Physics through Computational Thinking
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Module 07 Lecture 35
Linear systems – Insights from the phase space picture 1

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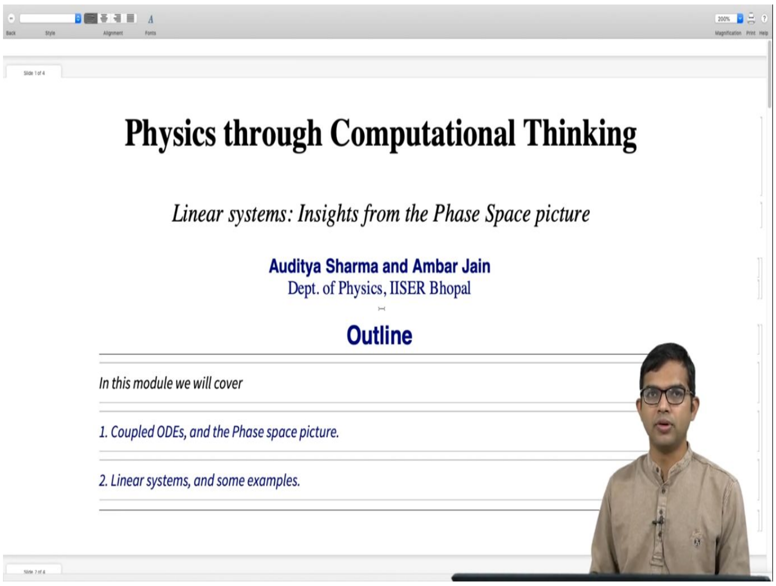

Hello everybody. So, today we are going to continue from ODEs, that we have been looking at. And then, so, we will develop a useful alternate picture, you know where we have some qualitative understanding of the systems can be pulled out, from an analysis of the so-called phase space picture, right. So, often times, you do not want to solve the full problem in a brute force way. But you just try to extract as much physics out of it, just by looking at the structure of the equations and so on. And so, this is where this analysis comes in.

And so, we will develop this within the so-called linear system approach, where an, an exact solution is possible. So, we will discuss how that happens and we will try to fit into this framework. You know examples which we have already seen. But then, what this does is it actually builds a nice platform, on which you know more difficult non-linear problems can also be analyzed, along these lines, where it is not possible usually to find a full analytical solution. So, whereas this, the insights from this phase space picture will be very useful.

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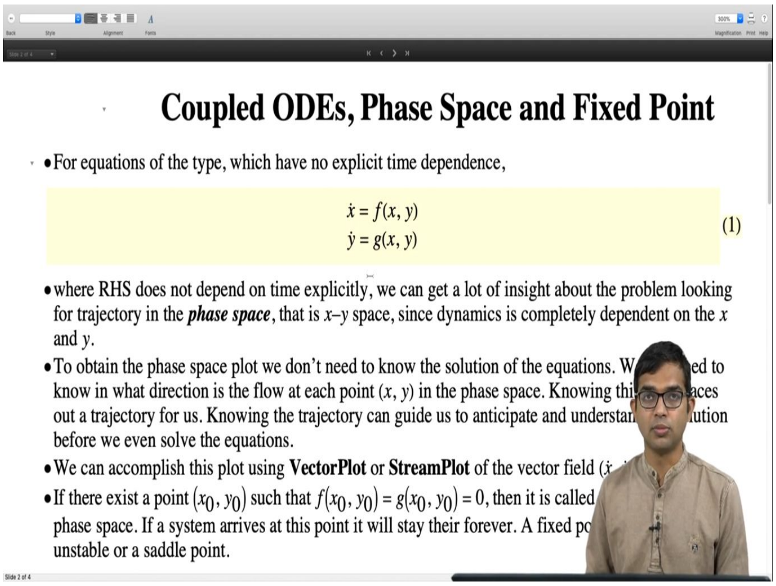

Okay. So, let us say that you have a system of equations involving 2 variables. I have taken 2 variables for simplicity, this can be easily extended to more number of variables. So, $\dot{x}$ equal to some arbitrary function of *x*, *y* and $\dot{y}$ equal to some arbitrary function of *g*. I am calling it *g*(*x,y*). So, where the RHS does not depend explicitly on time. So, we want to develop this phase space picture.

So, where we just look at; you know, I take any point in the xy plane and ask; what the point is doing there and where is it headed next. Suppose your system were caught at some time t, at some point xy. So, this; what is the time does not really matter, because you are considering a system, where the velocities are determined only by where you are. So, you are going to be able to extract a flow of where this system is standing. And then by studying the dynamics of this flow, a lot of very interesting and you know lot of insights can be drawn, you know from a qualitative study of these phase portraits, as they are called.

And Mathematica has a very useful plotting function, which is you know, we will use this thing called Stream Plot. You can play with Vector Plot and how; and and, you know basically carry out these kinds of studies. So, what we do is we will just look at the trajectory of this particle, based on the magnitude in that direction of the flow.

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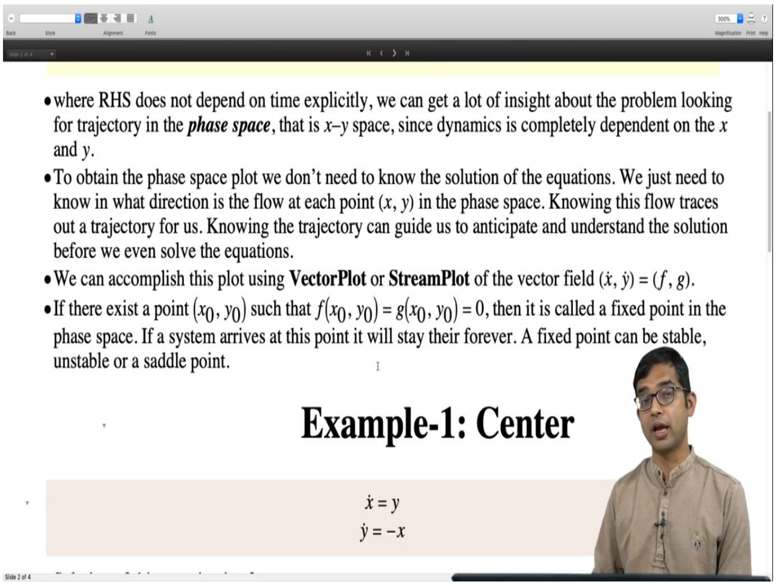

So, let us look at a very simple example first. Suppose I have a system, where $\dot{x} = y$ and $\dot{y} = -x$. Right; so, if you, you can pause the video for a moment and convince yourself that this a problem that you have already solved. What is this problem? Okay; so, now I am going to continue and tell you the answer. So, the solution is of course, first of all, I will write down the solution, which is just $x(t) = A\,cos(t) + t_0$ and $y(t) = -A\,sin(t) + t_0$.

