Physics through Computational Thinking
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Lecture 05
Radicals and Logarithms

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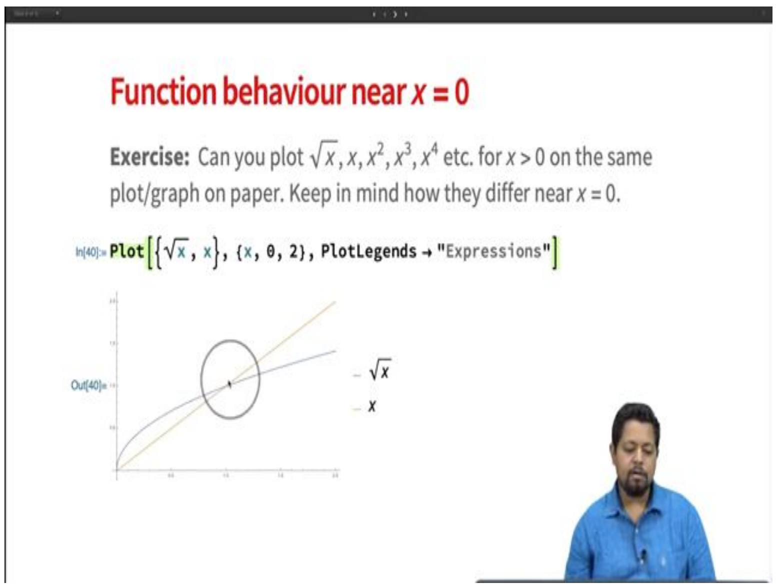

