Quantum Field Theory
Prof. Dr. Prasanta Kumar Tripathy
Department of Physics
Indian Institute of Technology, Madras

Module - 01
Free Field Quantization - Scalar Fields
Lecture - 07
Quantization of Complex Scalar Field

In the last lecture, we were discussing causality.

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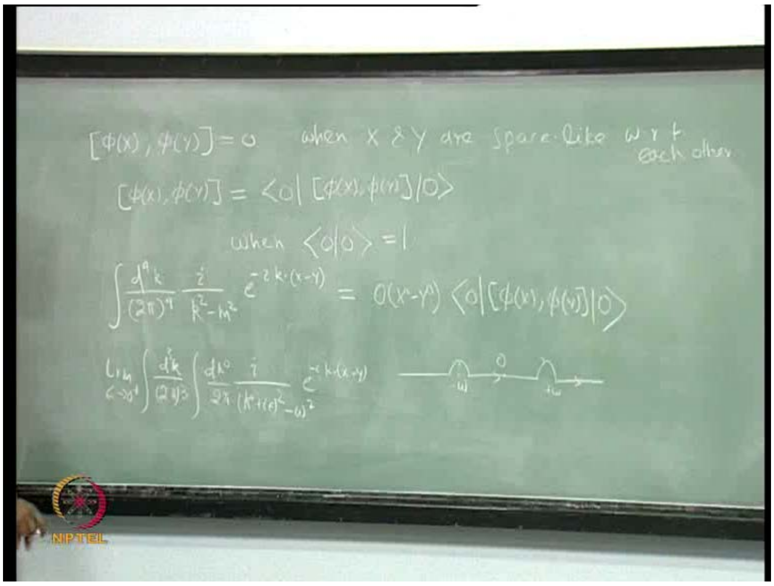

And then, we have seen that, the commutator phi of x, phi of y vanishes when x and y are space-like with respect to each other. We have also found an integral representation for the commutator. We have seen that the commutator phi of x, phi of y is the c number; it is a commuting number. And hence, this is also equal to the vacuum expectation value of itself – phi of x, phi of y; where, we normalize the vacuum such that it is ((Refer Slide Time: 01:27))

Then, we have seen that, if x 0 is greater than y 0, then this vacuum expectation value is the retarded Green's function for the Klein-Gordon operator. So, to be more specific, what we have seen in the last lecture is that, integral d 4 k divided by 2 pi 4 i divided by k square minus m square e to the power minus i k dot x minus y is equal to theta of x 0 minus y 0 times the vacuum expectation value of commutator of phi of x and phi of y;

where, this k 0 integration here is evaluated on the contour, which is taken like this. This is the origin; this is minus omega; this is plus omega.

Or, equivalently, we can use the epsilon prescription and then we can evaluate this integration by considering limit epsilon goes to 0 plus integration of d cube k over 2 pi cube d k 0 over 2 pi and i divided by k 0 plus i epsilon square minus omega square e to the power minus i k dot x minus y. So, you can consider this integration; you can evaluate it. And then, what we saw in the last lecture is that, when we evaluate the k 0 integration, this turns out to be equal to this expression. Today, we will discuss some more of it.

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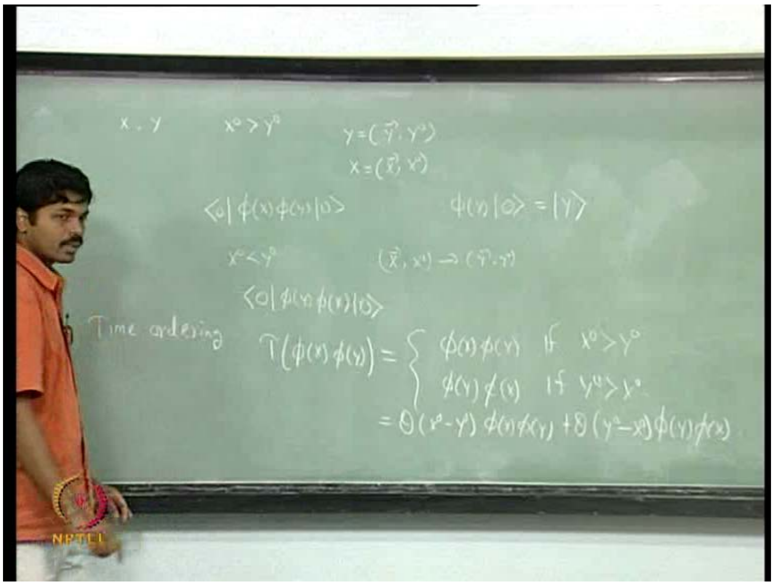