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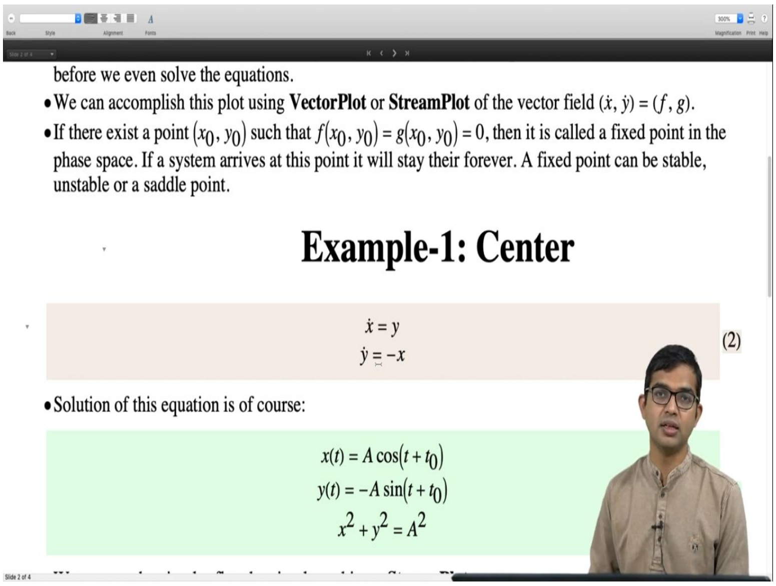

The solution is this, because it is really the harmonic oscillator. You can think of *x* as the position and *y* as the one-dimensional speed. Right, it is the 1D harmonic oscillator. A very simple problem. If you want to see this, you can write it as $\ddot{x} = -x$. And then you will see this. And so, the way it works is, to find out the trajectory in the xy plane, you want to eliminate this, you know these constants; you, you want to eliminate time basically.

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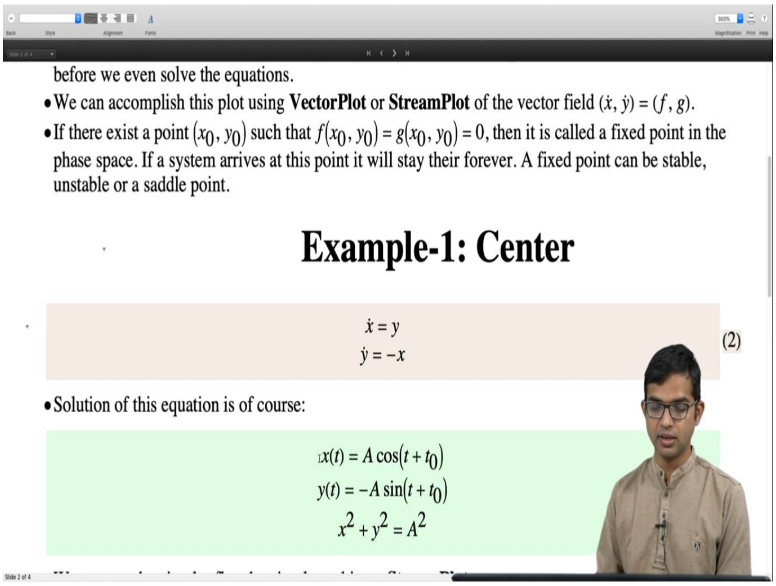

So, you want to write it as just functions of *x* and *y*. So, then you have $x^2 + y^2 = A^2$. And depending upon the magnitude of *A*, you are going to get a circular flow. Right; all of this is very nicely visualized, with the help of Stream Plot. So, the syntax for Stream Plot is simply; you must provide the flow, magnitude and the direction here, in terms of these, you know these 2 quantities have to be provided inside these flower braces; y comma minus x in our case. So, $\dot{x} = y$, $\dot{y} = -x$. And then you must tell the limits of *x*, that you are interested in and the limits of *y* that you are interested in. So, *x* goes from -1 to 1 and *y* goes from -1 to 1, is what I am looking at.

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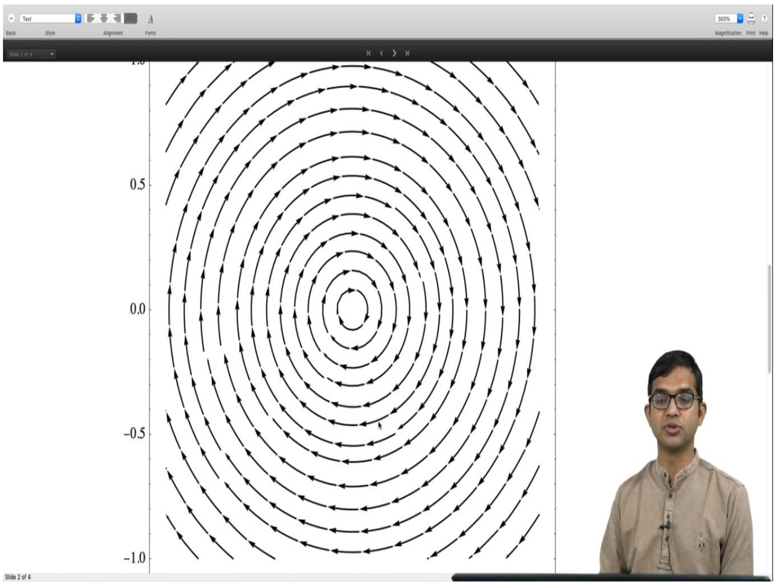

If I hit ‘shift enter’, then so, there you go. So, you see a nice flow diagram. So, this is what is called the phase portrait picture of the harmonic oscillator problem. So, it tells you that basically there is no loss of energy. Right; so, we know this. And the simple harmonic oscillator; the system keeps on going round and round, in exchanging kinetic energy to potential energy and potential energy to kinetic energy, again and again and again repeatedly.

