Quantum Field Theory
Prof. Dr. Prasanta Kumar Tripathy
Department of Physics
Indian Institute of Technology, Madras

Module - 2
Interacting Quantum Field Theory
Lecture - 8
Interacting Field Theory – I

So far we have been discussing free field quantization. However, there is nothing much interesting that we can do in free field quantization, like scattering of particles. Two particles come and then interact with each other, they scatter and then go out. So, processes like scattering or absorption of particle, one particle is a incident of some other particle and it is getting absurd or decay of a particle of a nonstop particle and so on. None of these physical processes you can understand in free field theory. So, almost everything that is interesting is not explained in free field theory, for that you need to introduce interactions.

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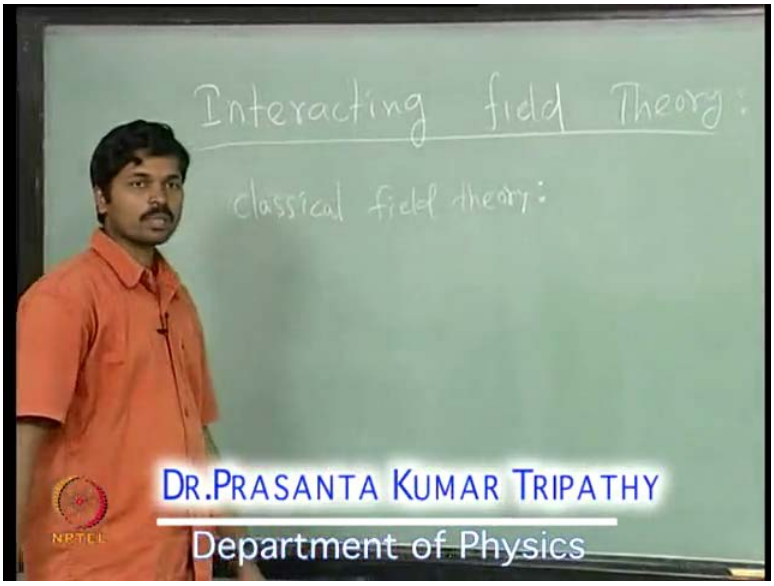

So, we need to actually study is interacting field theory. So, what we will do is will introduce interaction in classical field theory in a very simple example. Then I will consider interacting quantum field theories. So, let first consider how interaction appear in a very natural way, in classical field theory and then we will discuss how we can study interacting quantum field theories. One of the very popular example is a interaction of electromagnetic field with a charge scalar field.

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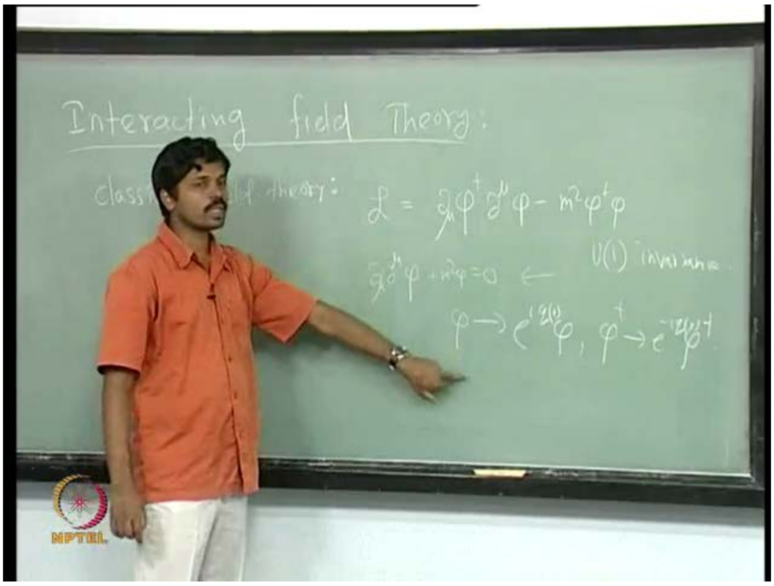

You already know what is a Lagrangian density for a charge scalar field. It is given by del mu phi dagger del mu phi minus m square phi dagger phi. We will see how interaction comes very naturally. In this theory as of now it is a free field theory, the equation of motion for this field theory is a del mu del mu phi plus m square phi equal to 0. It is a linear equation, there is no source and it is a free field theory. We will see how can I introduce interaction in a very natural way in this free field theory.

To do that, note that this field theory has actually a U 1, invariants a global U 1 invariance. That is if you consider this transformation phi going to e to the upper i q phi and hence phi dagger going to e to the upper minus q phi. Then the Lagrangian density is invariant under such a transformation.

Here, this parameter q is a constant, it does not depend on space time coordinates. So, one natural question that might asked is what will happen when you make this parameter q space time dependent? When you make this to this space time dependent, then you notice that the second term here is invariant under such a transformation. Its veritable this is phi dagger here, however the first term is not invariant under such a transformation.

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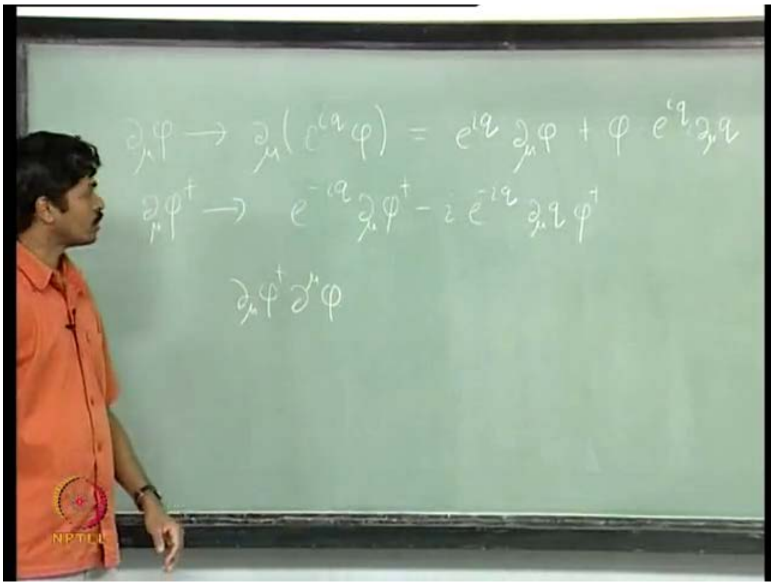

So, let us check that del mu phi goes to del mu e to the upper i q phi. So, this is e to the upper i q del mu phi plus phi times e to the upper i q i times del mu phi del mu q. Similarly, del mu phi dagger goes to e to the upper minus i q del mu phi dagger minus i e to the upper minus i q del mu q e to the upper phi dagger. So, when you consider this term, del mu phi dagger del mu phi. It does not remain invariant because of the presence of this second term and you get additional term, which depend on del mu q.

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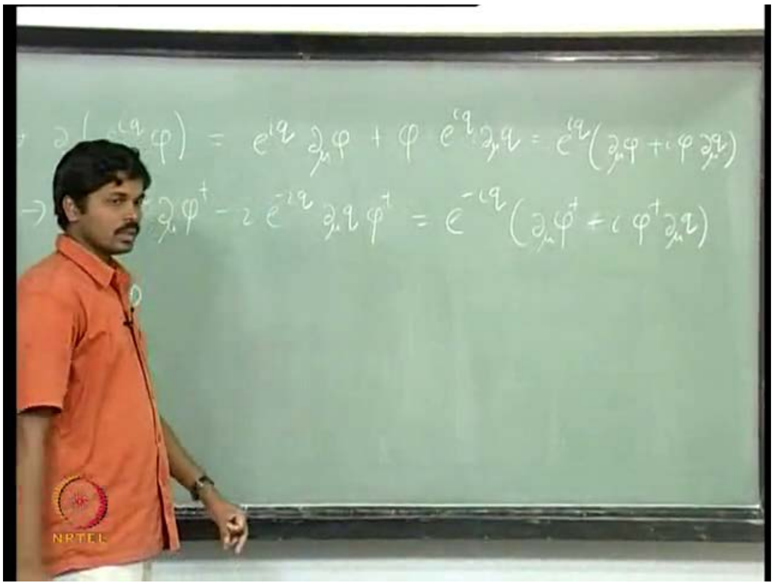

You can write this term here as e to the upper i q del mu phi plus i phi del mu q. Similarly, here this is e to the upper minus i q del mu phi dagger minus i phi dagger del mu q. If this term was not there, then you can see that if you multiplied these two terms that we will e to the upper i q here. There is e to the upper minus i q here and hence this term would be invariant, however it is this which makes it non invariant.