Let us consider two points: X and Y such that X 0 is greater than Y 0. So, in this case, you can talk about a particle propagating from this space-time point Y from the point Y at time Y 0 to the point – to the space-time point – this X to X 0. And, the question that you might ask is what is the amplitude for propagation of this particle at space-time point Y at Y 0 to the point X at time X 0? This is what you can ask when X 0 is elated time than Y 0. And, what is the amplitude for this propagation? The amplitude for this propagation is phi of X phi of Y – vacuum expectation value. Remember – phi of Y has two components: it has one positive frequency component, one negative frequency component. So, phi of i x – phi of Y especially acting on the vacuum will describe the… We can loosely call this to be Y; the positive frequency component will annihilate the vacuum and the negative frequency components will act on the vacuum to create a particle at Y. So, the amplitude for propagation from Y to X is given by this quantity.

Similarly, if ((Refer Slide Time: 06:21)) X 0 is less than Y 0, then you can talk about propagation of the particle from X, X 0 to Y, Y 0. And, the amplitude for propagation in this case is going to be phi of Y phi of X, 0. So, instead of talking about whether X 0 is greater than Y 0 or X 0 is less than Y 0, we can put the entire thing together by introducing what is known as time ordering. So, time-ordered product of two observables: phi of X, phi of Y is given by phi of X phi of Y if X 0 is greater than Y 0; and, this is equal to phi of Y phi of X if Y 0 is greater than X 0. Equivalently, this time-ordered product is also equal to theta of X 0 minus Y 0 phi of X phi of Y plus theta of Y 0 minus X 0 phi of Y phi of X; where, theta is the expansion. So, obviously, if X 0 is greater than Y 0, this term survives and the time-ordered product is equal to phi X times phi of Y; whereas, if Y 0 is greater than X 0, the first term vanishes and the second term survives and has the time-ordered product is phi of Y times phi of X. We can evaluate the vacuum expectation value of time-ordered product. That will describe the propagation of particle from X to Y or Y to X.

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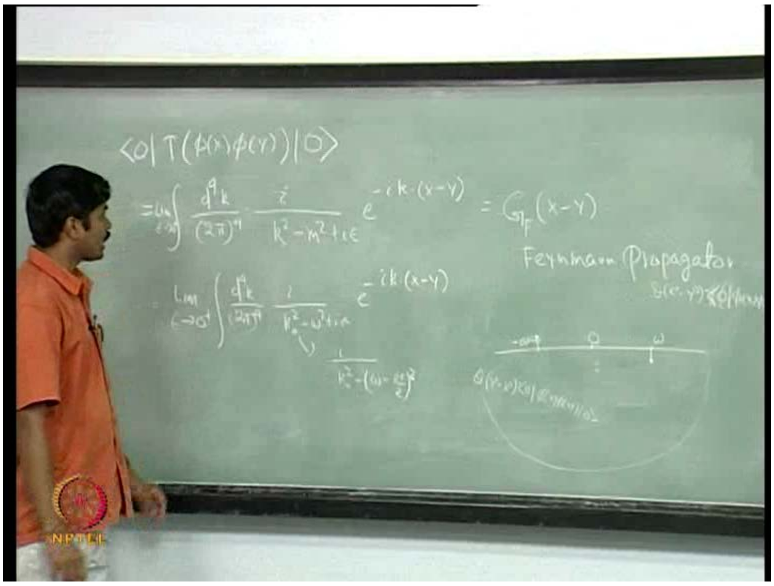

