drawn showing the direction of the flow. So, stream plot does not care about the length of the vectors, it simply draws all vectors of equal length showing you simply in which direction the flow is.

So, for a fluid flow electric field lines, magnetic field lines a stream plot is useful way of demonstrating than a vector plot. In this case you see that something funny business happening in the center that is in the center stream plot function is not able to plot very clearly what is exactly happening over here. And that is because, over here it is very difficult because x and y both vanish, the sense of direction is not clear.

As soon as you go away from this sense of direction is radially outward and at x = 0 what its sense of direction is not clear. In fact vector field is vanishing at that point.

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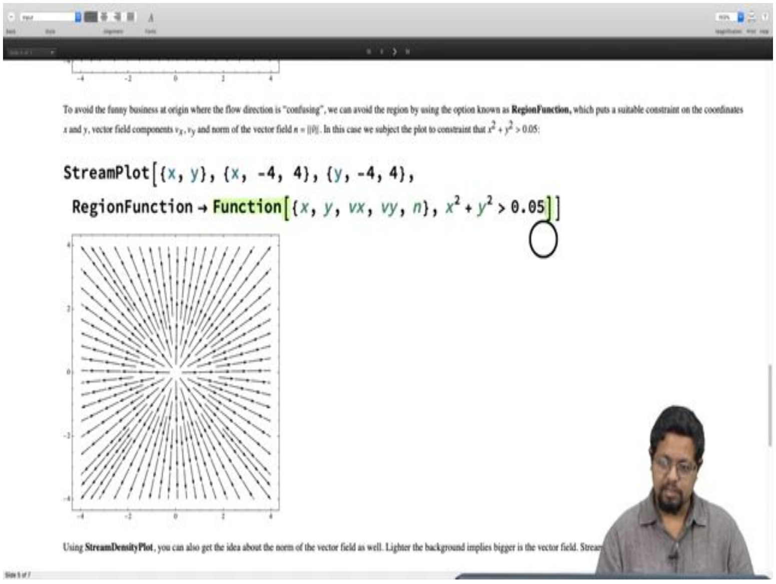

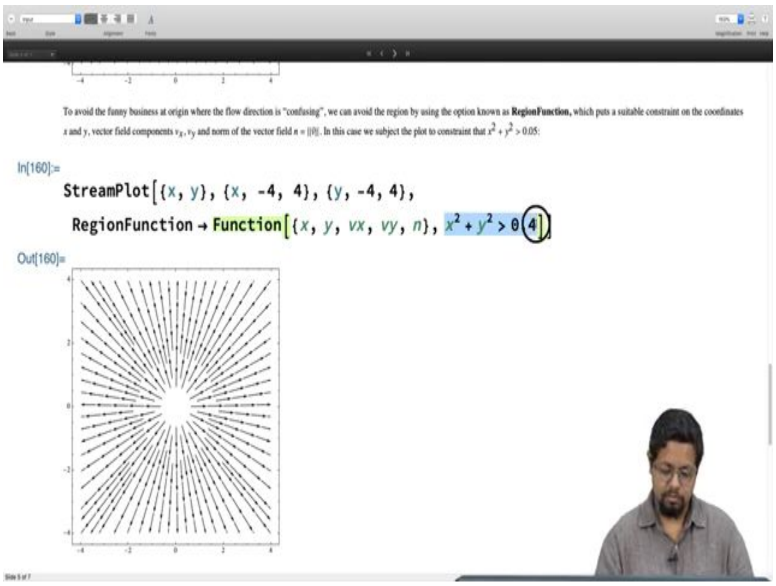

In order to avoid such a situation we can do a small trick. We can apply an option called region function on the stream plot. Region function is this, is given by this particular line of code at first it may appear something complicated but you just have to understand its usage which is very simple.

So, you define the region function tells you, in what region you should plot, you should make the stream plot and what region you should exclude. In this case you define the region function as the function which has 5 arguments xy, the x component is the vector field, the y component of the vector field and the norm of the vector field and then you specify a condition on these 5 components.

So, in this I have given a condition that $x^2+y^2 > 0.05$ that means make the vector plot only in the region where $x^2+y^2 > 0.05$. I have completely ignored what is happening with these other 3 quantities and as a consequence the vector field is, the stream plot is only drawn for the region $x^2+y^2 > 0.05$ as a consequence there is nothing over here.

That avoids the problem that we were seeing in this plot that was unclear what is happening at the center over here. And that makes the plot look nicer, okay. We can go ahead and play around with this and we can make this 0.1 which will make this region bigger, we can make it slightly more bigger may be 0.4. We can also do something else, for example depending on your need.