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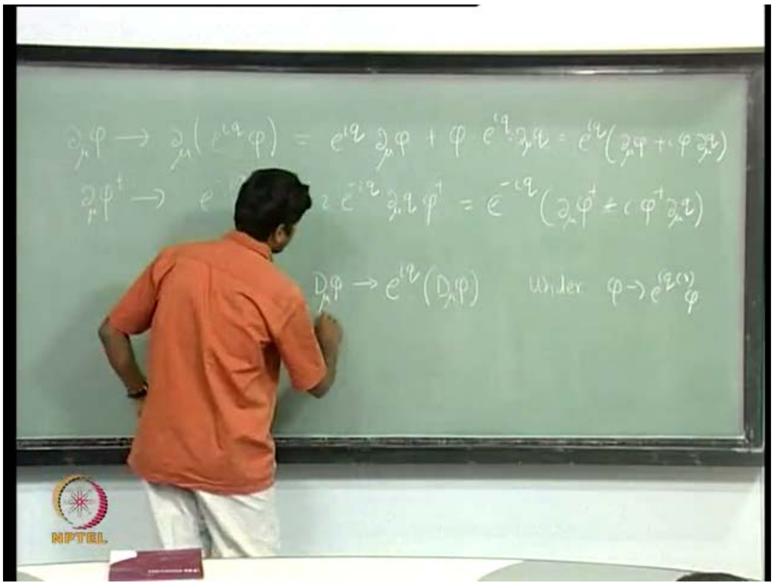

So, why do you need is actually something which is known as covariant derivative which I will denote as d mu phi, such that this quantity will transfer like e to the upper i q D mu phi under the transformation phi going to e to the upper i q of x phi.

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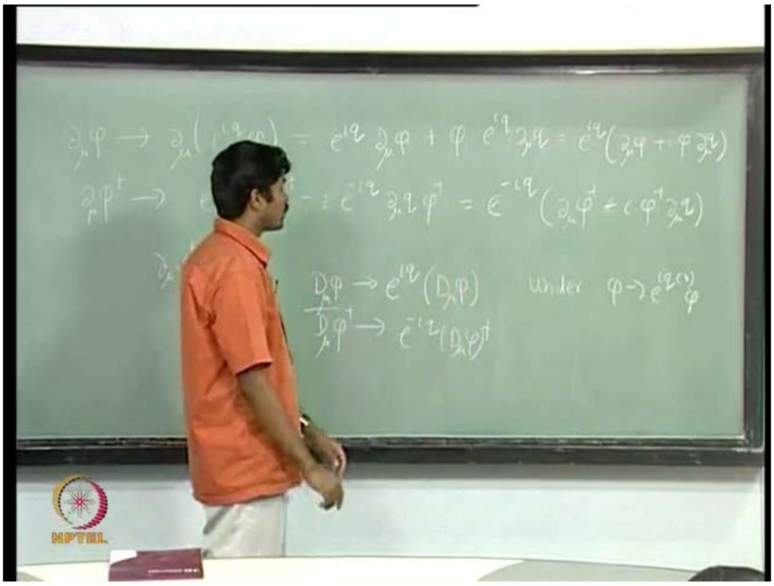

Its Hermitian constituted d mu phi dagger will transfer e to the upper minus i q D mu phi dagger. So, that D mu phi dagger D mu phi will remain invariant under this transformation.

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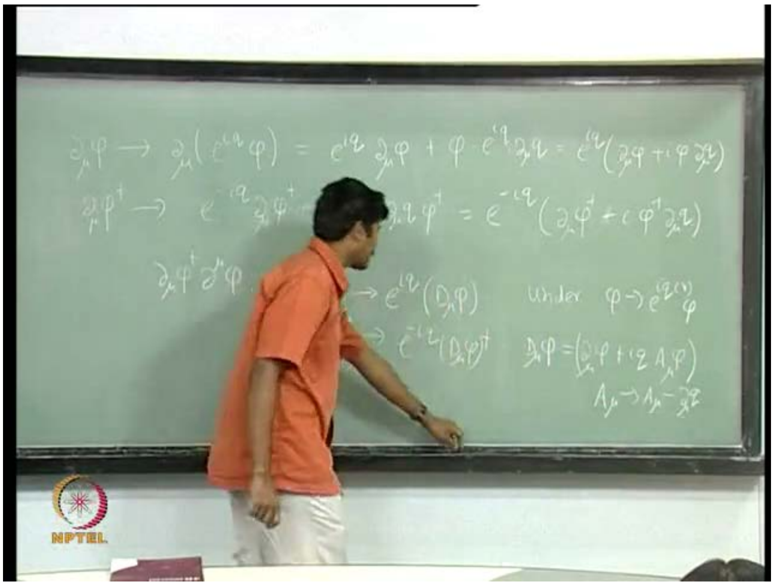

So, what is this d mu phi? Of course, with a single phi you can add have such a covariant derivative. However, if you define D mu phi to be del mu phi plus i q A mu phi, then you can see that if this transformation is supplemented by something a mu going to A mu minus del mu q. Then this d mu will transfer according to this, so just check that.

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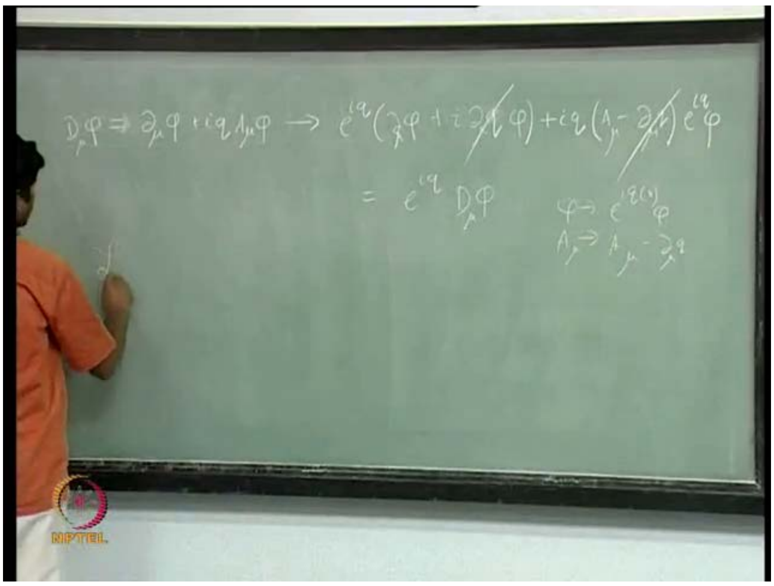

So, you have D mu phi is equal to del mu phi plus i q A mu phi and this transfers like del mu phi goes to e to the upper i q del mu phi plus i del mu q phi and when this term here transfers like plus i q times a mu goes to A mu minus del mu q and then this phi goes to e to the upper i q phi. So, this is how D mu phi transfers as you can see this term here cancel with this term, because of this minus sign all other terms remain same. So, this what you get is e to the upper i q times D mu phi. So, this D mu phi transform covariantly under the transformation, phi goes to e to the upper q of x phi and A mu going to A mu minus del mu q.

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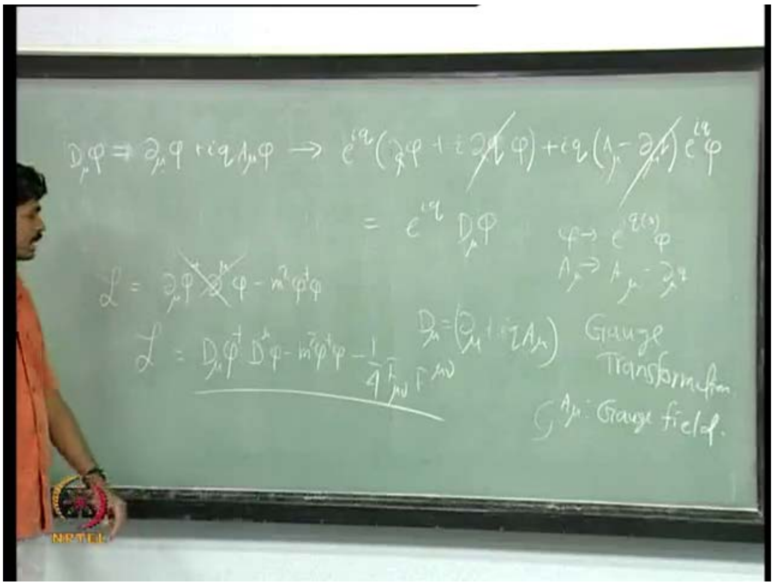

So, instead of taking the Lagrangian density to be del mu phi dagger, del mu phi minus m square phi dagger phi. If you take eth to be D mu phi dagger, D mu phi minus m square phi dagger phi, where d mu is given by del mu plus i q A mu. Then this quantity will be invariant under such a transformation, this is known as a gauge.