The vacuum expectation value of time-ordered product of phi of X phi of Y; we have already evaluated what is the vacuum expectation value of phi of X phi of Y. So, you can substitute it here. And, it so happens that, this also can be represented as a Green's function of the Klein-Gordon operator. And, this is equal to integral d 4 k over 2 pi to the power 4; so, in the limit, epsilon tends to 0 plus. This is known as the Feynmann propagator; and, this is represented by G F X minus Y. It is straightforward to evaluate thus the k 0 integration here. You have to remember that, this integration in the limit epsilon tends to 0 plus is also equal to limit epsilon tends to 0 plus d 4 k over 2 pi to the power 4 i divided by k 0 square minus omega square plus i epsilon e to the power minus i k dot X minus Y. And, here you can write it as i over k 0 square minus omega minus i epsilon over 2 whole square in the limit epsilon tends to 0. You can ignore all the terms of order epsilon square in the denominator; and then, to linear order in epsilon, this – the denominator here is equal to the denominator here. Therefore, both these are same. So, what is the point here?

What happens is you can again evaluate the k 0 integration here; only thing is that, the pole is shifted if this is the horizon and suppose this is plus omega and this is minus omega; then, here the pole is shifted downwards, because k 0 equal to omega minus i epsilon over 2 is the pole. And, here the pole is shifted upwards. So, if X 0 minus Y 0 is greater than 0, then this pole here will contribute; you can close it downwards and this one is going to contribute. Whereas, if X 0 minus Y 0 is less than 0, then you can close the upper half semicircle – semicircle in the upper half plane. And, this pole is not going to contribute; this is the one, which is going to contribute to the contour integration. This is the way you can evaluate and then you can show that, this in fact… So, one of these terms will give you theta X 0 minus Y 0. So, this first term is the one, which will give you theta X 0 minus Y 0 vacuum expectation value of phi X phi Y. Whereas, the second one when you close the contour in the upper half plane, is going to give a term, which is theta Y 0 minus X 0 vacuum expectation value of phi of Y phi of X. And, residue theorem says that, this integration is actually sum of the contribution from both these terms. So, that way you can have this. So, I will leave it as an exercise for you to evaluate this; and then, show that, this is equal to the vacuum expectation value of time-ordered product of phi of X and phi of Y. So, with this, we will close our discussion on real Klein-Gordon field.

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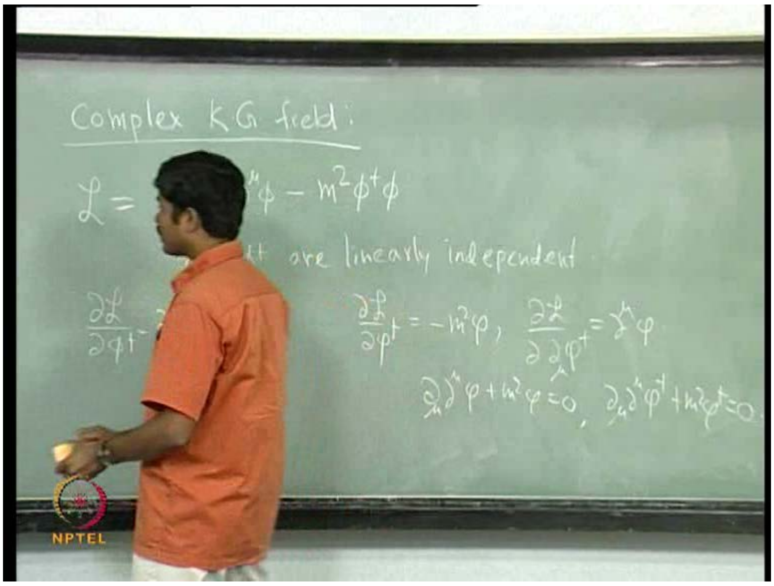

We can now discuss about the complex Klein-Gordon field. Most of the analyses in this case are quite similar to the real case. So, I will omit most of the steps and then summarize the main results in the quantization of a complex Klein-Gordon field. The Lagrangian density for the system is given by del mu phi dagger del mu phi minus m square phi dagger phi. Phi dagger is the Hermitian conjugate of phi. And clearly, phi and phi dagger are linearly independent; you can treat them as two independent fields.