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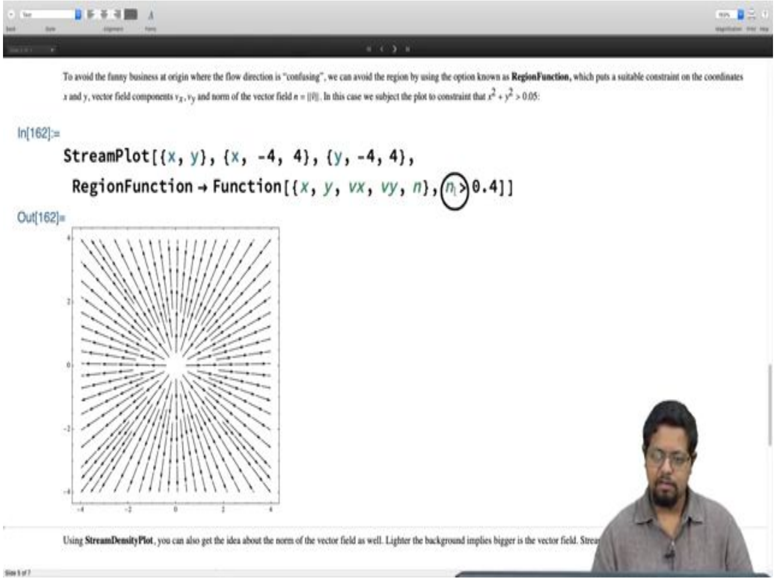

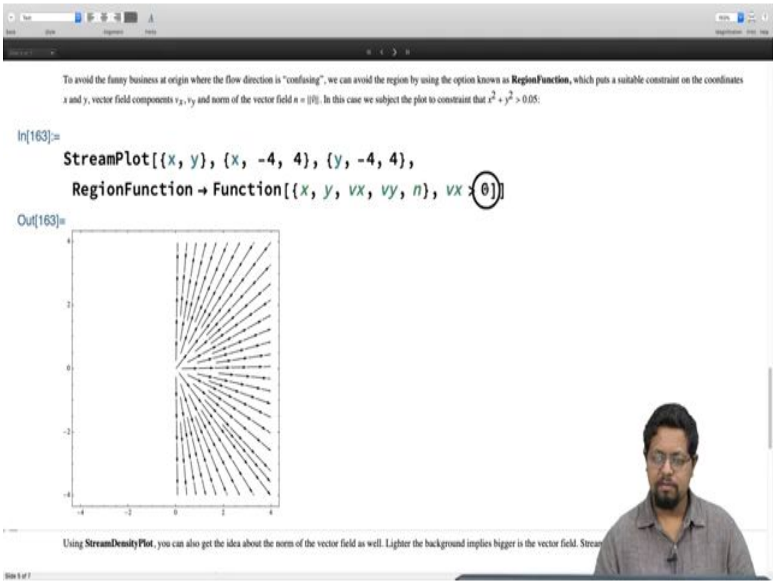

But, just play around with it. I want to demonstrate to you that we can also say that n is greater than some number like 0.1 that is if the norm is greater than 0.1, norm means $\sqrt{(vx^2 + vy^2)}$ if norm is greater than 0.1 only the then you plot and that also avoids certain region.

I can go and make it slightly bigger and that also does the same job effectively. Alright we can go ahead and play a little bit more with this if you wanted to only plot for certain for positive values of vx, the x component of the vector field. We will see that the plot will only be made for the positive values of the vector field.

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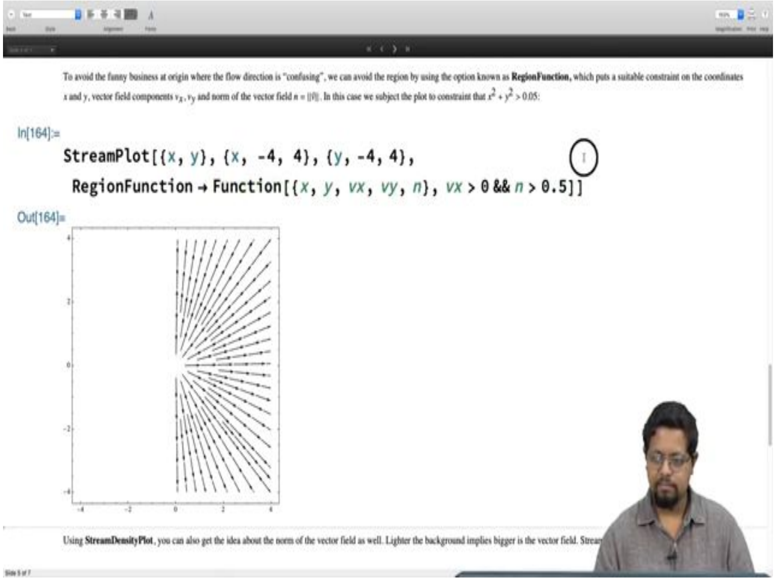