This is known as the gauge transformation and this A mu is known as the gauge field. The zeroth component of A mu is the scalar potential and the ith components they constitute the magnetic vector potential. So, this is the electromagnetic field, if you want dynamics for this A mu field also you can add a kinetic term for the gauge field which will be given by F mu nu, F mu nu. So, this Lagrangian density involves the gauge field and the complex scalar field. Now, you look at the equation of motion for this field, you can see that in the Lagrangian, if I expand this Lagrangian.

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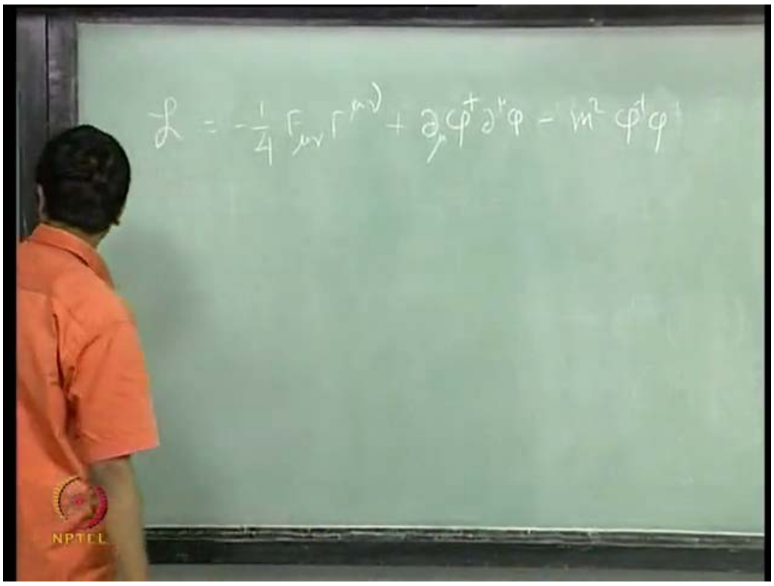

Then this will have a kinetic term for the gauge field and it will have a kinetic term for the complex scalar field, which is del mu phi dagger del mu phi. It has the mass term for the complex scalar field, which is m square phi dagger phi, but there are additional terms involving A mu as well as phi.

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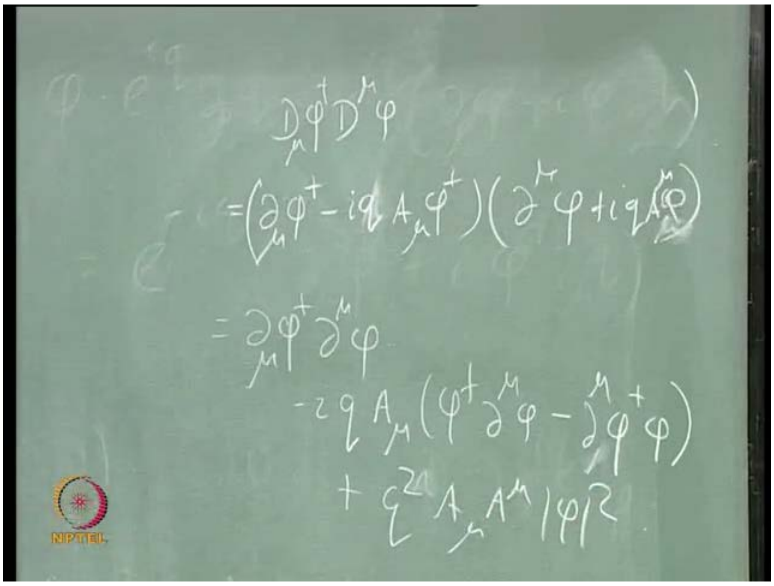

So, what are these additional terms. If you look at this D mu phi dagger D mu phi, then this is del mu phi dagger minus i q A mu phi dagger and del mu phi plus i q A mu phi.

So, this is del mu phi dagger del mu phi and minus i q A mu times phi dagger del mu phi minus del mu phi dagger phi and then plus q square A mu A mu mod phi square.

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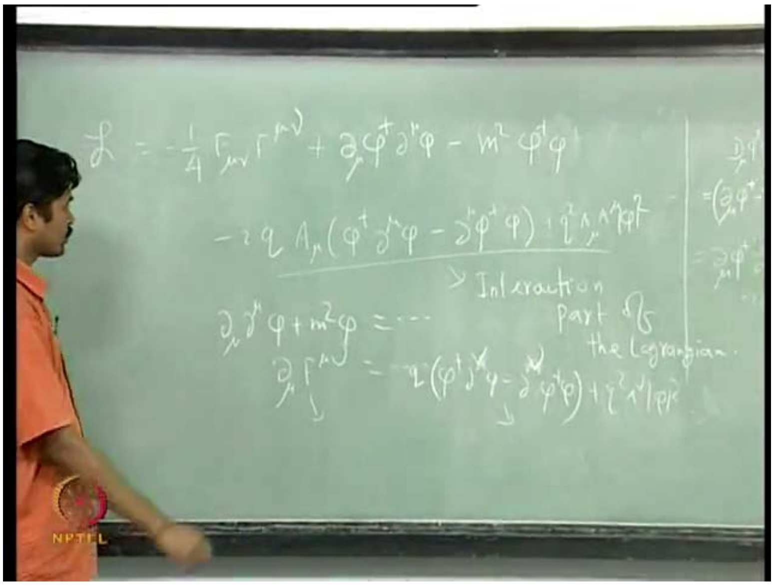

So, the additional term that you can get here is of this form, minus i q A mu phi dagger del mu phi minus del mu phi dagger phi. Then q square A mu A mu mod phi square. So, if this term was not there, then you would have got del mu del mu phi plus m square phi is equal to 0 for this scalar field equation. And del mu f mu nu equal to 0 for the gauge field equation of motion, but because of the presence of this term you will get additional terms in these equations of motion. So, you will have additional term here. So, what is the use of these additional terms? For example, if you look at the vector field equation, here you will get something like q times phi star del mu phi minus del mu phi dagger phi and then you will get A. I think its nu q square A mu more phi square terms like that you will get.

So, this simply means that if this term was not there then it could have behave like a free electromagnetic field, but because of this scalar field because of the presence of scalar field. The dynamics of the gauge field actually changes, therefore these terms are responsible for the interaction of this scalar field with the gauge field. So, these terms are known as the interaction part of the Lagrangian, alright?

So, engine railway what happens is when you consider a field theory which involves many fields, you have the free part of the Lagrangian for each of these fields. Then there is an interacting part of the Lagrangian which involves the interaction of various fields. Here, you can see all these free parts are quadratic in the fields and they are derivatives. This is quadratic in phi dagger phi, this is quadratic in the derivative of phi and this term here F mu nu, as you know is del mu A mu minus del mu A mu. So, this term here is quadratic in del mu A mu.

So, all these are quadratic terms. Whereas, the interacting terms are not quadratic, they are of higher order here. This is cubic in the field because there is a A mu, there is a phi and there is a derivative of phi. So, this is a q vector. Whereas, this is a quadratic term, there are two A mu's and there are two phi's.

So, these interaction terms are usually higher order in the fields, any field theory which have a quadratic term in the fields is basically a free field theory, whereas if you have cubic or higher order term, then those are the terms which introduce interaction in the theory. In fact you need not have an additional gauge field here.