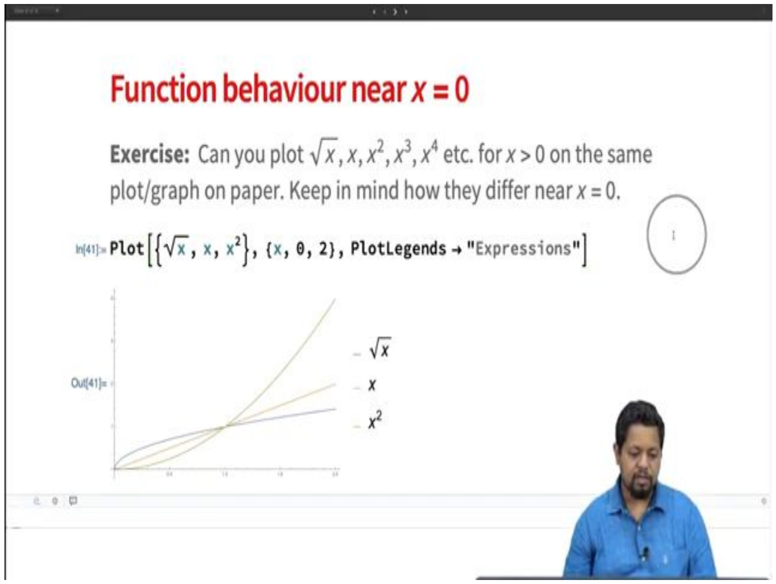

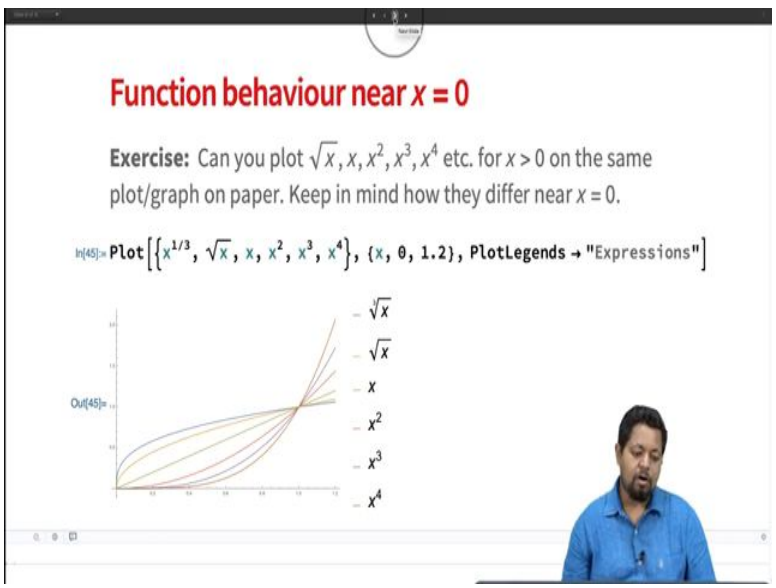

Let us go forward and understand further how do functions behave near x = 0. Whenever we have to plot a function, we should understand that certain special points, for example the minimas, the maximas, x equal to 0, even when we are plotting by hand we need to capture the behaviour of those functions over there properly and even if you are not plotting, you need to understand how does a function behave near x = 0.

Because by further understanding, you can infer many things about the function, if you have an understanding about how do the function behaves near x = 0. So in this exercise, we are asking can you plot a function $\sqrt{x}$, x, x^2, x^3, x^4 etc for $x > 0$ on the same plot or graph on the paper. So can you try to do this on a paper, pause the video for a minute.

Think about how these functions are going to look on the paper, can you plot it? How you have to keep in mind how they behave near $x = 0$ and that is the key here. All these functions have different behaviour near $x = 0$. So can you capture the gist of plotting this by going back to paper and pen and try plotting it. And after that we will execute this in Mathematica and see what it looks like.

Okay let us go back to Mathematica and plot this, so I want to plot $\sqrt{x}$, unless x is positive, so x I can make it from 0 to 2, you can plot it, this is what I get. Let me add more functions to this. Ofcourse $y = \sqrt{x}$ is same as $x= y^2$ that means it is a sideways parabola. So x is linear, that is the orange curve let me go ahead and add PlotLegends over here.

Let us go ahead and add more functions, so you see the behaviour: how $\sqrt{x}$ behaves at near $x = 0$ and how x behaves, x is a linear function it goes like that, $\sqrt{x}$ varies much more slowly compared to this. This goes fast, on the other hand this one rises up fast for small x and then for larger x it does not rise up fast enough. On the other hand the linear function, it is slow compared to $\sqrt{x}$ for $x < 1$.

But eventually picks up the pace and becomes faster than $\sqrt{x}$ and they intersect at $x = 0$ and $x = 1$ and for that matter all these functions will meet at $x = 0$ and $x = 1$ because all these function are 0 at $x = 0$ and they are all 1 at $x = 1$. Therefore they all are going to meet at this point and this point but so their behaviours would be different before and after.

Now, this is not a news for you, so let us go ahead and quickly check it out. Adding a quadratic function, a cubic and a quartic. Notice the difference near $x = 0$ let us zoom into the x axis a little bit more and make it 1.2. The higher powers of x are suppressed for small x but they are dominant for large x, when x is much greater than 1 or bigger than 1, the higher powers are more dominant and when $x < 1$ the higher powers are suppressed compared to the lower powers.

We can go ahead and and try more things we can add $x^{1/3}$ also and we can anticipate how does $x^{1/3}$ goes. $x^{1/3}$ will dominate $\sqrt{x}$ over here and then for $x > 1$ will be subdued by $\sqrt{x}$. Let us check that hypothesis, yes indeed it is true, the blue is $x^{1/3}$ and orange is $\sqrt{x}$.

So $\sqrt{x}$ is subdued by cube root over here but eventually $\sqrt{x}$ subdues over $x^{1/3}$. So this is understanding the behaviour of functions near $x = 0$ and this is something very important when we are plotting we should know. When you look at a function, we should have an anticipation of how the function is going to behave for small x and large x? And when we plot it actually either by hand or using a computer like what we are doing over here we should cross check our understanding.