And depending upon the, this as a radius of the circle that you are in, is a measure of that amount of energy that is trapped inside this system. Very familiar stuff, but from an alternate picture. Now let us look at another example. Suppose I have x dot equal to x and y dot equal to y. So, then again you can solve this equation. And the answer is just x of t equal to A times… They are uncoupled. So, $\dot{x} = x$ and $\dot{y} = y$. So, x is going to keep on increasing; y is also going to keep on increasing.

And so, in fact, $y = B/A\, x$. So, that is the, you know when time has been eliminated, so, you get this kind of a curve in the xy plane, which is nicely shown here with the help of this Stream Plot. So, you see that all these, you know trajectories are straight lines, which are all running away from the origin. So, if you let a particle be anywhere other than at the origin. Right so, if you are at the origin, of-course it will stay at the origin forever. But slightly away from the origin, depending upon which direction you are headed in. It is going to just keep on running away to infinity, along a straight line. So, that is what the flow directions are.

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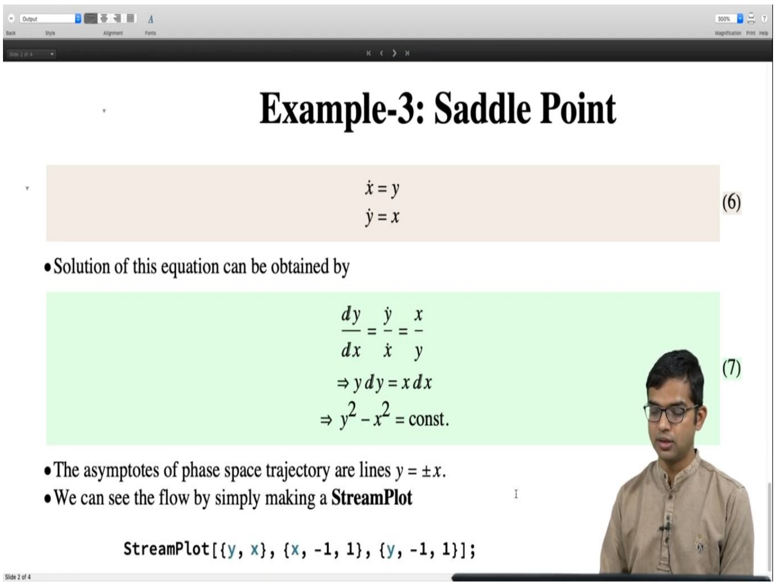

And then finally we look at the third kind of example, where you have $\dot{x} = y$ and $\dot{y} = x$. So, here, you can solve for it analytically. And so that answer turns out to be; you know after elimination, you get $y^2 - x^2$ is equal to constant. And here, the asymptotes of phase space trajectory are, are the lines; plus, or minus *x*.

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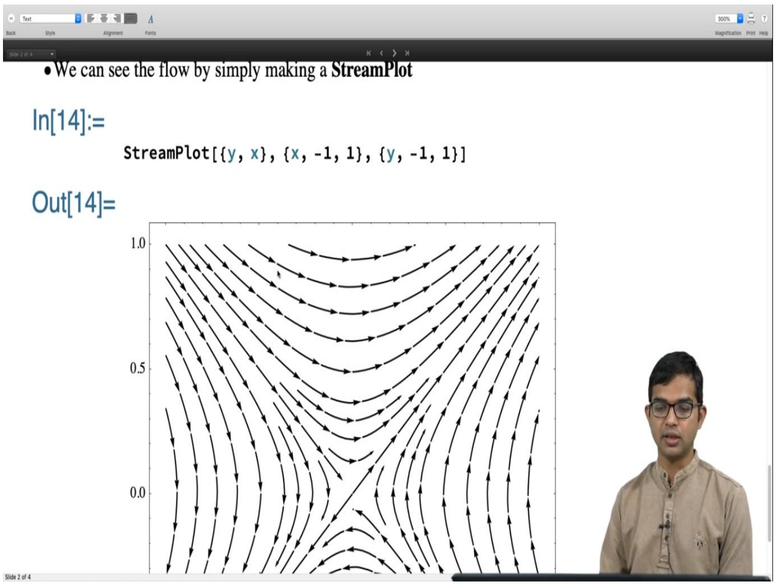

And now, you see that qualitatively the the flow is actually quite different from the previous case. And so, you see that for long times. If you are in one of these trajectories above, you you typically start from somewhere close to the line $y = -x$, if you are coming in from, if you are starting at a large negative time. And for large positive time, you are going to approach the $y = x$ region, if you are… It does not matter in which quadrant you start. It is eventually going to go to $y = x$. And for large negative times, you would be very close to the $y = -x$ line.

So, this is what is called the saddle point. So, these are; in fact, these three are very characteristic plots, which represent the so-called stable node, unstable node. And, so this is actually a very special type. You know, when you have circles all around, this is an unusual kind of a fixed point. There is a fixed point sitting at the origin. So, I will talk about this in a moment. But just to step ahead a bit, so, there is something called a fixed point, which is sitting at the origin. Basically, if you are at the origin in any of these differential equations, you will remain at the origin forever.

But if you are slightly away from the origin, in one case, the first case, you are at what it is called a neutral fixed point. It is basically going to keep on being at the same distance forever. But if you are in the second case, you are going to run away to infinity. And the third case, you would run away to infinity, if you are; you know if you are along, a certain direction you are going to run away. But in another direction, you know if you are carefully located along this line; y equal to minus x, then you would actually come back to the origin. Right? So, this is what is called as saddle point.