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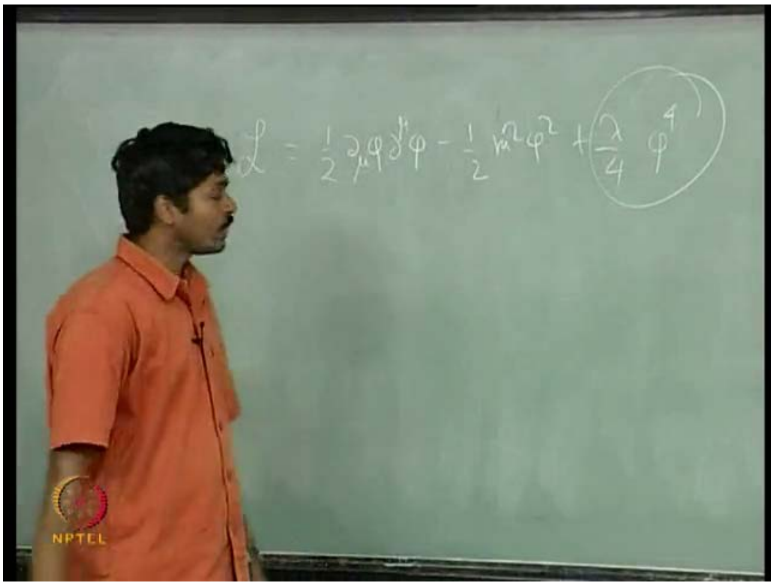

For example, if you consider let say a real scalar field theory with the Lagrangian half del mu phi del mu phi minus half m square phi square plus lambda over 4 phi 4, a phi 4 theory will be like this, then this term will be the interacts interacting term, because in this the equation of motion for this will be…

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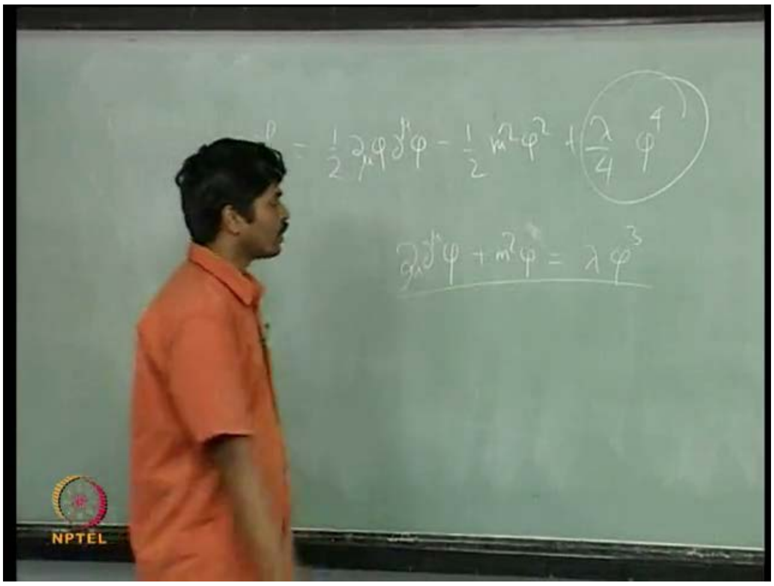

Del mu del mu phi plus m square phi square minus this is equal to lambda phi cube. It is look something like that and as you can see this is no longer linear in the field phi. So, this term is a, this is due to the interacting term and the field is no longer a free field. It is a self interacting field, the field interacts with itself.

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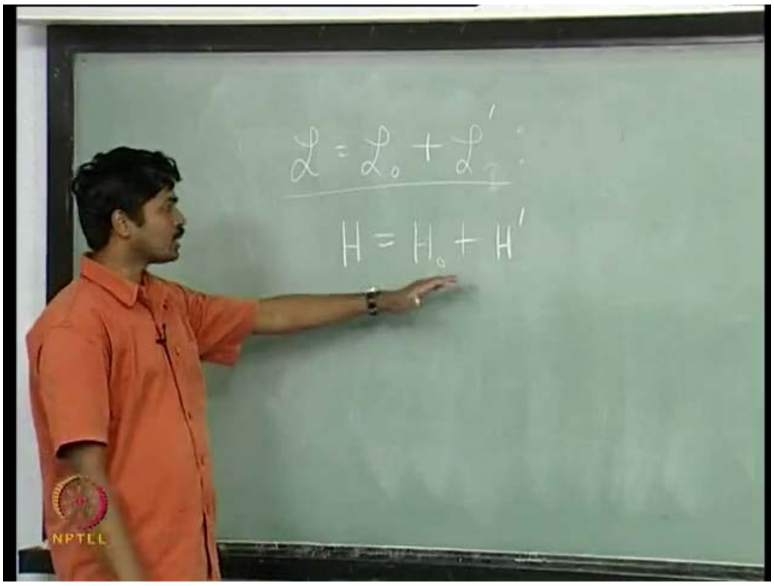

So, typically what you will have is in general the Lagrangian will have a free part which I will denote as L 0 and it will have an interacting part which I will denote as L prime. That is denoted as L prime to be in interacting part of the Lagrangian. The question is how to quantize such a theory. Correspondingly if you consider the Hamiltonian, the total Hamiltonian then you will have the free, you will have a free part of the Hamiltonian. Then you will have an interacting part of the Hamiltonian which I will call as a H prime. So, our task is to quantize such an interacting theory to find this spectrum, and then the question this we would like to ask is suppose you have a state.

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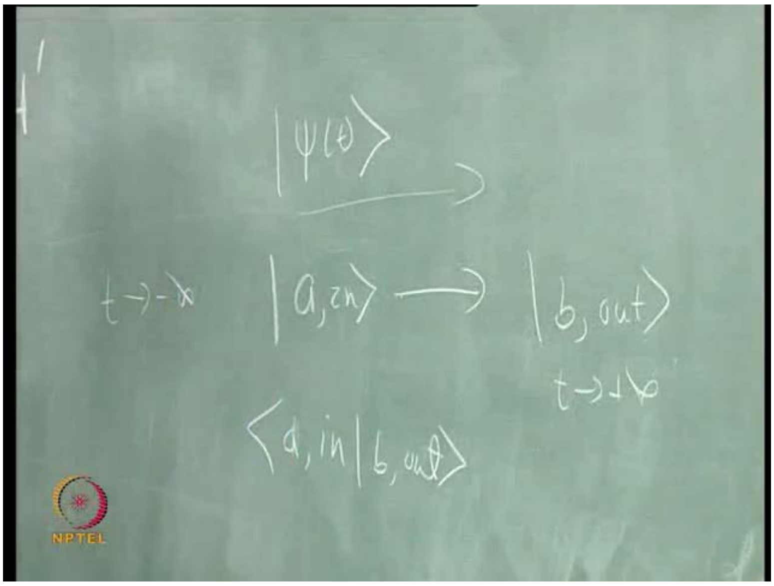

Psi of t its sometime t in presence of such an interaction, what is the evolution of such a state or in other words suppose at t equal to minus infinity. The system was in a state which I will call as a in. What is the probability of the state evolving two, some other states which I will call as b out as t goes to plus infinity. What is the probability amplitude? The probability amplitude is this and this is what we like to evaluate. So, let us consider such an Hamiltonian and then we will see how to proceed and how to quantize such a theory. Before that I would like to introduce you various pictures in field theory with using which you can quantize.

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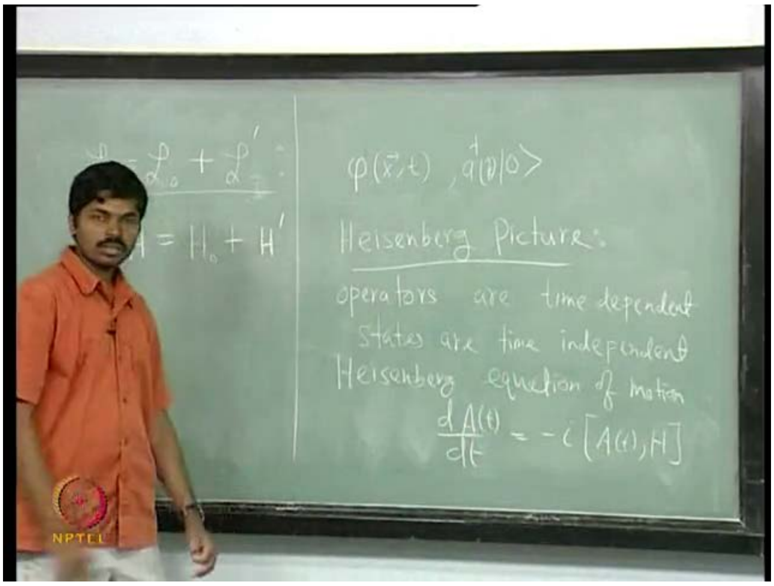

We have although I have not emphasized so far, what we have been doing in the free field theory until now is we are assuming the observables or the operators to be time dependent. So, we have in the relent complex scalar field, we have seen that this field phi has the time dependence. Whereas, the states are time independent, you consider vacuum state and then you consider a dagger k creation operator acting on the vacuum. And then you get the entire spectrum this way.