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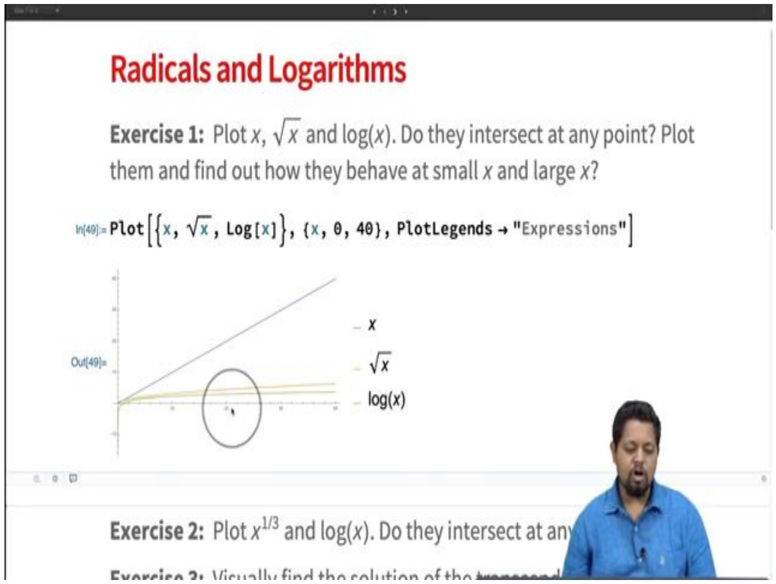

Let us go ahead and do an exercise. In fact this exercise is broken into 3 parts, let us talk about radicals and logarithms. So the first part of the exercise is to plot x, √x, log(x) and find out where do they intersect, we already know that x and square root of x intersect at 0 and 1, where does the root of, log(x) intersect with these functions. Plot them and find out how they behave at small x and large x.

This is an exercise you can very easily do computationally because you know how to do that. So let us go ahead and plot that and what is important here is how do you want to compare log(x), which is also also a built-in function and this log is built-in function Mathematica starting with a big L and we will set the range since log(x) is defined only for positive x, we are going to set the range from x goes from 0 to let us say 2.

And there we go, let me add the PlotLegends but you can guess which one of them is the log function. The green one is the log function, okay. You see the log(x) does not intersect with these two over there, the reason is that for large x, x is going to go to infinity much faster than log(x) and the √x is also going to infinity much faster than log(x).

In fact log(x) goes very-very slowly, if you do not believe me go ahead and plot it from x equal to 0 to 20 and you will see the behaviour. The orange curve is going faster than the green curve and so on, so you can increase the range further and orange and the green are never going to cross. The point of this exercise was to notice that √x and log(x) never intersect. So there is no root of the equation √x = log(x).

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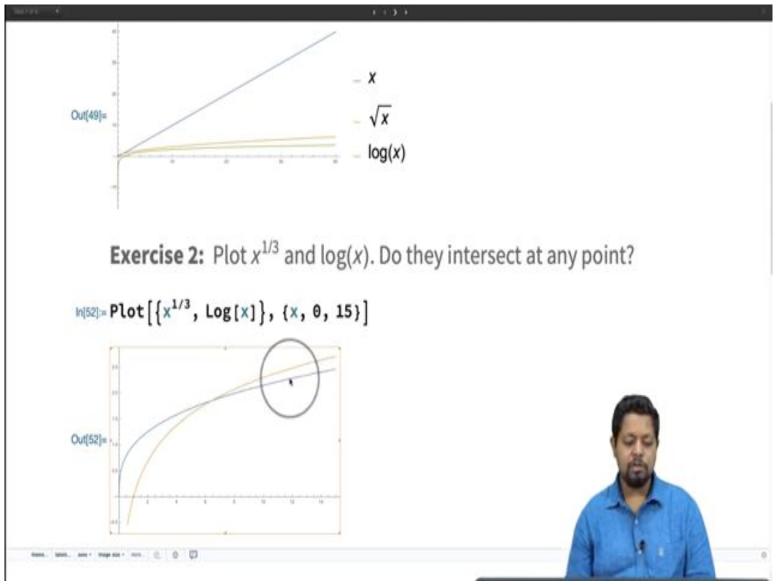