And it supposed to remind you of a saddle, which is we put on a horse and you see that, along one direction it is a minimum. But along the other direction, it is a maximum. So, that is where this terminology has its origin. Okay; so, let us make, we will make these notions a little more precise as we go along. But this is just a qualitative understanding of… So, let us move ahead and look at a few more examples. And then we will go into the general theory.

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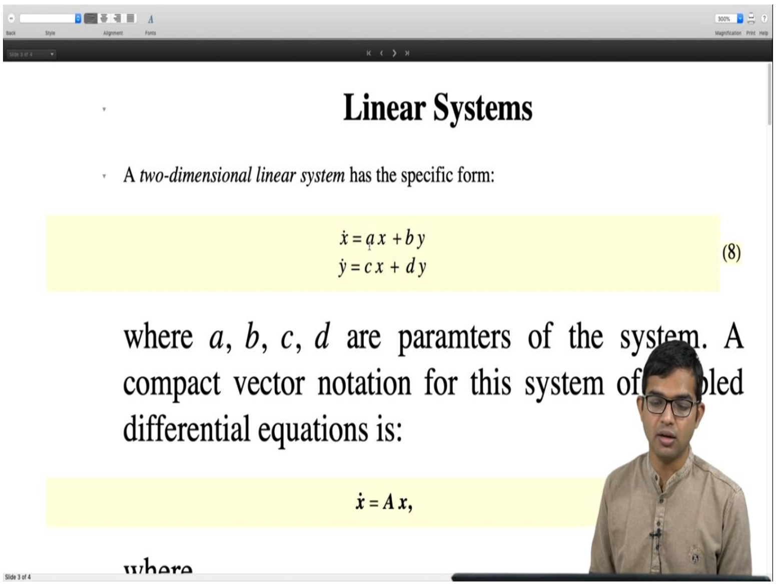

Okay, so, linear systems in general, you know with; if I have 2 degrees of freedom, x and y are available. So, I can think of a general linear system of this kind; $\dot{x} = ax + by$ and $\dot{y} = cx + dy$. a, b, c, d are parameters of your system. So, there is a compact way to write this, in terms of a vector equation. So, $\dot{x} = Ax$. Now this matrix A is of-course a, b, c, d. So, we can ask; what will be the, there is clearly a fixed point at the origin, always.

When $x = 0$ and $y = 0$, for sure, your system is not going to move. Its dynamics is null. It is just going to remain at the origin forever. But we have seen that, depending upon this matrix, the precise values inside this matrix A, the nature of the dynamics can be very different, around the origin, slightly away from the origin. That is what we are going to work out.

So, before we do that, let us quickly identify, you know which equations correspond to a linear equation. So, any linear equation necessarily has this form; underlying the linear equation or the linear system, is the so-called superposition principle. Basically, what it tells you is that, if you are able to find 2 solutions, then any linear combination of these 2 independent solutions is also a solution. So, this we have seen. And we have surely some intuitive idea of this.

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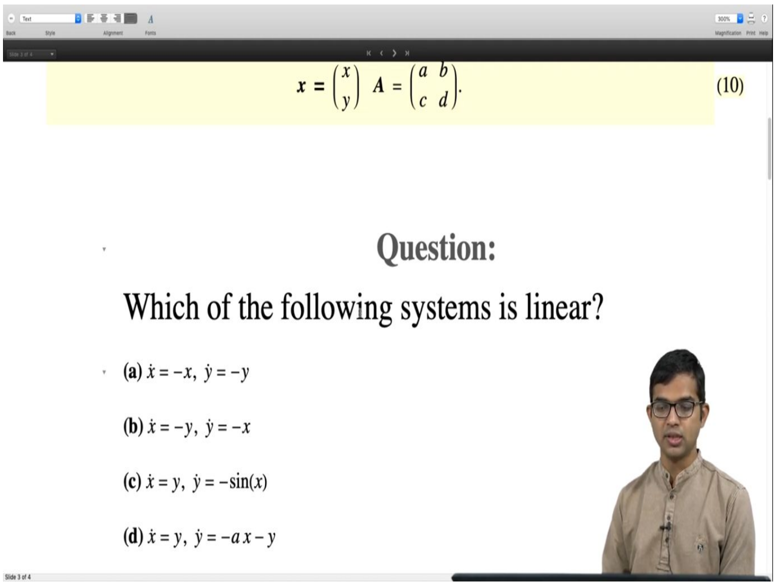

So, let us quickly go over this question. It is like a quiz. If you wish, so, you can pause the video for a moment and try to answer for yourself. Which of these four systems is linear and which of them are not? And I will give out the answer now. So, the answer is of-course ‘a’ is linear, b is linear. Yeah, I mean in fact, we have already looked at these examples. They are definitely of the form, which is described above. ‘c' is not linear, because $\dot{y} = -\sin(x)$ is involved. So, it is not linear. And ‘d’ is is also linear. It involves *x* and *y*. That is okay.

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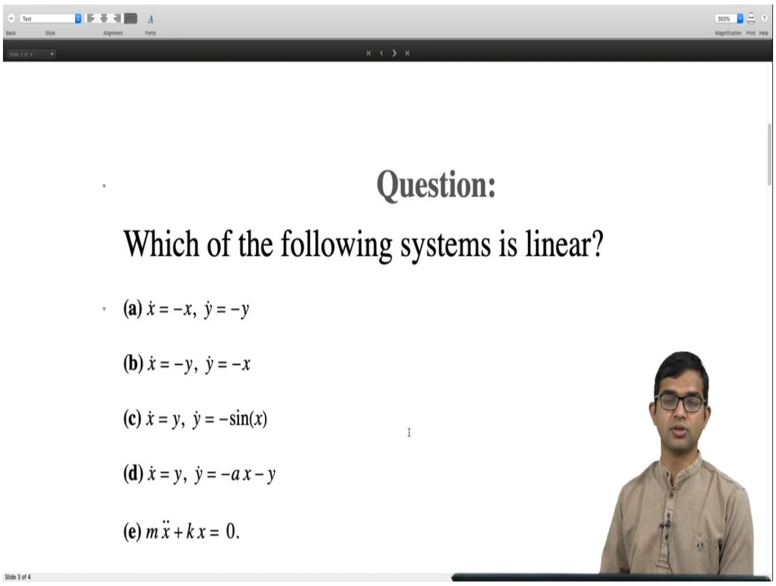