So, these are time independent, whereas the operators or observables are time dependent. This is known as the Heisenberg picture. So, in the Heisenberg picture the operators are time dependent, the states are time independent and the time evolution of the operators are in fact govern by the Heisenberg equation of motion, which basically says that if a of t is an operator then d a over d t is minus i commutator of a of t n, the Hamiltonian. You can also consider the operators to be time independent and the state vectors to be time dependent.

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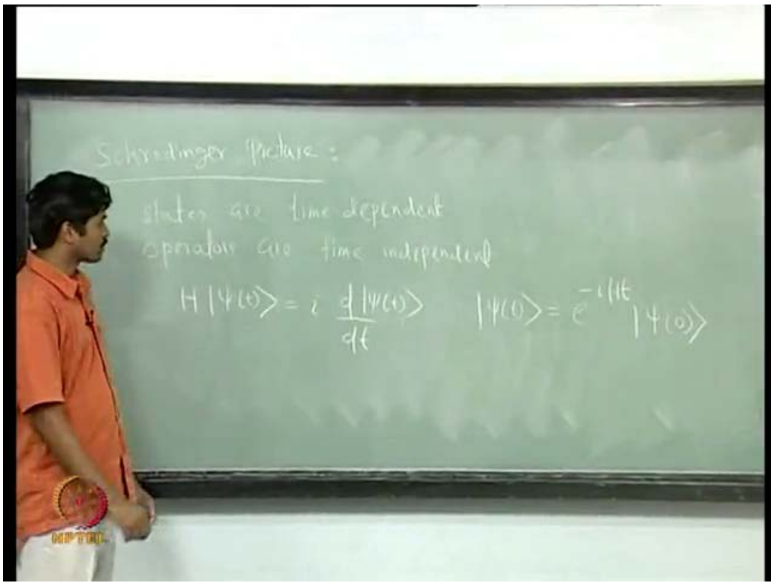

This is what is known as the Schrodinger picture. So, in this Schrodinger picture the states are time dependent, whereas the operators are time independent. Time evolution of the states are govern by the Schrodinger equation which is. So, suppose psi of t is a state vector, then h psi equal to i d psi over d t and hence you can get psi of t equal to e to the upper minus i H t psi of 0. You can relate the operators in the Heisenberg picture with a operator in Schrodinger picture by this relation.

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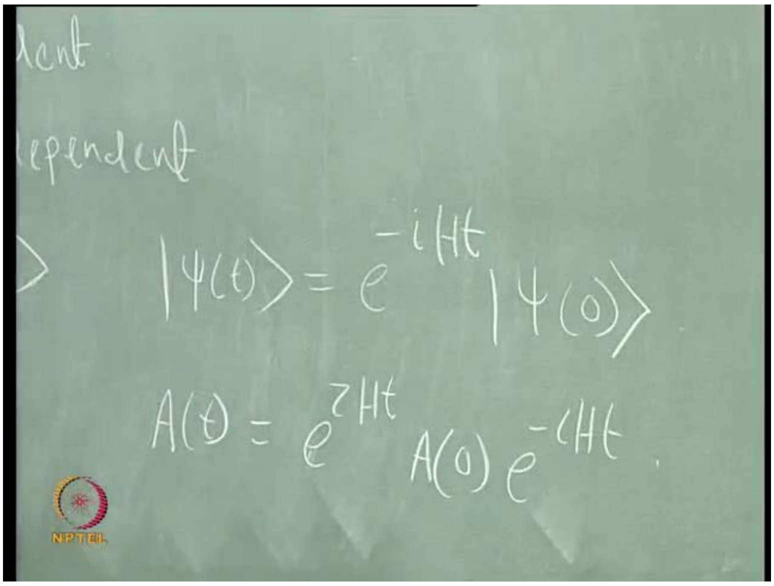

You can consider A t, this will equal to e to the upper i H t A 0 e to the upper minus i H t. So, you can either work in the Schrodinger picture by considering the states to be time dependent and the operators to be time independent and the time evolution of the state. State vectors are generated by the Hamiltonian and this is a how time evolution of the state vector is or you can consider the operators to be time dependent and the state vectors time independent. This is how the time evolution of the operators is given in presence of interaction. It is neither the Heisenberg picture nor the Schrodinger picture is convenient to work with.

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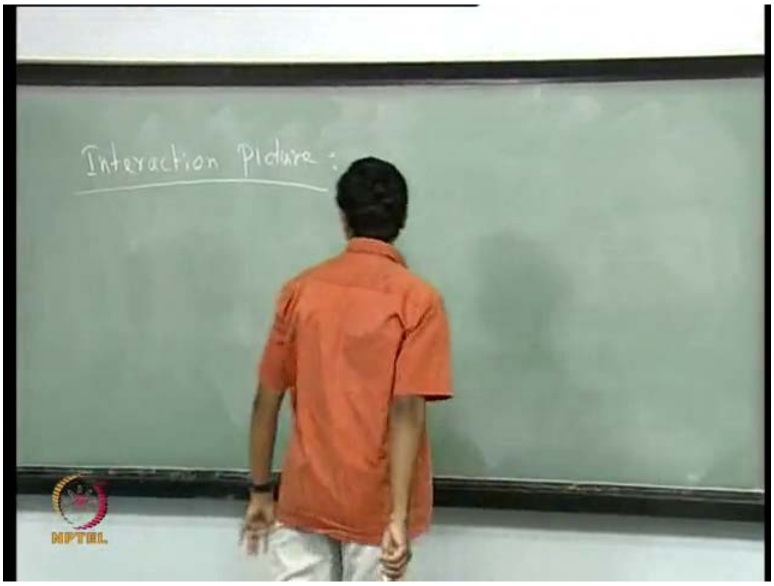

There is something called as a interaction picture, and it is quite convenient to work in the interaction picture when there are interacting terms. In fact we will throughout our lecture we will work only with the interacting picture.

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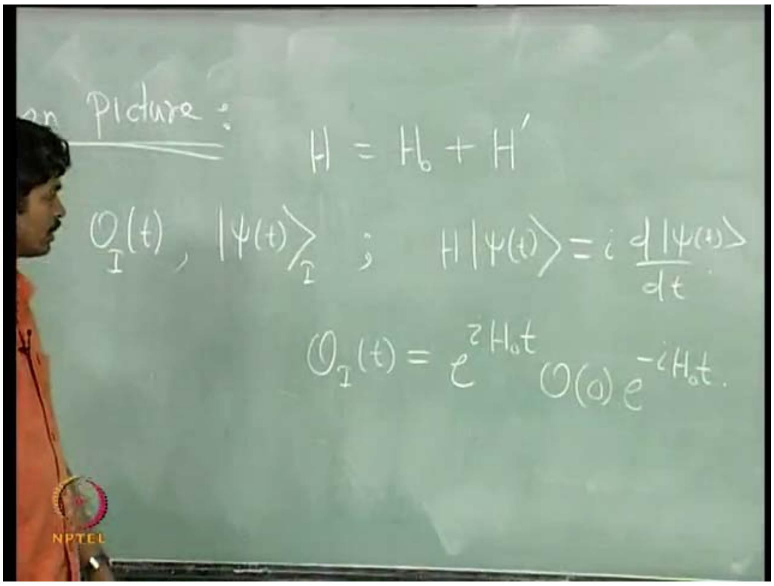

In the interacting picture, what happens is you have the full Hamiltonian, which I will call as H zero which is free, the part of the Hamiltonian which involves only the kinetic terms and the mass terms of the fields involved. The rest all of the terms in the Hamiltonian which I will call as the interacting part of the Hamiltonian and I will denoted as H prime.

