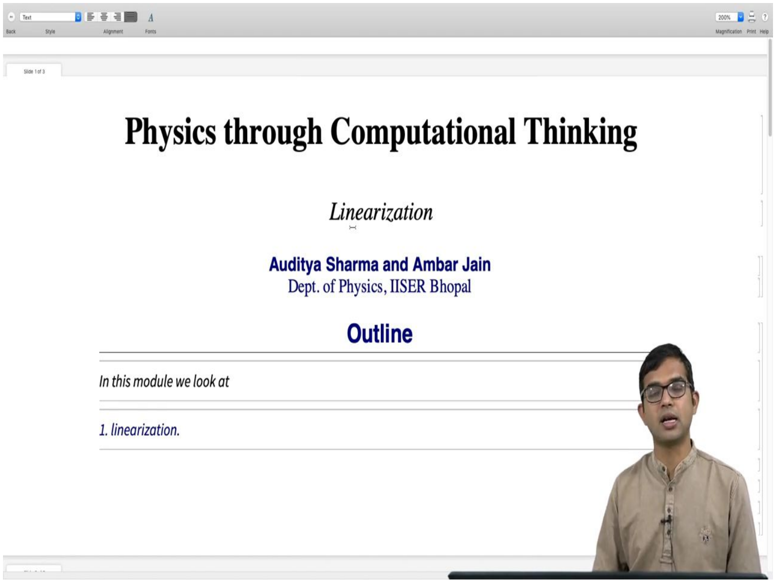
Physics through Computational Thinking
Professor Auditya Sharma and Professor Ambar Jain
Department of Physics
Indian Institute of Science Education and Research Bhopal
Module 07 Lecture 37
Linearization 1

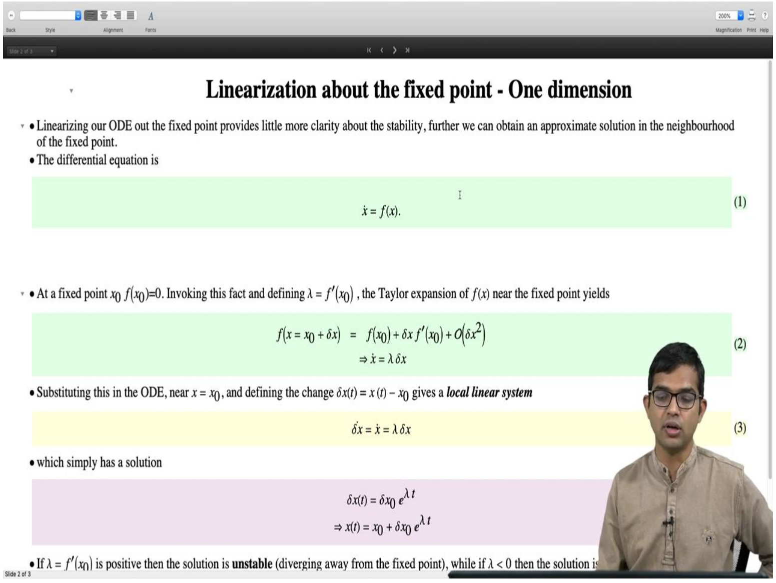
Hello guys. So, we will continue from where we left off last time, which is about, we introduced the idea of linear systems. We saw how they can be solved. We saw the usefulness of the phase space picture and then we saw how, you know how this, the general theory of these linear systems in terms of tau and delta play us out, with many examples, including that of the Romeo Julio Juliet problem of Strogatz. So, now, you might be thinking why we spent so much time on such very simple linear systems. Of-course, you can just solve it analytically. Why bother to study all this?

(Refer Slide Time: 1:10)



So, here, we will hopefully try and answer this type of question. So, which is the idea of linearization. Right, So, the vast majority of problems, which arise out of real life in Physics are actually not linear in nature, they are nonlinear in nature. And they are typically very hard problems to work with. So, we will look at how linearization plays out and how this helps us, And how insights from linear systems can be applied to other kinds of systems.

(Refer Slide Time: 1:35)