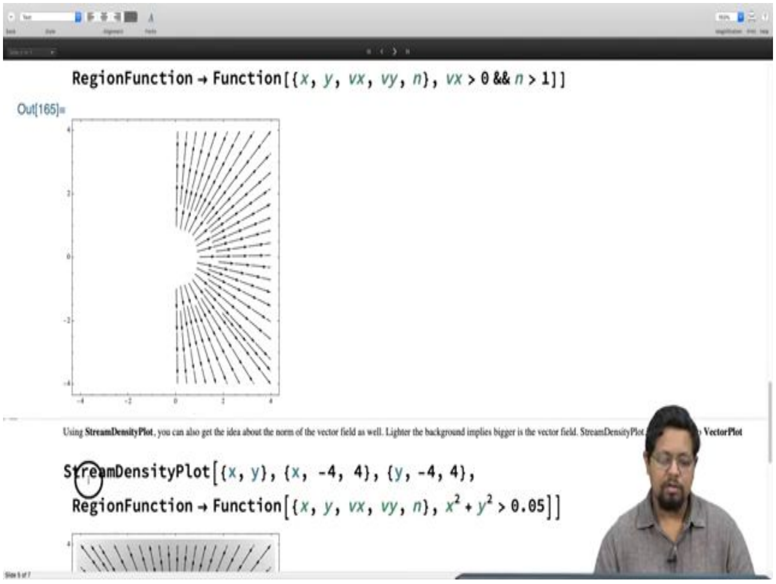

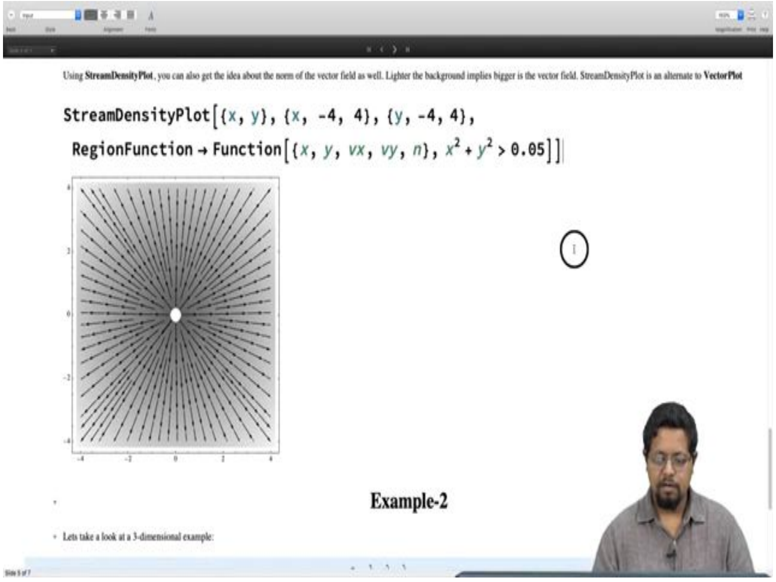

And if you wanted to add another condition to it that v of x or norm should be greater than 0.5, let us make it even bigger, 1, you can get a plot like that. So go ahead and play around with that, if sometimes you want that on the stream plot you also want see the length of the vectors you can use something called as stream density plot.

Stream density plot function works exactly like stream plot except that it also plots the density according to the length of the vectors. So, density, it will plot a grayscale density as for example in this case, as the vectors become longer and longer or the vectors become more strong and stronger, it is more white and as the vectors are weaker and weaker it shows you a black color.

Apart from that stream density plot and stream plot work exactly the same way stream density plot is sometimes useful to show the strength of the vector field.

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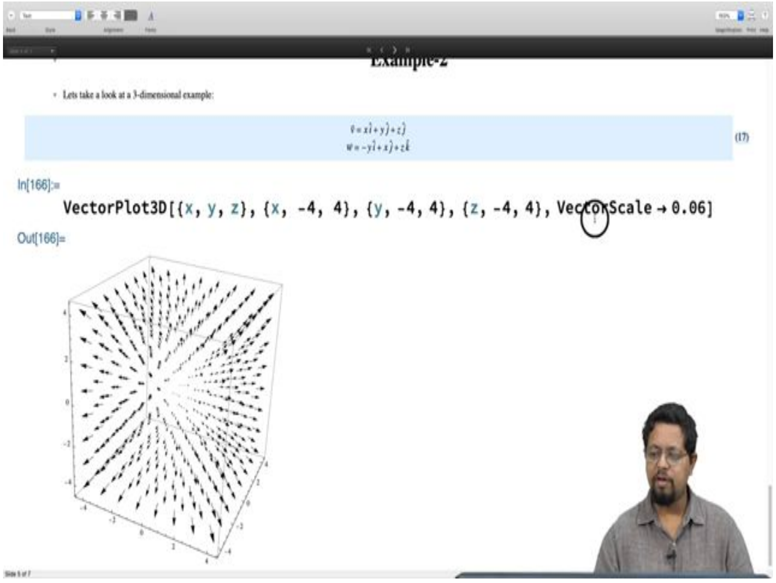

Let us go ahead and look at the 3 dimensional example of vector field. Here are here are two vector fields v and w, v is $x\hat{i} + y\hat{j} + z\hat{k}$ and w is $-y\hat{i} + x\hat{j} + z\hat{k}$. Let us go ahead and like a vector plot of this in 3 dimensions.

So, the first vector field is simply the 3 components are x, y and z so I put the 3 components x, y and z. Use the function, vector plot 3D because I want to make a 3D plot give the ranges of x coordinate, y coordinate and z coordinate. And I will go ahead and plot it. I have also given an option vector scale equal to 0.06. You can go around and play with it.

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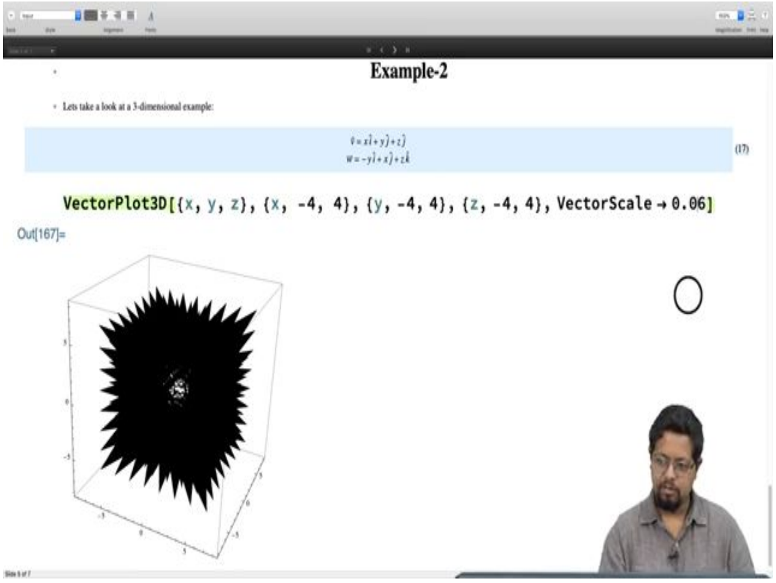