And what about *e*? Is it linear? It is linear. So, this is a place, where a lot people get confused. Linearity and order of the differential equation are two different things. So, linearity has only to do with you know, what is the power of your x or $\dot{x}$. So, in this case, you can just define $\dot{x} = y$ and then you can recast it in this canonical form. And it is still going to be linear. So, if you had $m\ddot{x}^2$ or $sin(x)$ or something more complicated than that, then it will be non-linear. And then the dynamics is generally much much harder to understand. Okay, let us move on.

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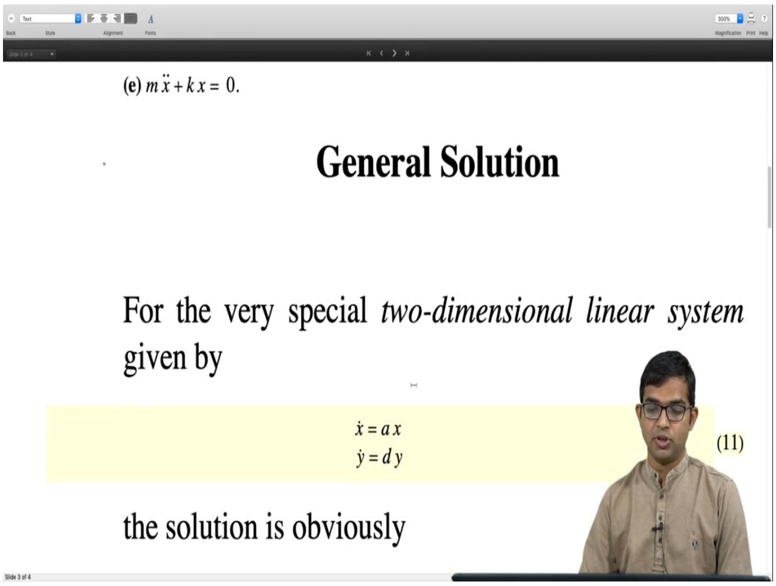

So, if you have a general 2D linear system of this kind, it is possible to go ahead and solve for it. So, we know; suppose you have this very special case; $\dot{x} = ax$ and $\dot{y} = dy$, which is uncoupled differential equation, then obviously the solution is just $x = e^{at}$ and $y = e^{dt}$. So, we ask whether the general solution can be of this form, can we take some vector.

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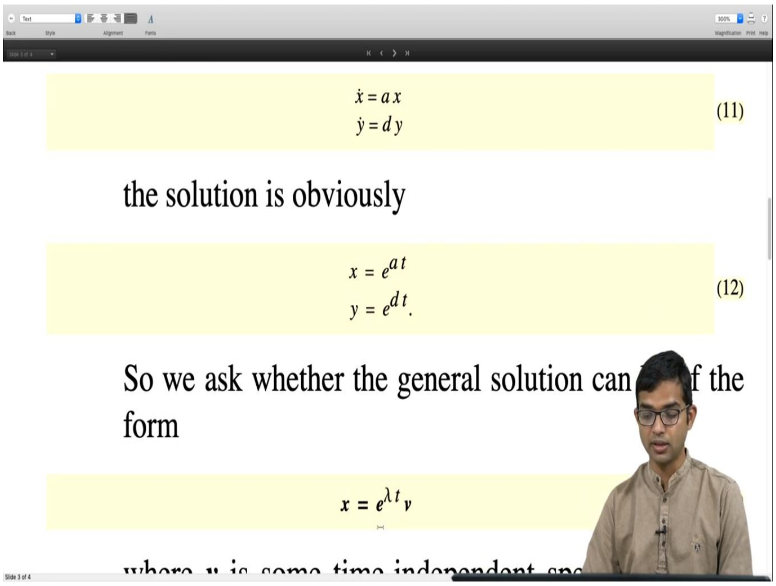

So, the general solution must be such thing, that just $e^{\lambda t}$ times some vector. Is there some special direction, which will still yield such simple solutions? And so, it turns out that when we make this as an ansatz and plug this into our differential equation, you know these vectors, we actually get a matrix eigenvalue equation. It becomes an eigenvalue problem. So, as you can see, if you plug this $x = e^{\lambda t} v$, into your original problem; $\dot{x} = ax$, so, $\dot{x}$ is of-course is going to give you $\lambda e^{\lambda t} v$. And then if you plug this, you really have to solve for $A v = \lambda v$.

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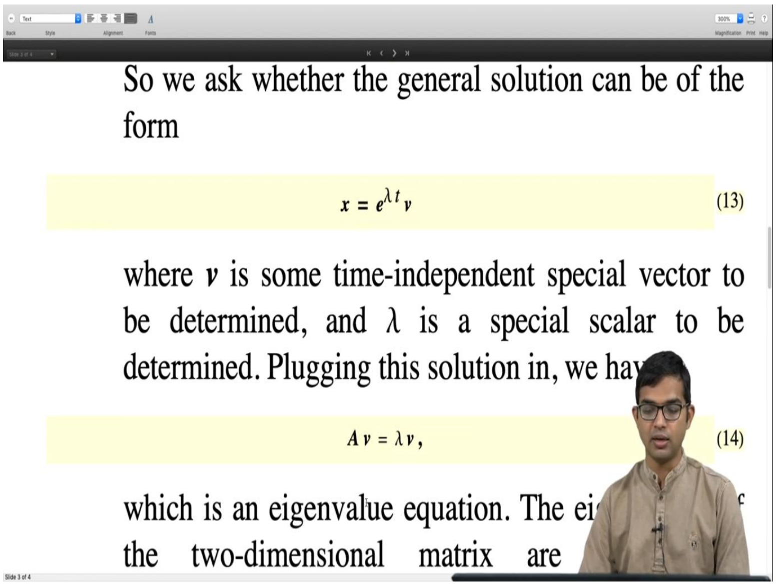

So, v is nothing but the eigenvector. And λ is the eigenvalue. So, if you can find λ and if you can find all the v's and the λ is corresponding to them, basically you would have solved this problem. Let us go about doing this systematically. So, the eigenvalues of the 2-dimensional matrix are conveniently expressed in terms of the trace and the determinant of the matrix.