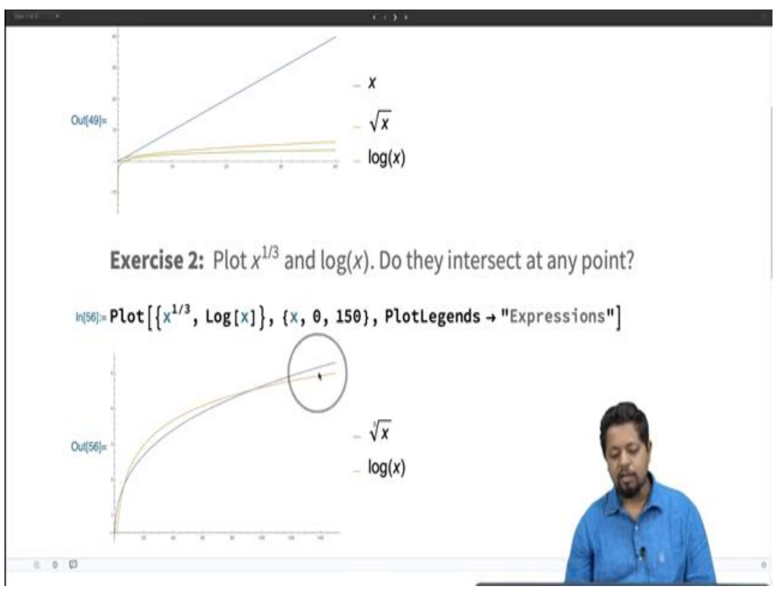

Let us go to the second exercise, we are going to plot $x^{1/3}$ and log(x) and see if they intersect at any point. So we want to plot $x^{1/3}$ and log(x). You can say x is between 0 and 2, I do not know the answer of this question you may not know answer this question but by plotting or visual thinking we can very quickly find out if that is happening or not and it looks like I have plotted, I have taken the range of x from 0 to 3 over here.

It looks like that they might intersect, so let me go and increase the range, okay. Increase it further, there we go, they intersect somewhere over there, okay. Now we can go and solve this equation $x^{1/3} = \log(x)$, equate this and that okay and solve it either analytically or numerically, whichever way you like. But the more important question is that $\sqrt{x}$ never intersected with log(x).

But cube root of x intersects with log(x), so what is the radical, what is the power of x for which you get the first time as 0 for log(x) and square root of x. That is one question, the second question which I find more interesting over here is that you know that $x^{1/3}$ will eventually dominate log(x) because log(x) is a very very slowly growing function. So $x^{1/3}$ will eventually grow faster than log(x).

But over here what do we see, we see that log(x) is subdued, log(x) is subdued $x^{1/3}$. In case you are confused let me add PlotLegends, so we see that log(x) has subdued cube root of x however we know that as x becomes very large this has to dominate. So how do we see that, let us go ahead and increase its range to may be, let say 45 and that still you see on over there curving that mean they are coming closer.

So let me go ahead and increase this further, there we go. So somewhere around 90, they cross again and beyond this beyond this you see log cube root of x will dominate log(x). So there are two crossings for these functions, $x^{1/3} = \log(x)$, there are 2 roots of this equation, one a small root where cube root of x was dominating log(x), then eventually log(x) started dominating cube root of x and finally cube root of x start dominating log(x) again.