What happens in the interacting picture is that the in the interacting picture both the states as well as the operators are time dependent. However, the time dependence of the operators in the interaction picture is govern by the free Hamiltonian H 0. Whereas, the state vectors are also time dependent, but the time evolution of the state vectors is determined by the interacting part of the Hamiltonian.

So, how is that consistent? We can let say you are in the Schrodinger picture, you know what is the Schrodinger equation of motion. It is the full Hamiltonian, H acting on psi equal to i d psi over d t. What we can do is we can first introduce the operators in the interacting picture, by saying that is the operators in the interacting picture is e to the upper i H 0 t e to the upper minus i H 0 t.

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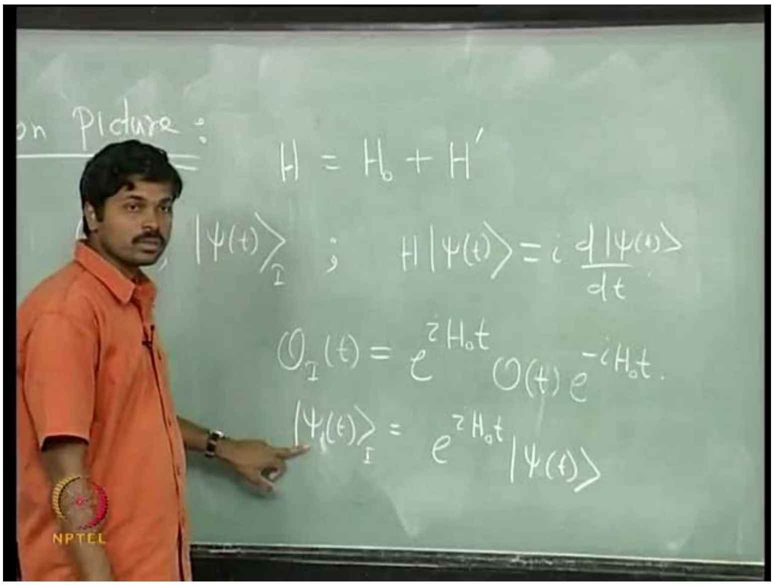

I will introduce the state vectors in the interacting picture as psi i t, which is e to the upper i H 0 t psi t. This is just a definition. So, you start with this definition for the state vectors and you used the fact that psi t, in fact is the state vector in the Schrodinger picture. Hence, psi t obeys the Schrodinger equation of motion with the full Hamiltonian. So, when you do that? Let us now see what do we get for d psi i of t over d t.

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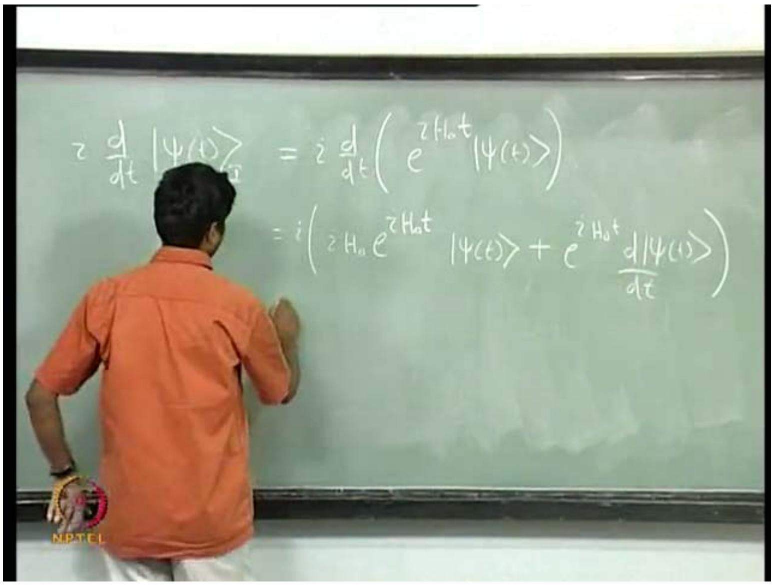

So, you consider i d over d t acting on psi t i. According to the definition of the psi I, this is i d over d t e to the upper i H 0 t, right? Now, you can see this is nothing but i times i h

0 e to the upper i h 0 t, because here H 0 is the free Hamiltonian in the Schrodinger picture, it does not dependent on time. So, therefore you have this and then you have an additional term which is given by e to the upper I H 0 t d psi t over d t. So, I can multiply this i here, what i and i can pullout, e to the upper i H 0 t.

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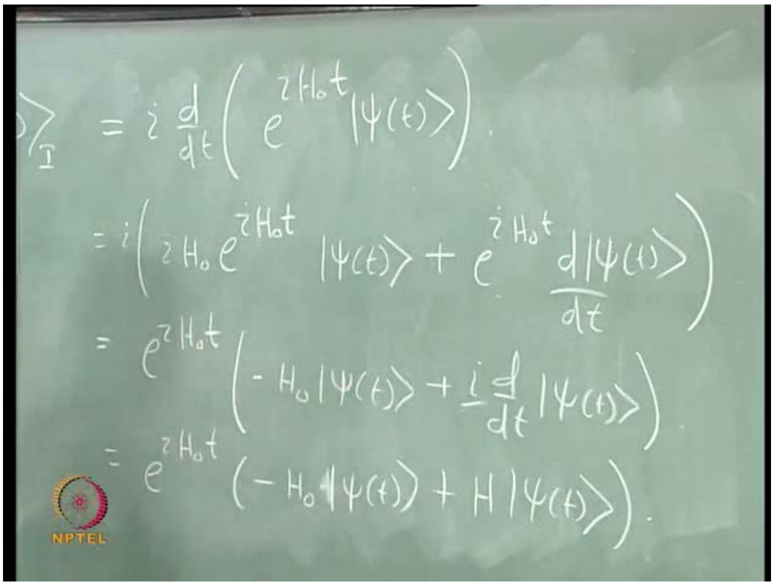

Outside the naught I will get is e to the upper i H 0 t times minus H 0 acting on psi t plus i d over d t acting on psi t. Now, again I can use the fact that psi t obeys the Schrodinger equation of motion in the Schrodinger picture. So, i d over d t acting on psi gives me H psi, where H is the full Hamiltonian. So, this is e to the upper i H 0 t times minus H 0 plus minus H 0 acting on psi t plus the full Hamiltonian, H acting on psi t. So, this is nothing but.

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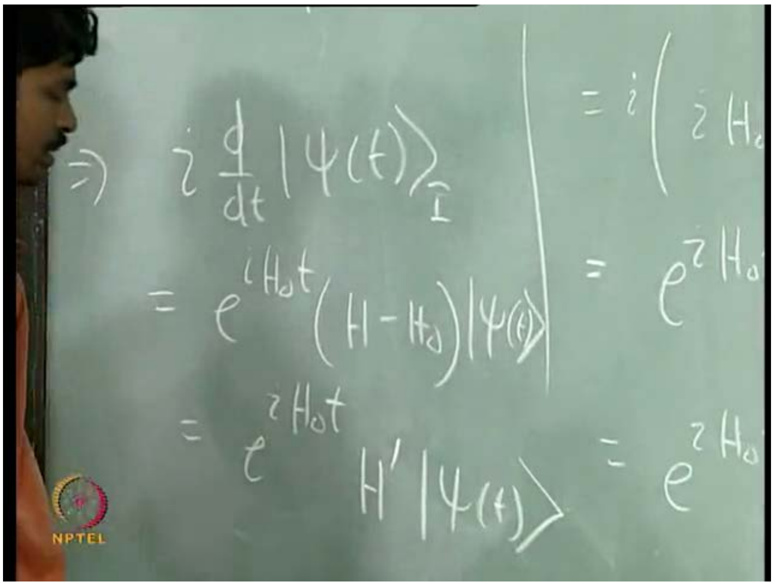

So, this implies i d over d t acting on psi t, in the interact interaction picture is e to the upper i H 0 t H minus H 0 acting on psi t, but H minus H 0 is H prime. So, this is nothing but e to the upper i H 0 t and then H prime acting on psi of t. So, we are not yet done.