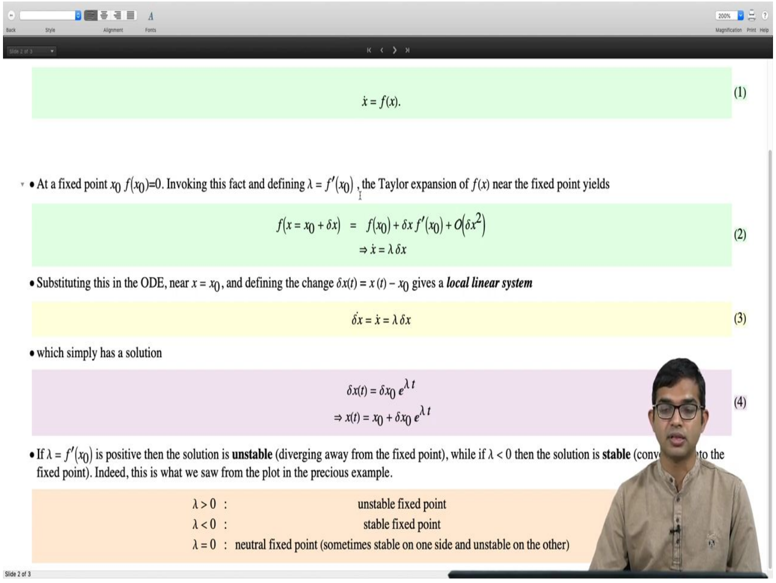
Okay. So, the idea here is linearization. Right. So, suppose you have some arbitrary differential equation. For simplicity, I will look at just 1 dimension for now. So, I have the differential equation $\dot{x} = f(x)$. Now, $f(x)$ can be a very complicated function. There is no requirement that the system has to be linear. So, it turns out that $f(x) = 0$ is you know points x_0, such that $f(x_0) = 0$ are points of particular interest. Right.

So, finally I am defining for you the idea of a fixed point, which I have already used in my previous discussions of linear systems. So, but it is intuitively clear, what it means. So, basically if you are at a fixed point, there is no flow, your system is basically stuck there forever, if you are at the fixed point.

And in 1D, this is particularly restrictive. So, let us look at this and then go on to to 2D. Now, if you are at an x_0, which is a fixed point $f(x_0) = 0$. So, $\dot{x}$ is 0 and therefore x is going to be the same point x_0 forever, all time. Right, so, invoking this fact and if you define $\lambda = f'(x_0)$, it turns out that the derivative of this function $f(x)$ at x_0 gives you some information.

So, let us say the Taylor expansion of $f(x)$ near the fixed point, it will give you $f(x_0)$, plus some $\Delta x f'(x_0)$, some plus order of the 2nd order term. So, basically to, 1st order, your $\dot{x}$ is going to evolve. If you are at a slight x away from, away from this fixed point, it is going to evolve as $\dot{x} = \lambda * \Delta(x)$.

(Refer Slide Time: 3:18)

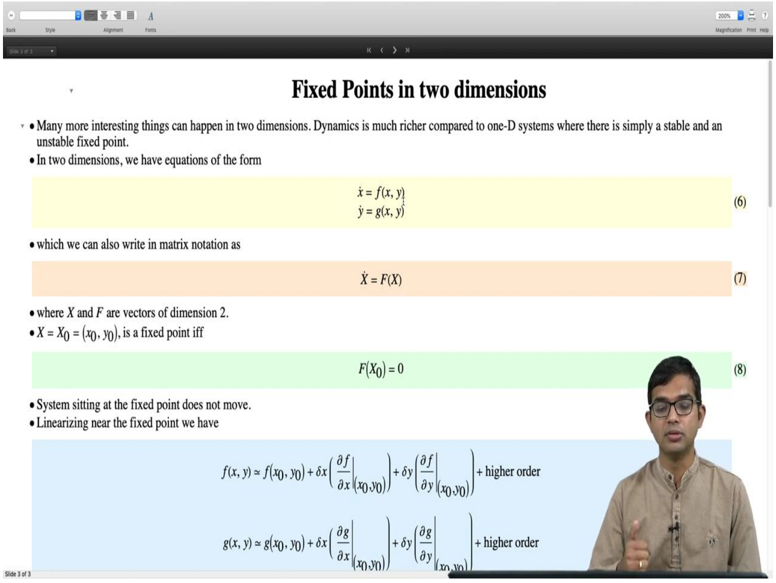


Now, whether your system is going to keep on running away from your fixed point or whether it is going to get merged into your fixed point, will rely on the sign of λ. Right. So, if λ is positive, then it is going to keep on increasing. Right. So, the solution is, $x(t) = x_0 + \Delta(x_0) * e^{\lambda t}$. And if λ > 0, so then you get what is called an unstable fixed point.

If λ < 0, you have a stable fixed point. And if λ = 0, this is you know what, sometimes it is called a neutral fixed point. So, this terminology must be used with care, because it is, you know the word 'neutral fixed point' is also associated with that of the harmonic oscillator, where you have you know these trajectories, which are circular trajectories around it.

So, maybe we should just, you know call this as λ > 0 is unstable, λ < 0 is stable. And then the λ = 0 type of fixed point. Right. We will move on to the case of 2D, which is significantly more complicated. The 1D situation is extremely restrictive. You basically have either a stable or an unstable fixed point, for all practical purposes and this pathology of λ = 0.