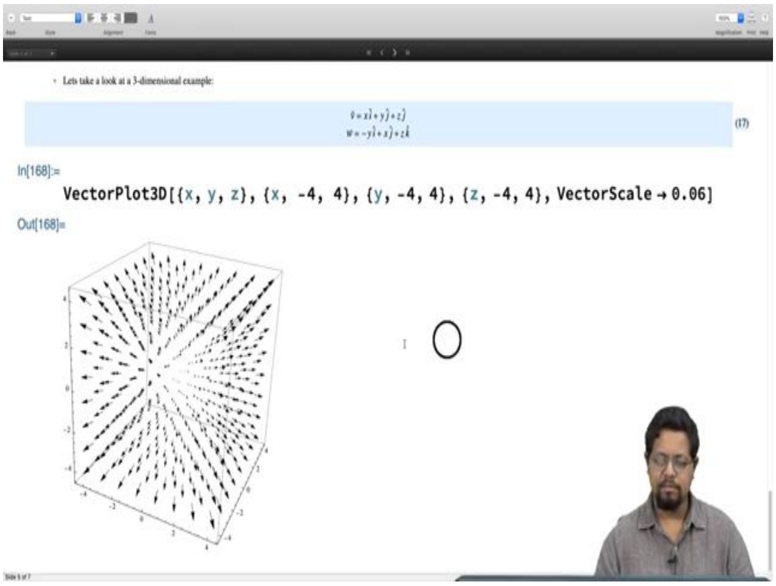

You see, if I make it too big the arrow height becomes really big, so I can fine-tune that by choosing it to get the right size.

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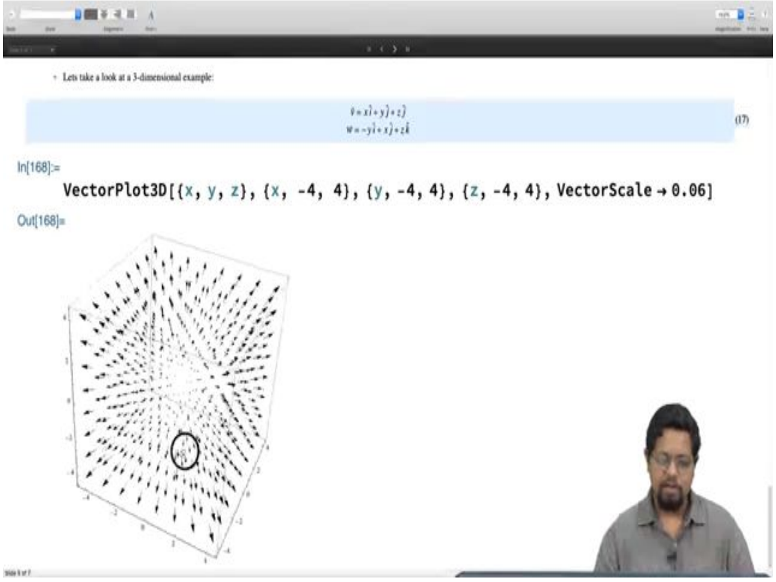

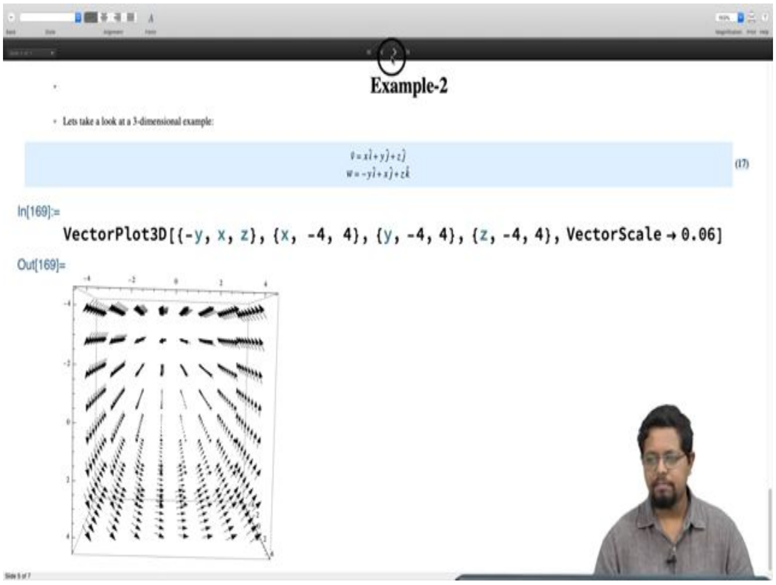

You can go ahead and rotate this and you can see what kind of vector field this is, again this is the vector field which has divergence everywhere. Let us go ahead and plot the second vector field by changing the x coordinate to -y, y coordinate to x, I will use the third coordinate as z and there we go.