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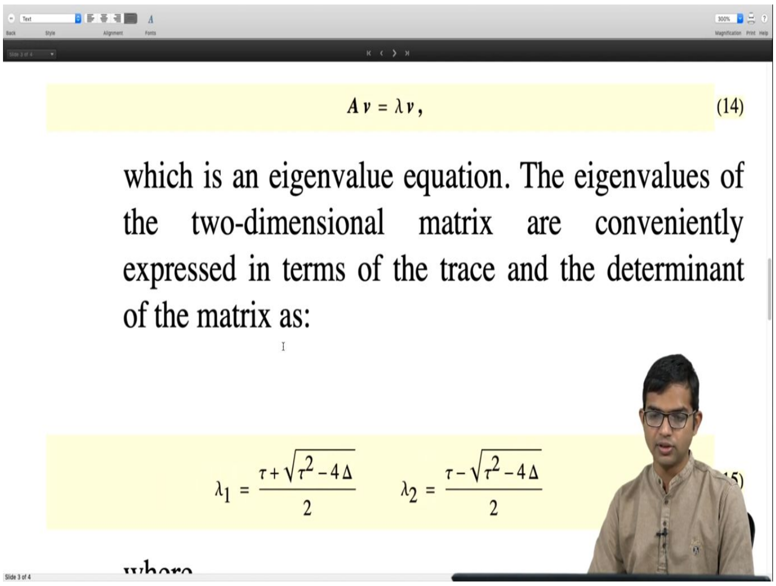

So, it is just a simple algebra. If you have not already seen this, I urge you to go back and verify this for yourself. So, if $\lambda 1$ and $\lambda 2$ are the eigenvalues of your (2 X 2) matrix, you can write this as $\lambda 1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}$. This is nothing but the root of a differential equation. You can verify this, solve this for yourself and check this. τ is the trace of the matric and Δ is determinant of the matrix. $\lambda 2$ likewise will be just $\frac{\tau - \sqrt{\text{determinant}}}{2}$.

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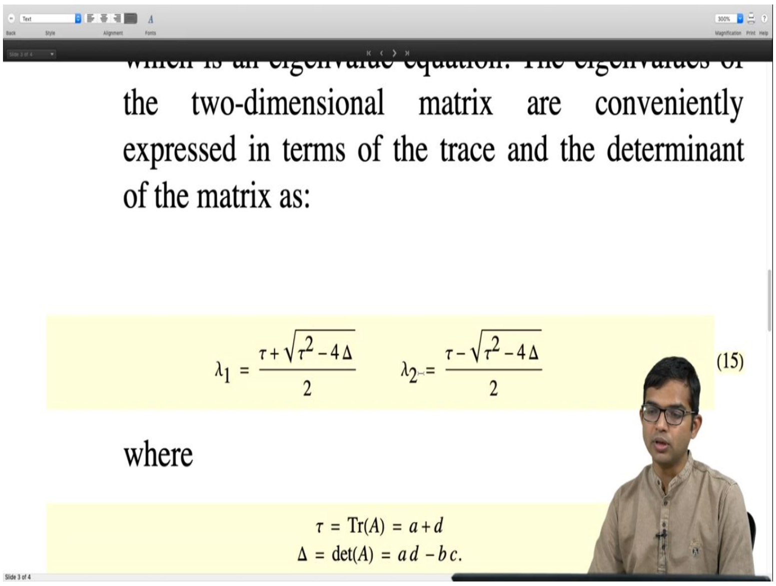

Now if $v1$ and $v2$ are, I guess we have called it $v1$. So, that is fine. So, $v1$ or $v1$; whatever you want to call this; are are respectively the relevant eigenvectors. So, in general $\lambda 1$ is not equal to $\lambda 2$. And this means that the 2 eigenvectors are linearly independent. And therefore, they span the space. Right; so, if you take a course on linear Algebra or in some Physics, Math methods type course, you will you may go into, you know some details of what or when the matrix is diagonalizable and when you can, it will give you eigenvectors, which are linearly independent and so on.

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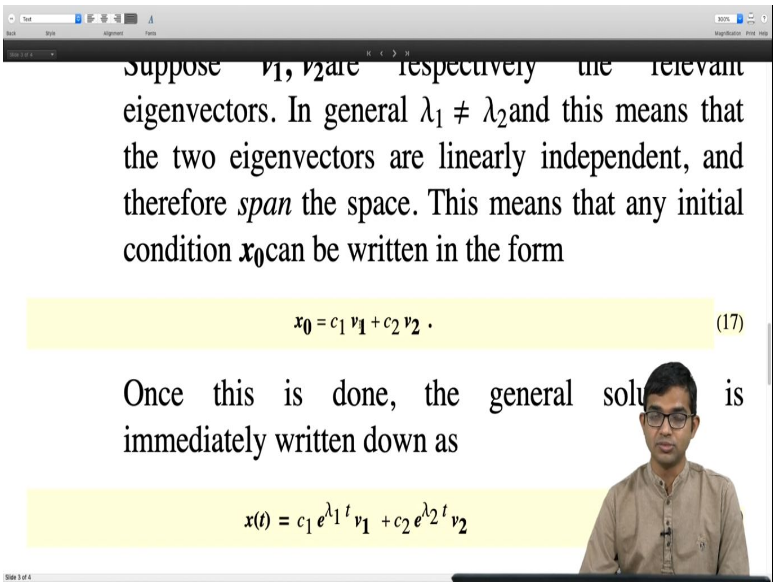

So, but let us not go into that discussion here. Suppose $\lambda 1$ is not equal to $\lambda 2$, then these 2 eigenvectors are linearly independent and therefore they span the space. Then it is possible to basically expand your initial conditions x_0 as a linear combination of these eigenvectors, independent eigenvectors; $v1$ and $v2$. It is possible to find these coefficients; $c1$ and $c2$. And once you do this, the general solution is immediately written down. So, it is just $x(t)$ is $c1 * e^{\lambda_1 t} v1 + c2 * e^{\lambda_2 t} v2$.