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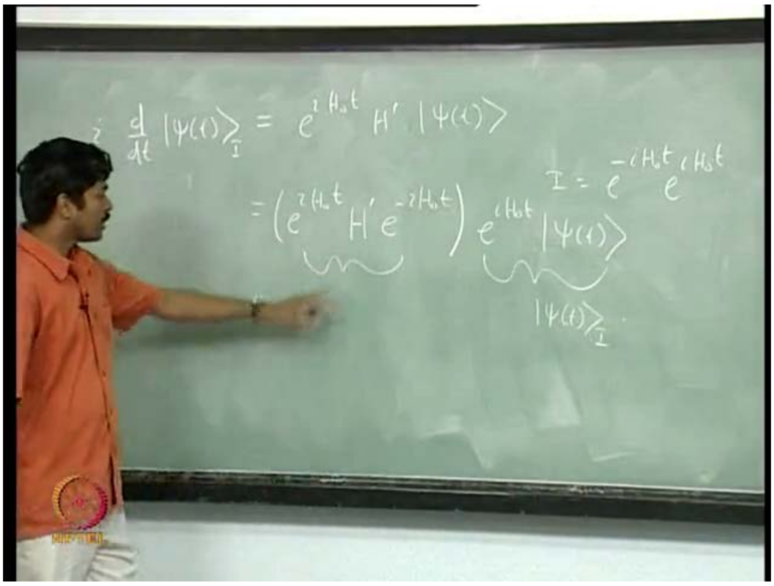

So, we have shown is i d over d t psi t. In the interaction picture is given by e to the upper i H 0 t H prime acting on psi t. Here, I will introduce an identity operator which is basically e to the upper minus i H 0 t times e to the upper i H 0 t. So, if I do that then what I have is e to the upper i H 0 t h prime e to the upper minus i H 0 t and then e to the upper i H 0 t acting on psi t. This is nothing but according to our definition, this is psi t in the interaction picture.

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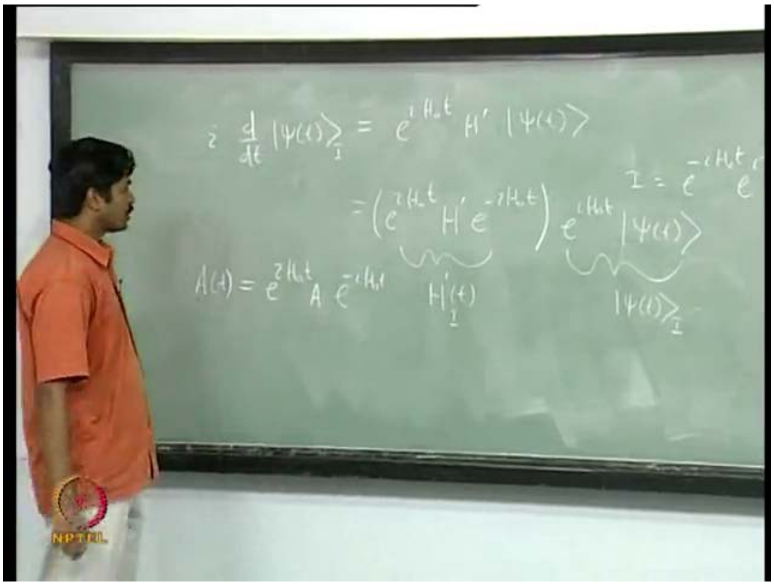

Whereas, this is the operator H prime of t which I will call H prime i of t because according to our definition the time dependence of an operator, any arbitrary operator you consider A t, the time dependence is govern by the free Hamiltonian. So, this is e to the upper i H 0 t and then A e to the upper minus i H 0 t. So, I will denote this to be the interacting Hamiltonian in the interaction picture. Therefore, if I use this definition that the state vectors in the interaction picture are defined like this.

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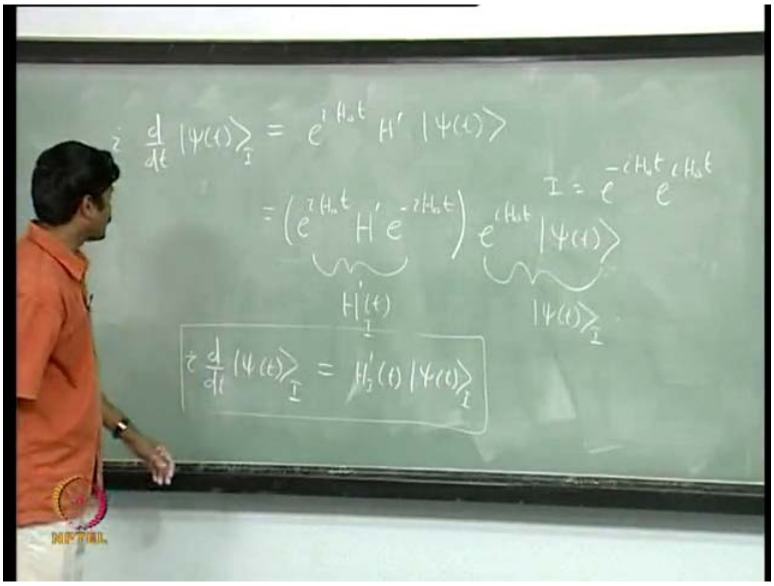

Then these state vectors obey an equation where the time evolution of the state vectors are actually govern by the interacting part of the Hamiltonian H i prime of t acting on psi t, where this interacting part of the Hamiltonian is in the interaction picture. So, state vectors, the evolution of state vectors is given by this equation.

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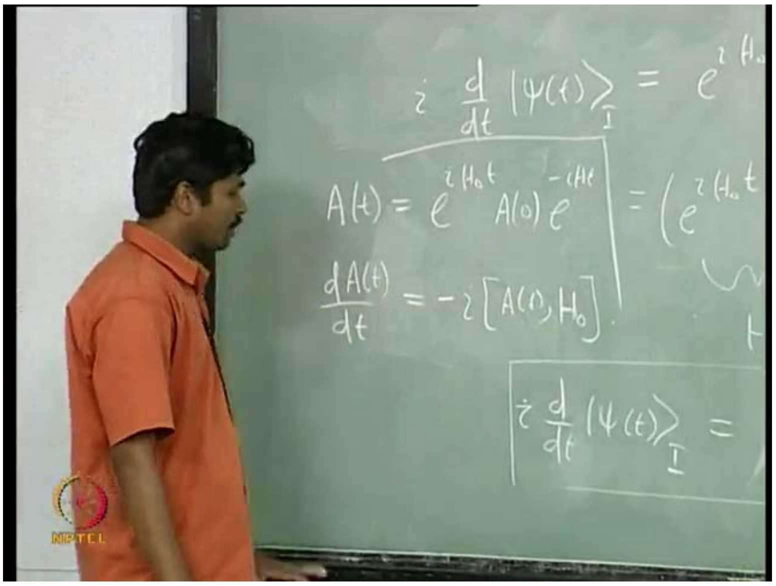

Whereas, the evolution of time evolution of operators is given by e to the upper i H 0 t A, in the Schrodinger picture I will call it as A 0 e to the upper minus i H t. Therefore, this A of t obeys a Heisenberg equation of motion as if the Hamiltonian is the free Hamiltonian.

So, d A over d t is minus i commutator of A n H 0. Whereas, 0 is the free Hamiltonian, so we will use this interaction picture and then we will consider the quantization of interacting field theory. So, as usual we will consider the fundamental commutation relation and then we will see what is this spectrum that we can find. This spectrum or not or how to proceed and all those things we will see.

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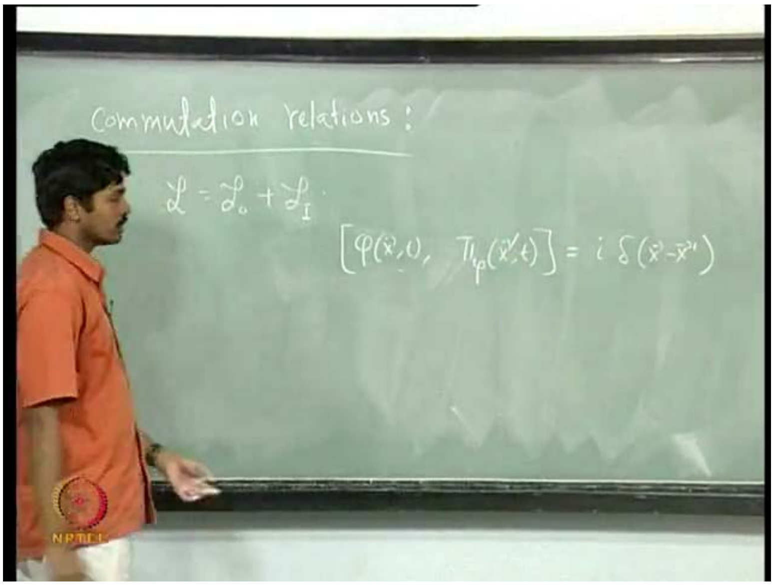

So, let us consider the commutation relations, you have the Lagrangian which have a free term and which have an interacting term. So, let say you have the field phi, you can ask the question what is the corresponding conjugate momentum phi of t. Let first assume that phi, phi of t they are all in the Heisenberg picture and then pi phi x t.