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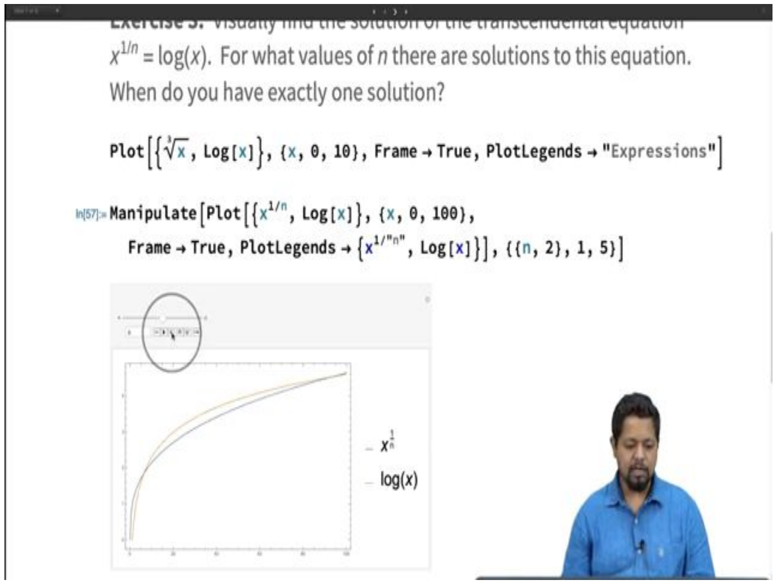