We get a field which has got some sense of curling around in fact if you align yourself with the z axis you will see that this is going to have a sense of curling. The projection reaction why x and y plain is a vector field which is minus y, x something there is the curl. So when I align myself with the z axis such that the z axis is pointing out you see a sense of curl, sense of rotation in this vector field.

So, go ahead and play around with vector plot and vector plot 3D and stream line for some of your own examples.

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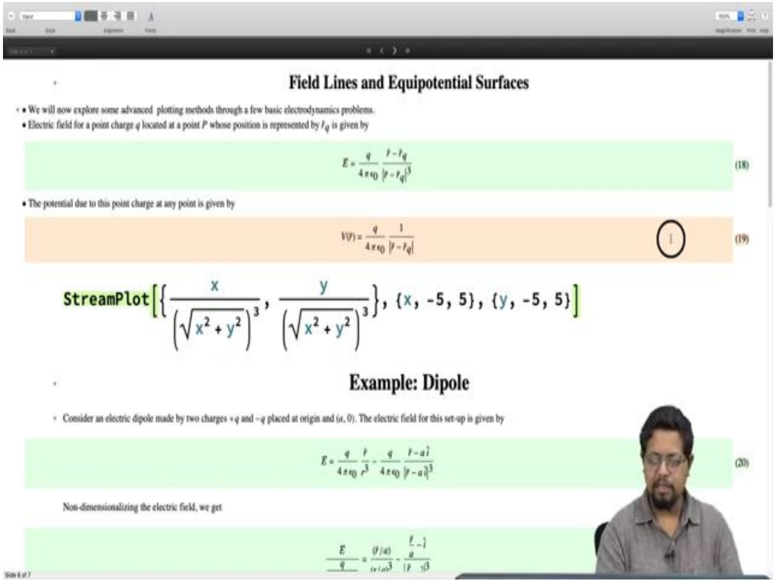

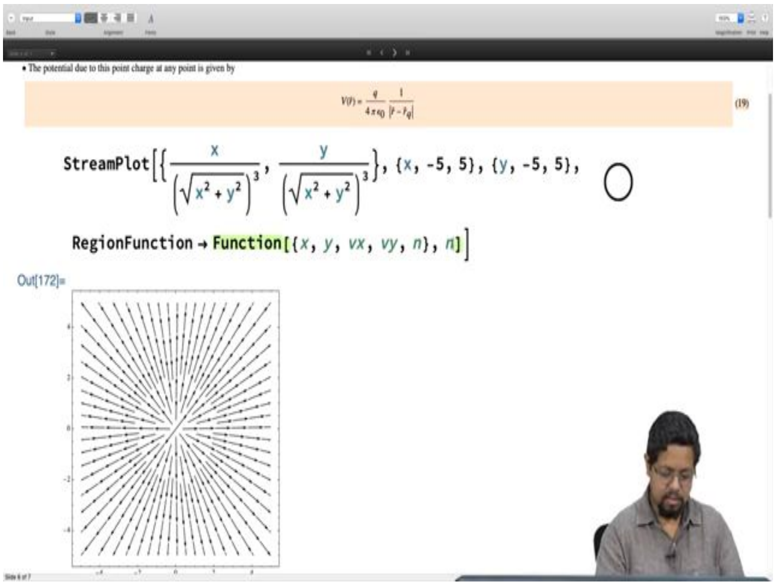

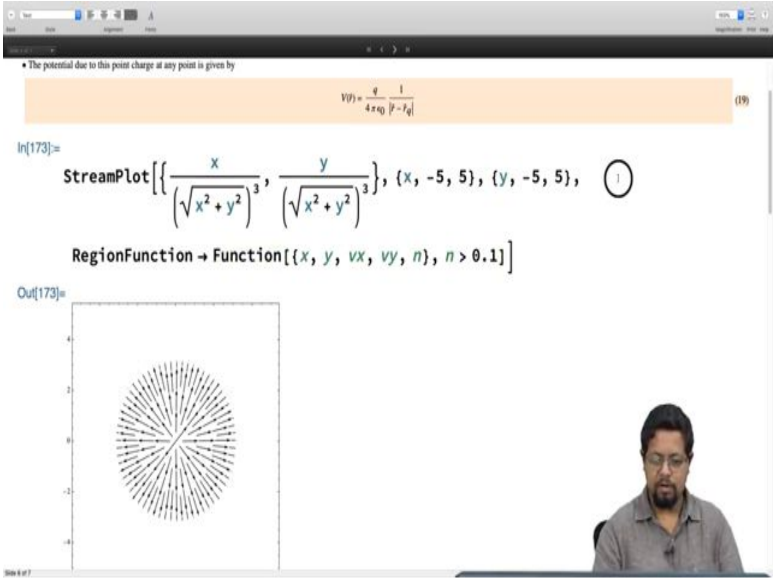

Let us apply this understanding or knowledge of vector fields and stream line plots that we have just learned to electrodynamics or electromagnetism where we study field lines and equipotential surfaces. So, in this example we will learn to plot the field lines and equipotential surfaces. Now as you all know that the electric field for a point charge located at the point r_q is given by this expression.

$q/4\pi\epsilon_0$ which is the overall constant the direction of vector field is $r - r_q$, r_q is the point with the charge particles placed divided by $(r - r_q)^3$ that is something you have seen in some course on electromagnetism or in some book on electromagnetism that is the electric field for a point charge at an arbitrary point r. The potential for the same is given by this expression where the potential goes like $1/|r - r_q|$ multiplied by this constant $q/4\pi\epsilon_0$.