So, you again would like to have a commutation relation in this Schrodinger picture which is i delta x minus x prime. However, as I stated we will like to work in the interact interaction picture. So, we like to consider these operators in the interaction picture, the question is whether the same commutation relation holds in the interaction picture. In general it will not hold, however we will we can make an assumption that this interacting part of the Lagrangian does not content any time derivative of the fields.

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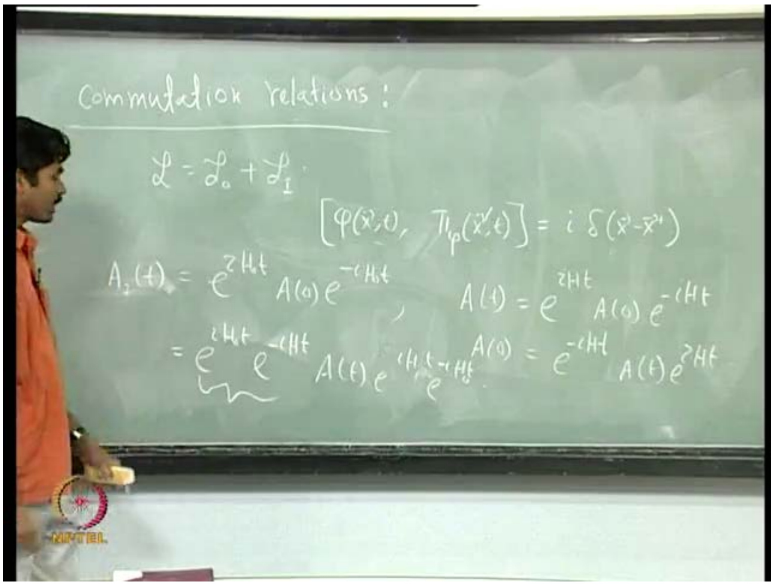

So, what we will assume that, what we will assume is A del 0 phi this interacting part does not depend on del 0 phi l i on. So, this commutation relation of course, will be same as the commutation relation in the free field theory. So, this will hold, now you can introduce the operators in the interaction picture. We already know what are the operators in the interaction picture, in the interaction picture A i. If I call as an operator A in the interaction picture, this is e to the upper i H 0 t A 0 e to the upper r minus i H 0 t. What we are interested is how the operators in the interaction picture are related to the operators in the Heisenberg picture, not the operators in the Schrodinger picture.

Of course, it is drivel to determine because we know the operators in the Heisenberg picture are related to the operators in the Schrodinger picture, by the relation e to the upper i H t A of zero e to the upper minus i H t, where h is the full Hamiltonian, right? So, you can invert this relation, this will tell you A 0 is e to the upper minus i H t A t e to the upper i H t. You can substituted here, when you substitute this here what you will get is e to the upper i H 0 t e to the upper minus i H t and then e to the upper minus i H 0 t e to the upper i H t. So, thank you e to the upper i H t will come first with a plus sign and then e to the upper minus i H 0 t, you can denote this to be you can define an operator U, so let us introduce.

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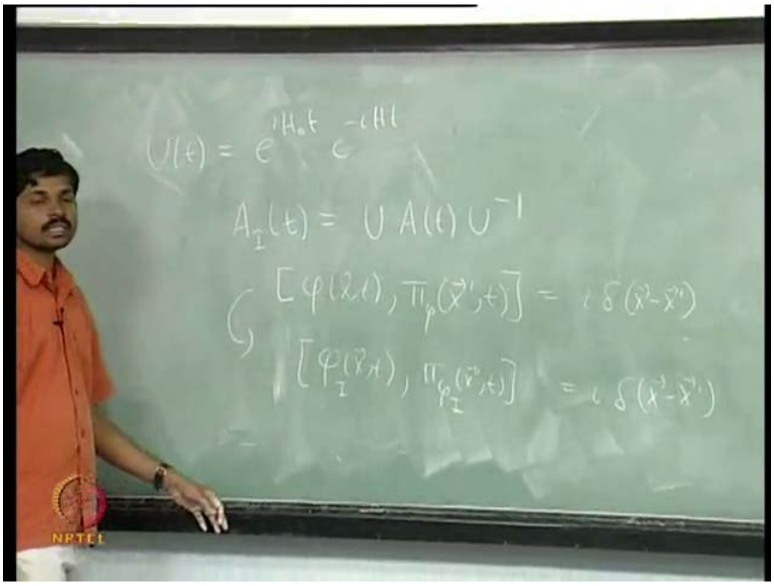

Let us introduce this operator U of t to be e to the upper i H 0 t e to the upper minus i H t, then A I t equal to U A t U inverse, right? This is what we get. Therefore, now you can see that if you consider this commutation relation phi x t pi phi of x prime t, the fundamental equal time commutation relation. This is i delta x minus x prime, this is the commutation relation in the Heisenberg picture. Then you can operate U inverse on both sides.

So, u phi x t pi phi x prime t U inverse, this is again going to be i delta x minus x prime, whereas this will give you the operators in the interaction picture. So, this is basically going to become phi I, where phi i is the operator in the interaction picture. Similarly, here pi phi i in the interaction picture. So, the commutation relations in the interaction picture also holds the same commutation relation holds for the interaction picture. So, as far as the commutation relation is concern we are perfectly fine, but then the next question is how to how can we write the general solution, what are the creation and relational operators and so on.

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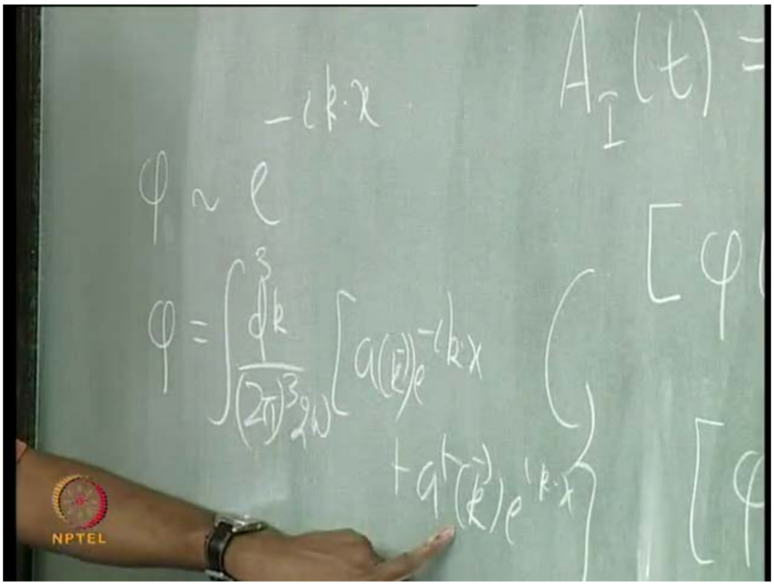

In the free field case at least you already we already knew that this field phi, it means plan wave solution which are e to the upper minus i k dot x. So, the most general solution you can write as phi equal to integral d q k 2 over pi cube to omega. Then a k e to the upper minus i k dot x plus a dagger k e to the upper i k dot x, and you can put it in the fundamental commutation relation. You can find what is the commutation relation between a and a daggers and then you know how to interpret these a.

These as to be these a as the n relational operators and the a daggers is creation operators. However, in presence of interaction we do not have the luxury to do this because we cannot in general construct a solution like in the most general solution in this for. So, we have to worry about how to proceed and then quantize and interacting field theory. So, we will discuss more on this in the next few lectures.