The equation of motion is obtained by using the Euler Lagrange equation, which is del L over del phi minus del mu of del L over del del mu phi equal to 0. And, this will give you similar equation for phi dagger. So, let us consider the phi dagger equation del L over del phi dagger is minus m square phi. And, this one here – del L by del del mu phi dagger is del mu phi. When you substitute these two in the Klein-Gordon equation, you see that you get del mu del mu phi plus m square phi equal to 0. Similarly, the Hermitian conjugate of this equation, which is del mu del mu phi dagger plus m square phi dagger, is equal to 0. These are the equations of motion. You can find the plane wave solution to the equations of motion.

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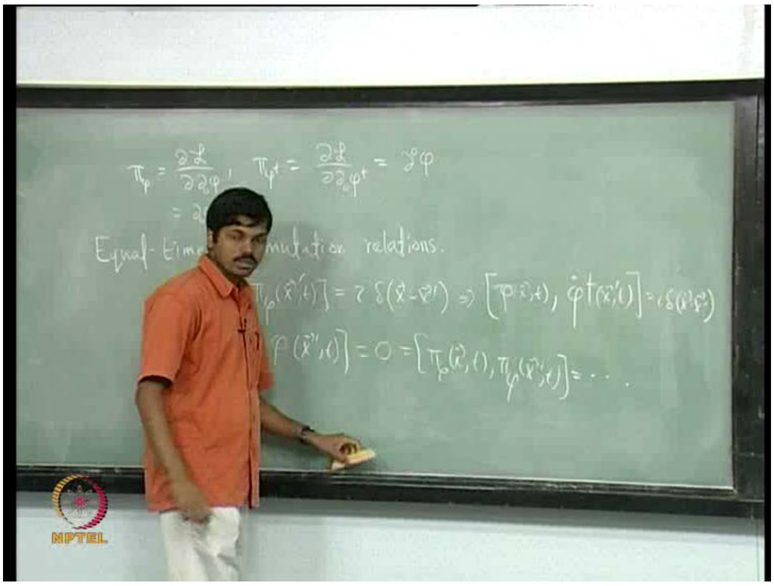

And then, the conjugate momentum, which is pi phi – conjugate momentum to the field phi is del L over del del 0 phi. Whereas, pi phi dagger – the conjugate momentum to the field phi dagger is del L over del del 0 phi dagger. So, this one we have already evaluated is del 0 phi. You can show that this is del 0 phi dagger.

So, del 0 phi dagger. So, the equal-time commutation relations as in the real case are given by… If you look at this phi of X t, pi phi of X t equal to i delta – X prime t – X minus X prime; and, a corresponding equation for phi dagger and pi phi dagger. And then, all other commutators are 0. Phi of X t, phi of X prime, t equal to 0 equal to pi phi of X t, pi phi of X prime t and so on. This equation can be written as phi of X t, phi dagger dot of X prime t is i delta X minus X prime.

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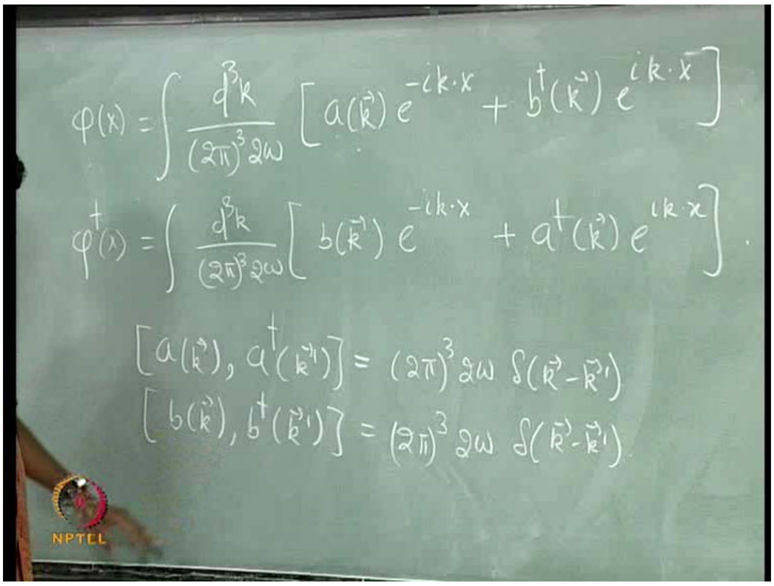