Alright, let us go ahead and plot stream lines for this electric field. In order to do that, I make use of stream line plot and this electric field goes like $\vec{r}/r^3$ where we have vector r in the numerator and r^3 in the denominator let us take the charge particle to be placed at origin. So, then we have got $\vec{r}/r^3$ which is simply x and y in the 2 dimensions, divided by $(x^2+y^2)^{3/2}$.

And similarly for y let set the range for x, from -5 to 5 and set the range for y from -5 to 5. Let us go ahead and make the stream plot and you see the electric field lines are going outwards again we have got some funny business happening at the origin for this I can go ahead and use the region function option and set that to function of xy, vx, vy and n and let the function be set that n is greater than 0.1.

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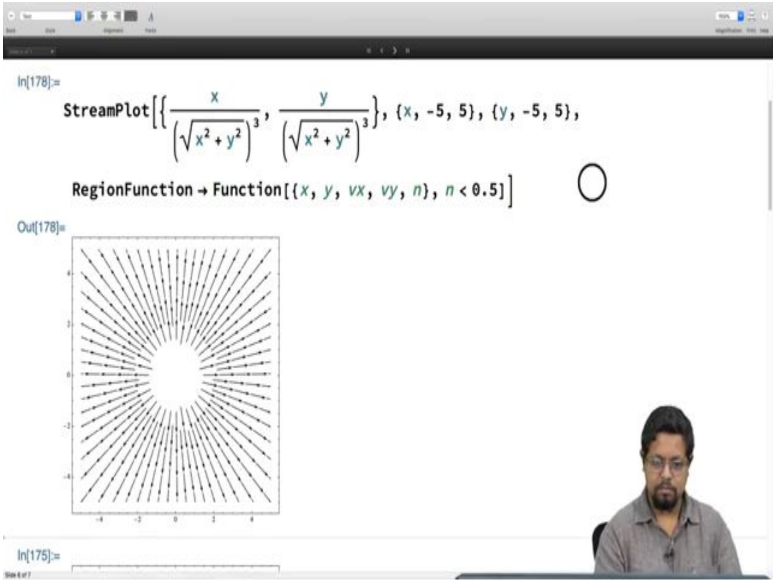

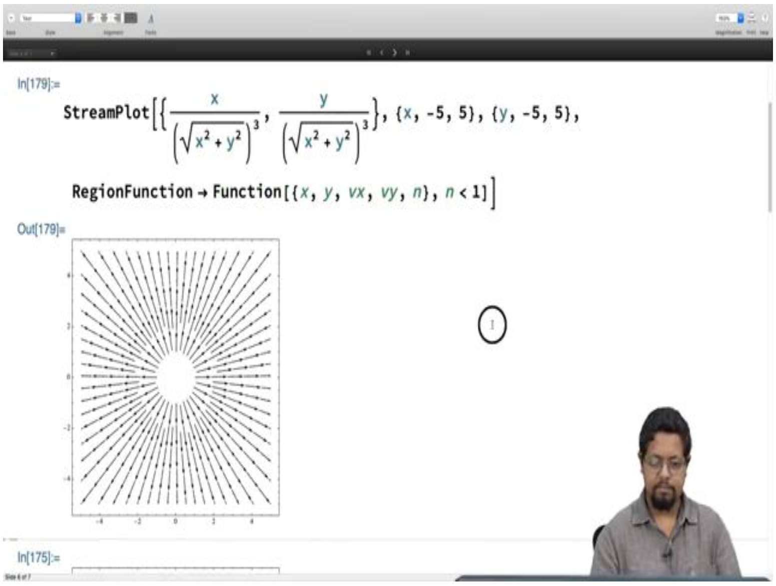

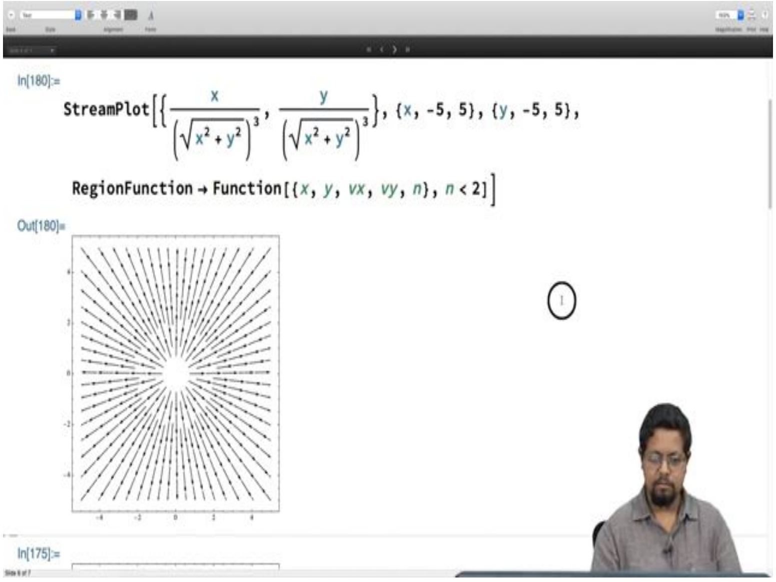

Yes, a bigger value, we have to go to n less than let say 1, there we go, so that is an example of electric field lines for a single charge particle where the charge particle is placed at origin.