So, this is something that you might have also seen in a 1st course on Quantum Mechanics. If you try to solve the Schrodinger equation, Schrodinger equation is a linear differential equation. So, if you are able to find, you know all the eigenvectors, so, what do you do, you take the initial state of your system, expand it in the, as a linear combination of the eigen functions of your Hamiltonian. And then you just simply, the, the time evaluation of your full wave function is given by you know these phases, attached to the eigenvectors and these coefficients, these crucial coefficients come from the original expansion of your initial state, in terms of the eigen function. So, this is something that you have seen.

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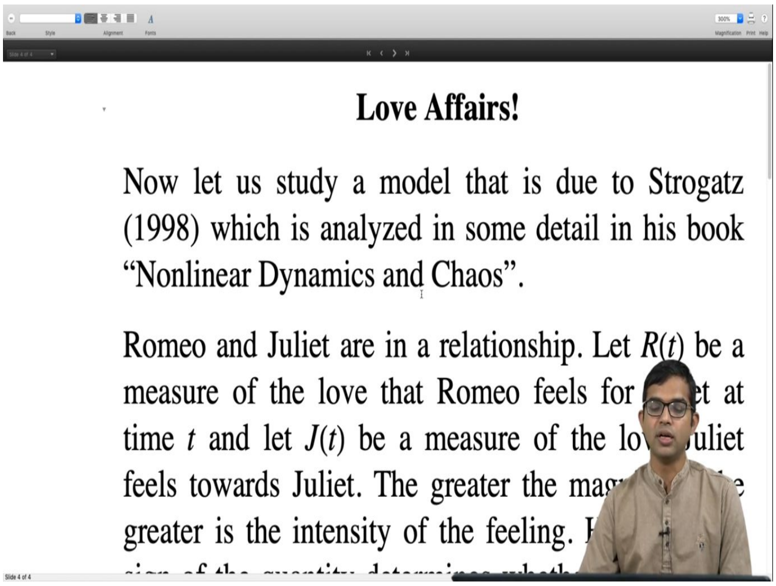

Okay. So, now what I want to do is, use this to describe a very interesting problem, which is given in Strogatz book on Nonlinear Dynamics and Chaos. So, this is about Romeo and Juliet and how their relationship can be modelled as a differential equation. So, let $R(t)$ be the measure of the love that Romeo feels for Juliet at time t and let J of t be a measure of the love that Juliet feels towards Romeo actually. It should be Romeo. So, let us change this. So, this is Romeo. Now the greater the magnitude, the greater is the intensity of the feeling. And however, the sign of the quantity determines whether it…So, the sign determines whether it is a positive emotion or it is a negative emotion.

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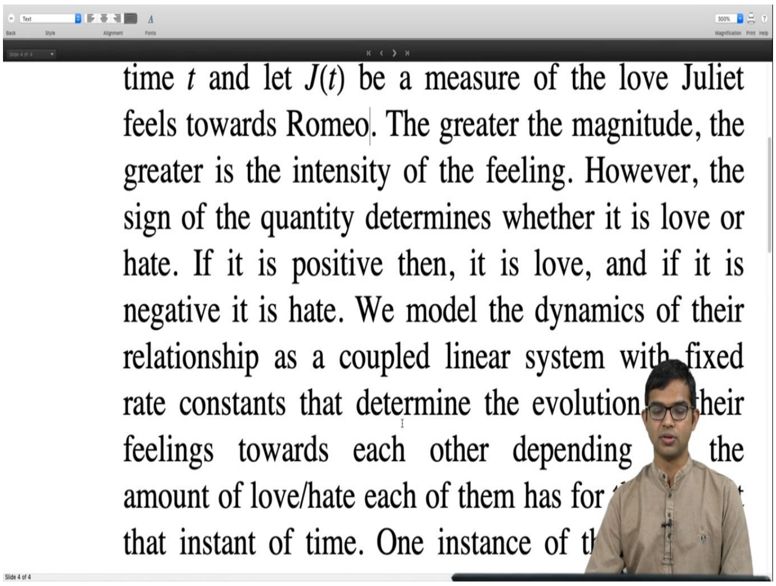

So, there could be also a repulsion and the repulsion would come from a negative sign. And the magnitude will give you, you know the the intensity of the feeling. It could be positive or negative. So, what you do is, you model the dynamics of their relationship as a coupled linear system, with fixed rate constants that determine the evolution of their amount of feeling for each other; positive or negative. One instance of this story has the following system.

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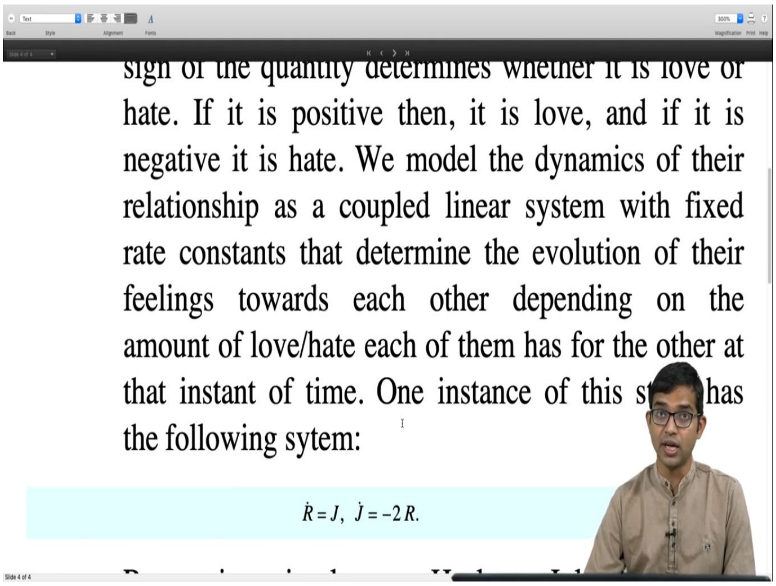

So, you say $\dot{R} = J$ and $\dot{J} = -2R$. So, this is an interesting coupled system. So, so Romeo being a simple man, he is, he feels greater intensity and he is; the more that, you know *J* increases, *R* also is directly proportional to it. And it is increasing in a positive manner. So, the more that Juliet comes towards him, the greater is his feeling for her. But Juliet on the other hand, is a little more complicated.