We can find the most general solution in terms of the plane wave solutions. And, this can be written as phi of X is again just like the real Klein-Gordon case. This is d cube k over 2 phi cube 2 omega for convenience and to make the integration measure Lorentz invariant; then, a k e to the power minus i k dot x plus b dagger k e to the power i k dot x. Remember in the real Klein-Gordon case, we had this operator a k here and here a dagger k; where, a dagger has Hermitian conjugate of a. This we have to take, because in that case, this field phi was a real scalar field. Here phi is a complex scalar field. So, there is no reason for us to consider this operator here to be the Hermitian conjugate of this operator. So, this one is some other operator is quite independent of this. And, I have denoted this to be b dagger k just for convenience. If this is the case, then phi dagger of X is of course the Hermitian conjugate of phi of X. So, this can be written as d cube k over 2 pi cube 2 omega times b of k e to the power minus i k dot X plus a dagger a plus k e to the power i k dot X.

Now, we can find ((Refer Slide Time: 22:57)) And, we can substitute it in the commutation relation here. And, just like the real Klein-Gordon case, we can evaluate the commuter; do some integration; or, we can express these a's and b's in terms of phi phi dot phi dagger phi dot dagger; and finally, evaluate the commutator. You can show that, in this process… I will not work them out here. When you evaluate the fundamental commutation relations, you will find that, a k a dagger k prime commutator is again as in the earlier case equal to 2 pi cube 2 omega times delta of k minus k prime. Here we have two sets of new operators b k, b dagger k prime. And, the commutator is again equal to 2 pi cube 2 omega delta k minus k prime. Then, all other commutators vanish.

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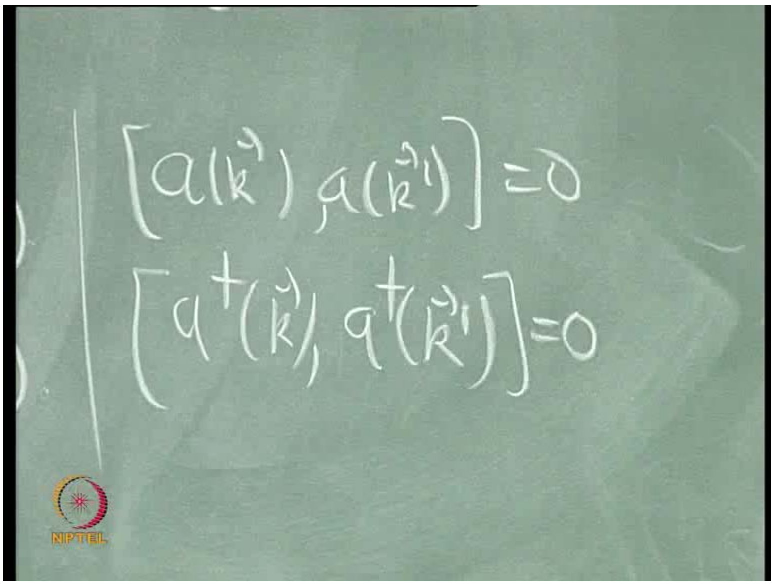

a k, a k prime commutor equal to 0; a dagger k, a dagger k prime equal to 0. Hence, similarly for b case and b dagger case. You can find the Hamiltonian and the momentum operator for the system.