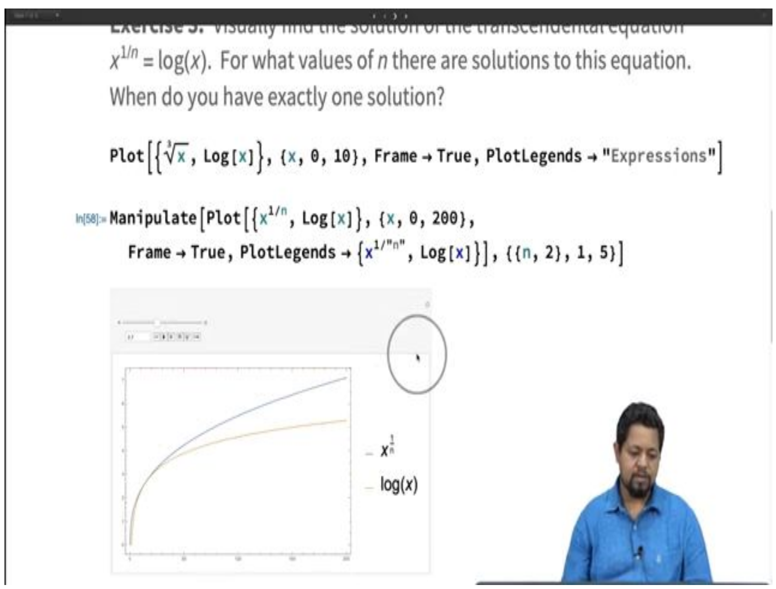

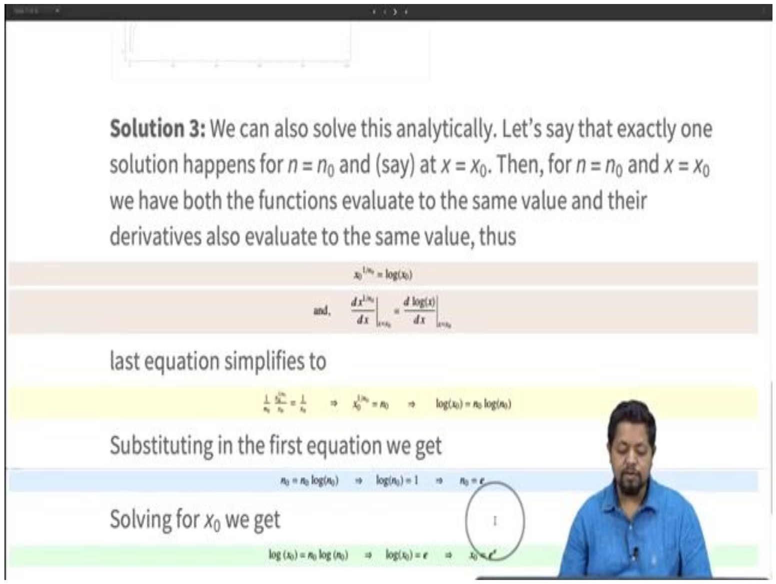

So the third exercise actually asked you to find out using visual thinking, just by visually plotting, figure out visually, find the solution of the transcendental equation $x^{1/n} = \log(x)$. So we know that $x^{1/2} = \log(x)$ has no solution, $x^{1/3} = \log(x)$ has two solutions. So there is some value of n between 2 and 3 for which this equation has got exactly one solution.

The question is what values of n there are solutions to this equation when do you have exactly one solution. So we need to find the value of n for which we have exactly one solution and then after that what happens is that we are going to always have two solutions. So in order to solve this exercise here is a line of code, this is something we have already executed, so let me go ahead and execute this line of code.

So what does this line code do is: we use Manipulate command to plot $x^{1/n}$ and log(x) and we vary n. So we start with n = 2 there is no solution, so the initial value of n we have chosen is n = 2. You can go make n smaller and we will see there is no solution, if you make n large beyond 2 there is no solution and then somewhere there, there we go. Now there are two solutions, so n is 2.78 we see there are two solutions and if you make it slightly smaller there is no solution.

So somewhere on 2.7, just start to see that there is a solution that there is exactly one solution somewhere around 2.7 whether it is 2.71,72, 73 that we have to work out but you see at 2.8 we have two solutions and 2.7 looks like this, there isn't any solution let us go back and fix this range, this is 100 so okay. So let us go back, go forward and see how it works, 2.8 there are two solutions, 2.93 there are two solutions and this distance between the two solutions keeps on increasing.

So if I go ahead and make this 200 and again go to n = 3, as I increase n now these solutions keep on going further and further away. In fact the bigger, the larger solution is receding very-very fast. Okay so let us find out that point where we exactly have one solution. Now visually we have figured out it is somewhere closer to 2.7 not exactly 2.7 somewhere closer to 2.7 may be somewhere greater than 2.7. Let us go and solve this problem analytically and find out where is that solution.

So here is my analytical solution, to solve this analytically, let us say that exactly one solution happens for n = n0, so there is one solution exactly for n = n0 and say that happens at x = x0, so the root is x = x0. Then for n = n0 and x = x0 we have both the functions evaluated to the same value and the derivative is also evaluated to the same value.

Because if they are just touching, for example in this graph over there at 2.7 when they are just touching the derivative of the two functions, this and this are matching, that is $x^{1/n}$ and log(x) their derivatives are matching and they also have the same value. So that is what we are going to use to find out that root. So therefore we assume that the root is x0 and their corresponding value is n0.

So the first condition says that x0 must be equal to the log(x0), the second condition says that the derivative should also match at those two points. So let us go ahead and and calculate the derivative, d of $x^{1/n0}$ over dx evaluated at x = x0 is 1/x0 which is shown over here, which simplifies to $x0^{1/n0}$ = n0 which you can also write as log(x0) = n0 log(n0).

Okay, now from the second equation we get the condition $x0^{1/n0}$ is n0, so if we substitute that in that equation we get, if we take this equation and substitute that into the first equation over here this will become n0 and log(x0) we have calculated over here which becomes n0 log(n0). So we are eliminating from this equation, we are eliminating x0 in favour of n0

On from both the left hand side and the right hand side when we do is that we get this equation n0 is equal to n0 log(n0) which says that log(n0) must be equal to 1 and that tells you that n0 is the Euler constant E which has a value of 2.7, it is close to 2.7 its exact value we can work it out numerically and we can also work out x0 , x0 is log(x0) is E and therefore x0 is e^E.