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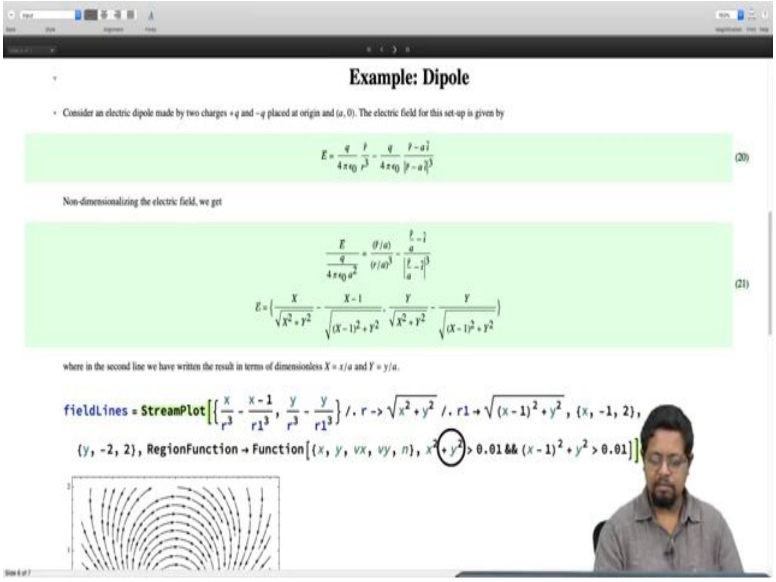

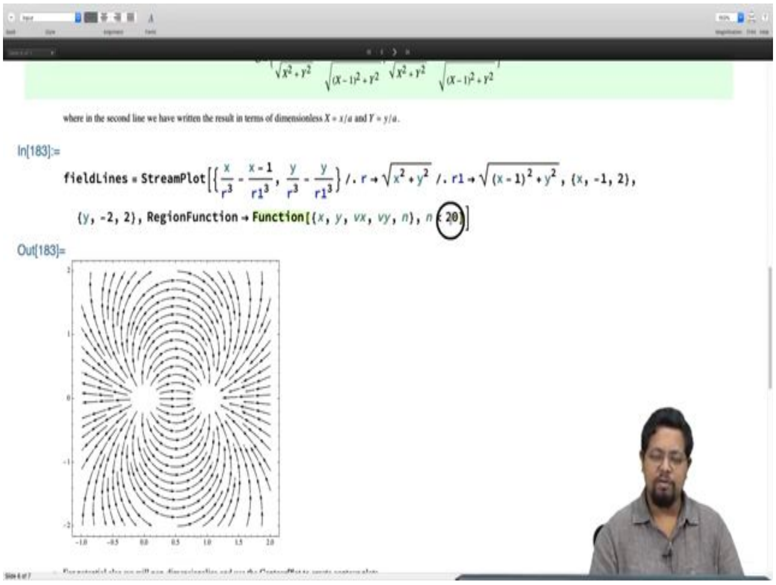

Let us take an example of a dipole, an electric dipole. Now if I consider a electric dipole made up of 2 charges +q and -q placed at origin, so +q is placed at origin and -q placed at (a, 0) that is on the x axis at distance 'a' away from the origin. The electric field for this setup is given by the electric field because of the positive charge and electric field because of the negative charge.

The positive charges placed at r = 0 so therefore, it is given by r/r^3 while the negative charge is placed at r = a therefore, its field is given by $(r - a\hat{i})/|r - a\hat{i}|^3$. We need to non-dimensionalize electric field so that we can actually plot it.

In order to non-dimensionalize we are going to divide r by a and electric field by $q/(4\pi\epsilon_0\, a^2)$ because the dimensions of electric field are given by $q/(4\pi\epsilon_0\, a^2)$. Doing that we simply get, r/a divide $(r/a)^3$ and $|r/a - \hat{i}|^3$. Taking r/a as (x/a, y/a) etc. I get the following components for the non-dimensionalized electric field where x is x/a and y is the y component of r/a.

So, we can verify that for the x component of the electric field I get this expression where x, $x/\sqrt{x^2+y^2}$ is coming from this part of the expression and $x-1/\sqrt{(x-1)^2+y^2}$ is coming from this part of the expression, similarly for the y component.

So, therefore this is my vector field that I want to plot, and to plot this vector field I have encoded the expression over here. To avoid right typing this expression again and again I have use the replacement rules over here. Which says that r is to be replaced with x^2+y^2 and r1 to be replaced with $(x-1)^2+y^2$. So, therefore my vector field becomes $x/r^3-(x-1)/r1^3$ and $y/r^3 - y/r1^3$.

In order to type this arrow key all you have do is type the dash key and then the greater than sign and type anything after that this turns into arrow. Slash dot is used for a replacement rule so, this is the replacement rule over here I am making the replacement in the previous expression over here with r goes to $\sqrt{x^2+y^2}$ and r1 goes to this expression.

Slash dot says that apply these replacement rules to this previous expression. I define the range of x from -1 to 2, and range of y from -2 to 2, and I also apply the region function and I can rather than choosing this complicated region function we can choose a simpler region function such as n less than 5 may be even bigger, may be even bigger there we go.

We have got the electric field lines for electric dipole where the +q charge is placed at this point and -q charge is placed at this point. The electric field lines are moving from +q charges to the -q charges. The norm is less than 20 just make sure that the stream plot is drawn only for the region where the norm of the electric field is less than the value of 20.

We can make it even higher as long as we do not try to draw this region where the directions is ill defined is perfectly fine. So, here this was an example of electric field lines for a dipole.