(Refer Slide Time: 4:40)

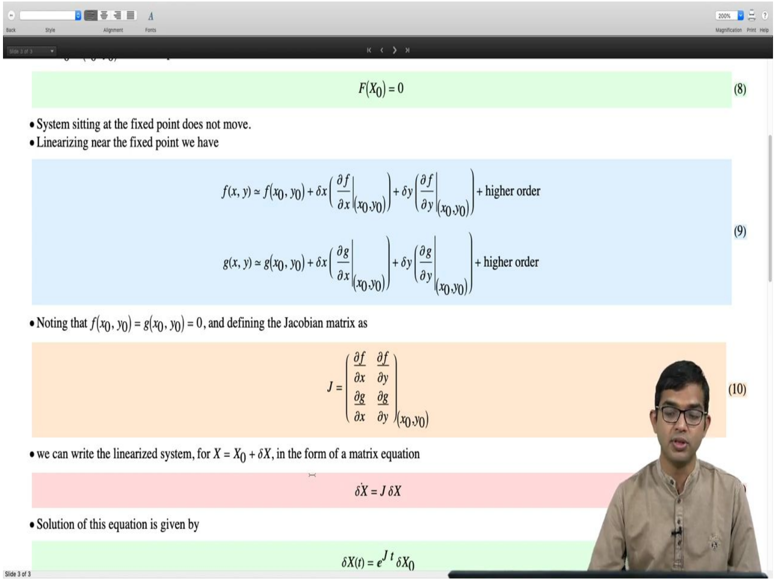


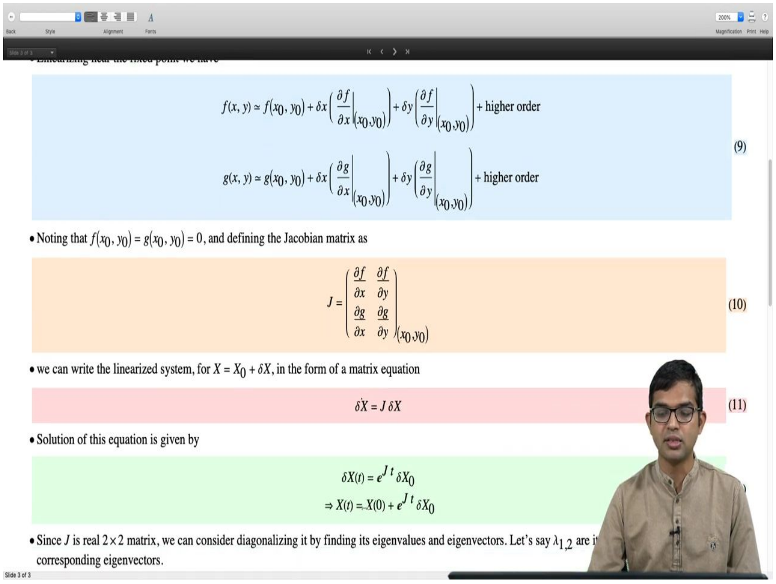
Fixed points in two dimensions. So, we have already seen this. But the point of this discussion is to argue that, in fact it encapsulates a broader set of systems. It is not necessary to look at only linear systems, but you can also linearize around a fixed point. That is the whole power of understanding linear systems very well. Right.

So, basically what you do is, you take some complicated systems, like $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, which you can write in vector notation as $\dot{X} = f(X)$. And then you just simply find all the fixed points of your system. So, in here, the fixed points are given by $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$. Right?

There should be no flow in either direction. It is not enough, if only one of these 2 functions is 0. Right. Both of them are 0. Basically, it means that your system is going to be stuck at that point, forever, x_0, y_0 together will constitute a fixed point. Right. So, this is the vector equation.

(Refer Slide Time: 5:41)



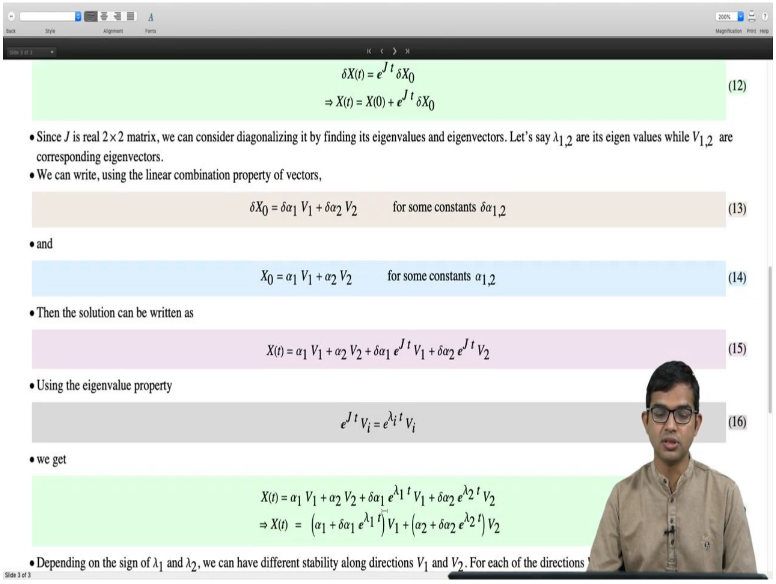


And then what you do is, you linearize around this, like just like what we did for, in the 1D problem. You can come up with a Jacobian, $f(x,y) = f(x_0, y_0) + \Delta x * \partial f/\partial x$ evaluated at that point, $+\partial f/\partial y * \Delta y$ evaluated at that point, in higher order terms. Likewise for g.