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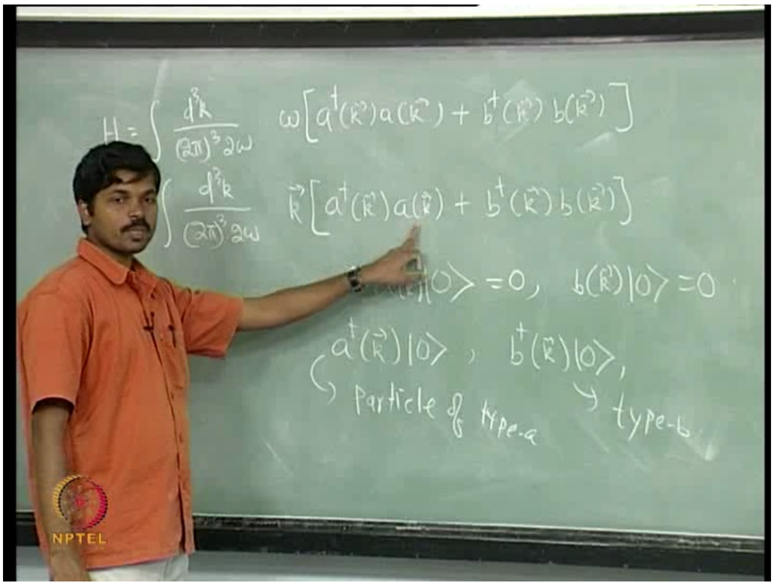

And then, you can express them in terms of the operators a’s and a daggers. It turns out that the Hamiltonian for the system when expressed in terms of a’s and a daggers is equal to d cube k over 2 pi cube 2 omega times omega a dagger k a k. But, now, we have also these operators b dagger and b. So, I have b dagger k b k. This is the normal ordered Hamiltonian. And similarly, the momentum operator is given by d cube k over 2 pi cube 2 omega times k and a dagger k a k plus b dagger k b k. Again we can show that, this a daggers and b daggers are creation operators; whereas, a's and b's are annihilation operators. And, there exist a ground state, which is annihilated by all the annihilation operators. So, a k acting on 0 equal to 0 and b k again acting on 0 is equal to 0. Then, the entire spectrum of the Hamiltonian is constructed by acting the creation operators on the vacuum. So, a dagger k acting on vacuum will give a one particle state. Similarly, b dagger k acting on the ground state will give another one particle state of a different kind. And, you can have two particle states. And, in general, you can have n particle states here. You can construct the number operator. So, the number operator…

Now, you can see that, there are two different types of particles. This one I will call as particle of type a; and, this one I will as the particle of type b. In a moment, we will see what exactly they are. But, for time being, let us call this as particle of type a; and, this is particle of type b. So, you have corresponding number operators, where particle of type a and also for particle of type b.

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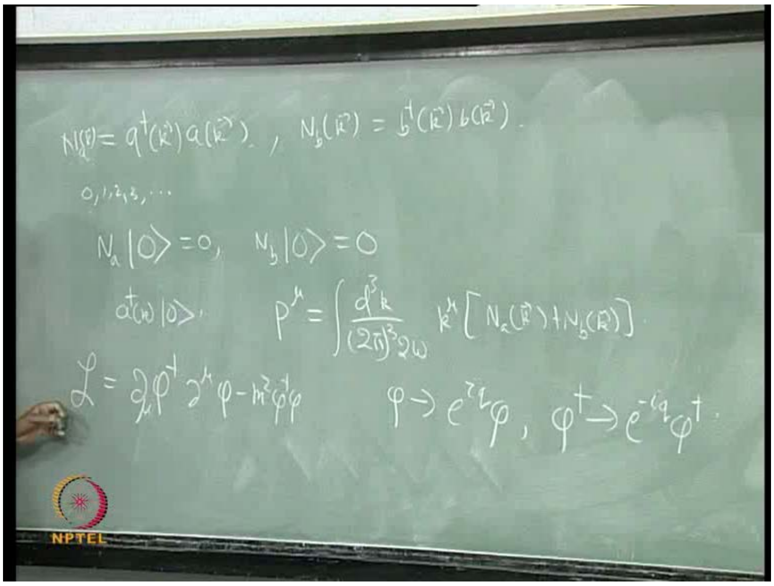

I will denote them as N a – this is a dagger k a k; and, N a of k; and, N b k – this is b dagger k b k. These have eigenvalues 0, 1, 2, 3 and so on. So, these are the occupation numbers. You can see that, the ground state has 0 occupation number. So, N a acting on this is 0 because there is an a to the right and a annihilates the ground state. Similarly, N b acting on this is also 0. Therefore, the ground state is the state of 0 particles. And hence, this is also the vacuum state. And, if you consider a k dagger acting on 0, this will have occupation number 1 and so on. So, this is one particle state, which coincides with our earlier definition and so on. The Hamiltonian and momentum can be expressed in terms of these occupation number operators.