So, she is, she feels attracted towards Romeo as long as he keeps, you know he keeps himself at a distance. If he throws himself upon her, so, she has that tendency to run away. So, this is Juliet's nature. And so, you can ask what happens to this set of differential equations and where, you know what is the fate of their relationship, based on our analysis of linear, linear systems.

Clearly, this is a linear system, because you have $\dot{R} = J$ and $\dot{J} = -2R$, in our very simple model. You can of-course have a real life scenario might be even more complicated. It does not have to be exactly *J* and and - 2*R*. It could be something more complicated. But let us say, we stick to this model, because we are familiar with linear systems. And then see what happens in this case.

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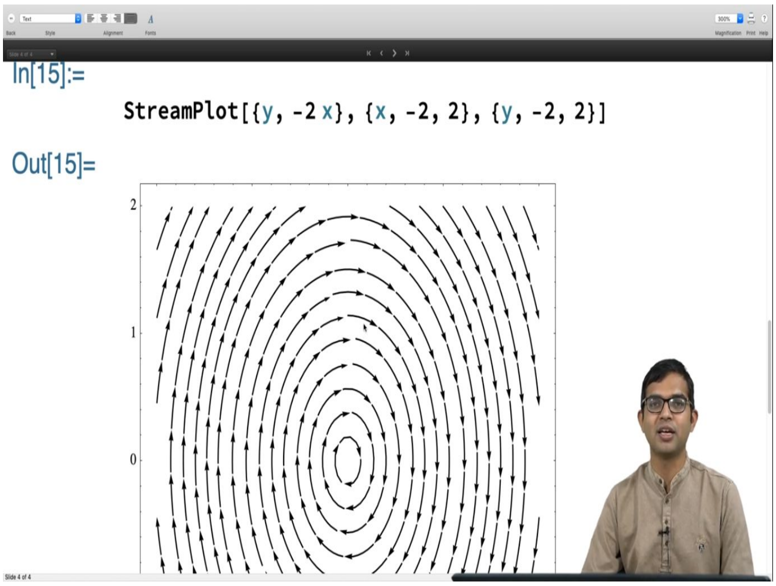

So, let us start by making a phase portrait. So, we have our familiar Stream Plot now. We make a phase portrait. And there you see; you have this picture, which is actually something very familiar. And if you think a moment, you will see that this is not a surprise at all, because this set of differential equation is something that we have already seen. This is nothing but the problem of the simple harmonic oscillator. It is a 1D problem.

So, there the dynamics of their relationship is an unending cycle basically. So, they are caught in some circle, which is determined by the total energy of your system, if you wish or the intensity of their feeling. But you see that in one quadrant, both; R and J are positive. So, both of them have a mutual interest in each other and both, it is positive in nature. But then there are these three quadrants, where at least one of them is negative.

So, one of them is going behind the other. But the other is unhappy. He is running away. And then there is the third quadrant, where both of them are basically running away from each other. So, it is a one fourth successful relationship. But somehow, at least one fourth of the time, there, there is mutual harmony and love. And it is a very, it is a cyclical dynamic all the time.

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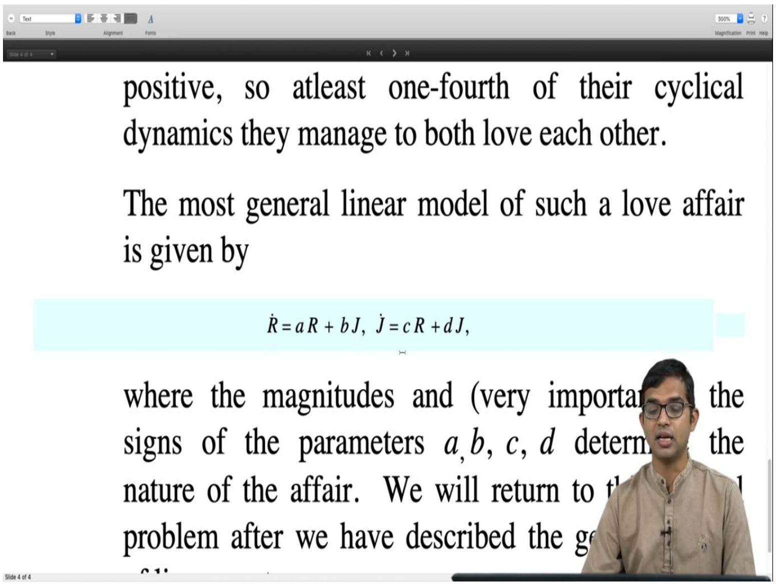

So, the most general such linear model that you can think of is, of-course $\dot{R} = aR + bJ$ and $\dot{J} = cR + dJ$. So, where you can play with all these parameters *a*, *b*, *c*, *d* and see what happens to their, you know relationship, what is the fate of their relationship based on all these parameters. This is an analysis that we can do. But before we do that, we will go and develop the full general theory of such a system.

So, I told you that, your eigenvalues are something that you can write down in terms of tau and delta. And in fact, all the qualitative nature of these dynamics is contained in these 2 parameters; τ and Δ. So, we will see how we can actually make a phase diagram, which tells you the nature of the fixed points underlining themselves; the nature of the dynamics if you wish. We have not, still not defined the fixed point. And based on just tau and delta; so, that is coming up next. Thank you for now.