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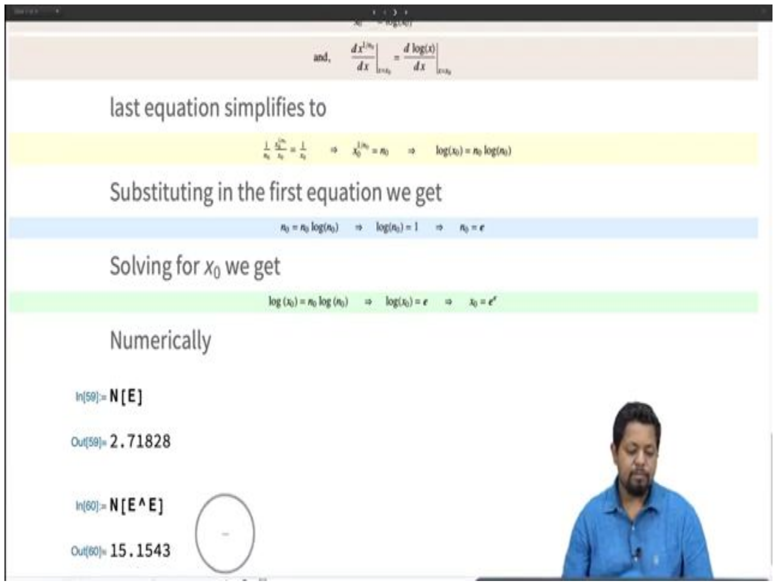

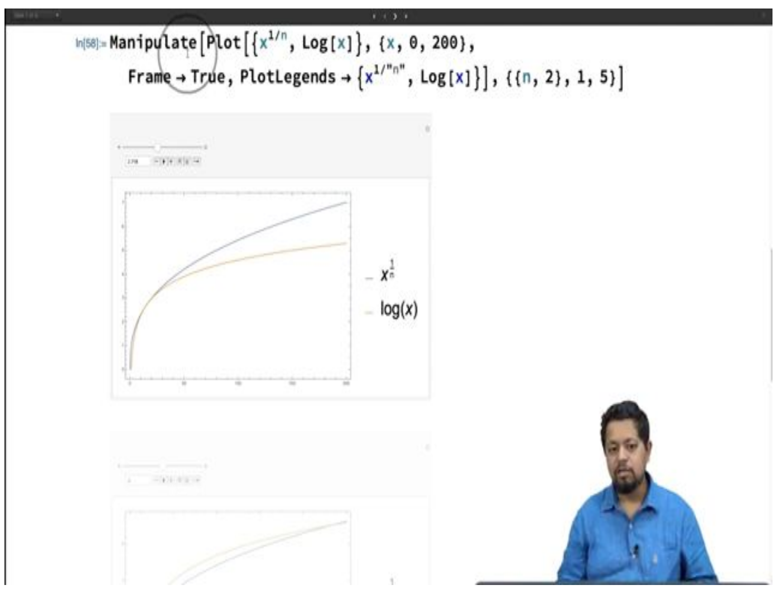

We can numerically work these things out, so to work out things numerically we use a numerical function. E is the Euler constant, in Mathematica capital E is the Euler constant we evaluate, N is a numerical function, so doing N[E] gives me the value 2.718 and e^E is the value of x0 which when I evaluate I get 15.1543. So let me go back to my plot here and put in 2.718 and there we go, that is the root of this equation for n = 2.718 we have exactly one solution log(x) and $x^{1/n}$ meet exactly at one point.

So by visual thinking before even solving these equations we were able to find out the approximate solution of this equation, we were able to develop some intuition of this. For this problem we were able to understand when there is no solution for x, for this equation and when there are two solutions for this equation and then doing an analytical algebra we were able to exactly find out the solution.

But we can very quickly get quick answers by simply using visual thinking, by simply plotting in various ranges changing the value of n, using the manipulative command makes it really easy, to vary the parameters and develop some understanding about the problem. So that was the main focus of today's lecture, we will continue with some more examples of visual visual thinking in the next one.