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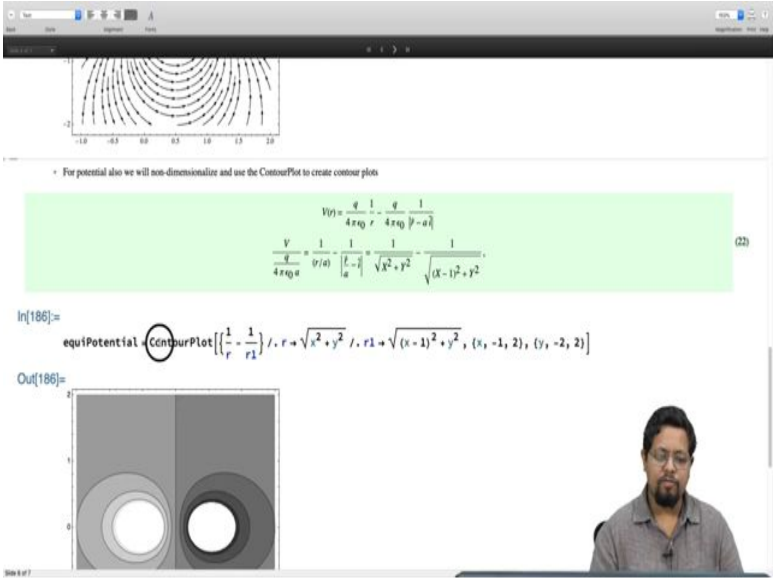

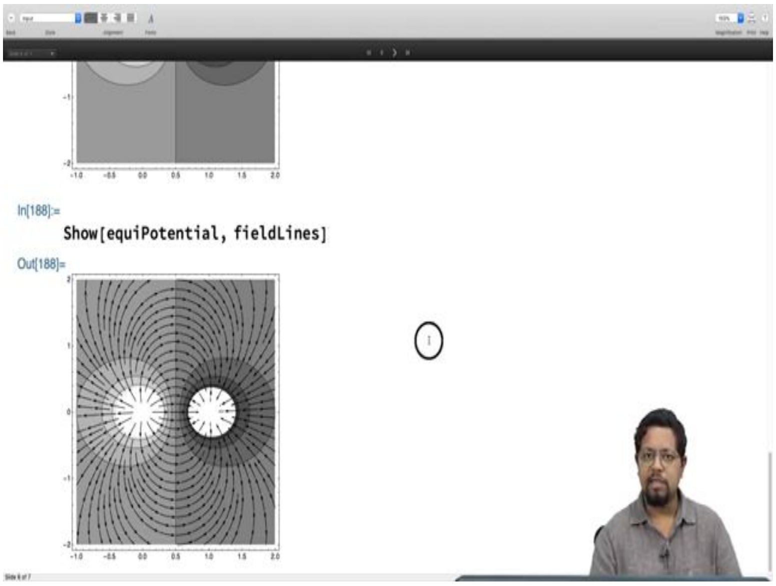

Now, I can also workout the potential of the dipole and non-dimensionalizing in the similar way I got potential for the dipole as $q/4\pi\epsilon_0 \quad 1/r$ - $q/4\pi\epsilon_0 \quad 1/|r - a\hat{i}|$. Non-dimensionalizing this expression in the same way as I did for the electric field except, for potential I am going to divide by $q/(4\pi\epsilon_0\, a)$ which is the dimension of the potential.

Then I am left with $1/(r/a) - 1/|r/a - \hat{i}|$. Convert this into the dimensional less coordinates x and y we are left with this expression. v is a scalar function now we want to plot this scalar function. But, this time we want to plot contours for the scalar scalar function.

In order to do the contours I am going to define, I am going to use the contour plot and this time contour for the contour plot I have got simply 1/r -1/r1, for the non-dimensionalized coordinates. So, 1/r -1/r1 I substitute r with $\sqrt{x^2+y^2}$ and r1 for $\sqrt{(x-1)^2+y^2}$ so, we go ahead and execute the contour plot.

When I draw the contour plot I get this particular function. Now, these contours show the equipotential lines contour plot which means that I am drawing a contour for all the points for which V(r) is constant. So, these contours correspond to various values of V(r). For example, this contour corresponds to V(r) = -0.5, this corresponds to V(r) = -1. This corresponds to -1.5 and so on.

On this side I have got positive values V(r) = 0.5, V(r) = 1, V(r) = 1.5 and so on. These contours showing my equipotential surfaces or lines of contours of equipotential, over here I have got stream plot for or electric field lines plot for the dipole which I have defined as field lines, this plot is called field lines because, I have, to the variable field lines I have assigned the value of this plot.

So, let me execute this again, for contour plot for equipotentials I have assign to a variable called equipotential and when I lay over this on top of the view using the show function, show equipotential comma field lines I can plot them on top of each other and when I do that I get this very interesting plot where I got a plus charge over here minus charge over here and this plot is showing you both the equipotential lines and the field lines and relative orientation of them.

You can verify that equipotential contours are perpendicular to the electric field lines at all points. So, for example for this contour you see the electric field lines enter or cross this contour at 90 degree angles and similarly, on this side which is what we have learned in electrody-, electromagnetism that the equipotential lines are always perpendicular to the electric fields and here by usual, by visual aid we can verify this directly. That will be all for today, and we will continue with more interesting things next time.