And then, noting that $f(x_0, y_0)$ and $g(x_0, y_0)$ are both 0. You can rewrite this whole thing, in terms of this Jacobian matrix, evaluated at x_0, y_0. And for this linearized system, you have $x = x_0 + \Delta x$. And then you can ask, what is the time evaluation of this Δx, if you are slightly away from your fixed point, what will be the dynamics. Right.

So, it turns out that the solution is very similar. Right. You can write a vector solution. Now $x(t) = x(0) + e^{jt}$, where j is now a matrix, this Jacobian times, $t \; \Delta x_0$.

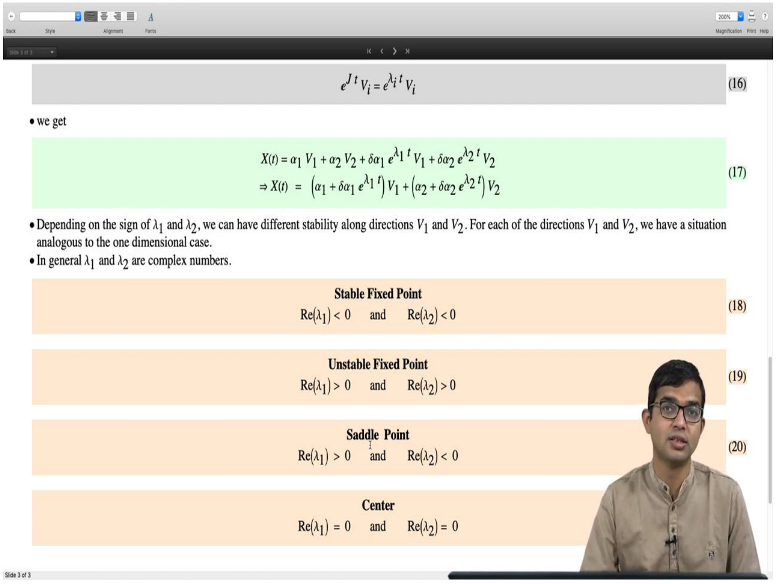
(Refer Slide Time: 6:46)



Now, we can write using the linear combination property of vectors. If v_1 and v_2 are the eigenvectors of your Jacobian matrix, you can write down the solution, $x(t)$ as, you know first of all, you write your initial conditions and expand it in terms of the linear, in terms of the eigenvectors of your Jacobian matrix. And then the full solution comes out to be just, $x(t) = \alpha_1\, v_1 + \alpha_2\, v_2 + \Delta\alpha_1\, e^{jt} v_1 + \Delta\alpha_2\, e^{jt} v_2$.

So, then the crucial point comes in here now, $e^{j\,t} * v_1$, is a very simple object now. It is just going to be a phase times that vector itself, because we have considered eigenvectors of j. So, $e^{j\,t}$ acting upon an eigenvector will just give you, e to the lambda i t, times that particular eigenvector. And so then, your final solution simplifies a great deal. And so, you have this as your final solution.

(Refer Slide Time: 7:49)



So, now of course in general, both these, λ1 and λ2 can be complex. Right? So, the nature of the solution and whether your system will run away from where you are or whether it will come towards this fixed point. Now it is no longer necessary that the origin is the fixed point in general. It is some arbitrary point. But basically, you can think of it as an exercise in shifting your origin. You linearize your system and go to whatever fixed point you are looking at, keep that as an origin and then you have a linear system.

And then we have actually worked out this theory of linear systems in great detail.

So, you can, you know you make use of this tau delta diagram, that we already have and where we have seen that, you know borderline cases will give you, you know degenerate node or centers and so on. But otherwise, for all values of, you know in the, for delta less than 0, you get a saddle point.

So, saddle points are the most common kind of fixed points. And in this whole classification, we have already looked at it. So, hopefully from this discussion, you will take home the idea that, you know linearization gives so much power to a very detailed understanding of linear systems, because even nonlinear systems, although you cannot solve them analytical exactly.

You may not be able to tell, exactly what $x(t)$ will be, $y(t)$ will be. But you can say, whether it is a fixed point is, what is the nature of the fixed point. And by linearizing around that point and getting the qualitative dynamics of the system out from this. Okay, thank you.