Trivially, I can write the energy momentum operator as this, which is d cube k over 2 pi cube 2 omega k mu times N a k plus N b k. So, all these things are very similar to our discussion in the real Klein-Gordon case; only thing is that, there is an additional particle type, which I call as the particle of type b. However, there is an important difference between the real Klein-Gordon case and the complex Klein-Gordon case. In the real Klein-Gordon case, there was no global symmetry; there was no continuous global symmetry under which the Lagrangian was invariant; whereas, in the complex Klein-Gordon case, we have seen that, the Lagrangian density is del mu phi dagger del mu phi minus m square phi dagger phi, which is invariant under a continuous global symmetry.

If you consider phi going e to the power i q phi; where, q is a constant parameter; and hence, phi dagger going to e to the power minus i q phi dagger; under this continuous symmetry, the Lagrangian is invariant. Therefore, Noether’s theorem tells us there exist a conserved charge here corresponding to this continuous symmetry; which is not there in the case of real Klein-Gordon field. So, there is a new observable in our complex Klein-Gordon case, which is not there in the real Klein-Gordon case. We will see what this name observable is. We can use Noether’s method to compute what this charge is. I think we have computed it in one of these earlier lectures.

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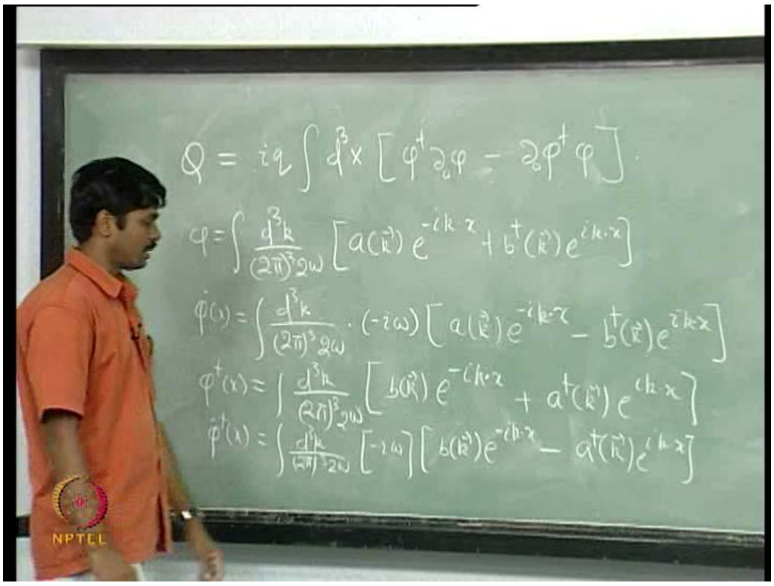

And, the Noether charge in this case is given by… I will denote this as Q. This will be equal to i q times integration d cube x phi dagger del 0 phi minus del 0 phi dagger phi. You have to do normal ordering to get a physically sensible answer. What we will do is that, we will put the mode expansion for phi and phi dagger here; evaluate the x integration; and then, we can express this Q in terms of the creation and annihilation operators a dagger a and b dagger b. So, let us do that.

Let us write again the expression for phi and del 0 phi; phi is d cube k over 2 pi cube 2 omega a k e to the power i k dot x plus b dagger k e to the power i k dot x. And hence, phi dot of x is equal to d cube k over 2 phi cube 2 omega times minus i omega times a k e to the power minus i k dot x minus b dagger k e to the power i k dot x. Phi dagger is the Hermitian conjugate of these. So, I can interchange a and b here. And, the same in this expression. So, let us do it anyway. Phi dagger of x is d cube k over 2 pi cube 2 omega b k e to the power minus i k dot x plus a dagger k e to the power i k dot x. And hence, phi dot of dagger of x is d cube k over 2 pi cube 2 omega times minus i omega times b k e to the power of minus i k dot x minus a dagger e to the power i k dot x. So, now, we can substitute all the four expressions here in the expression for Q. And then, we evaluate the x integration. So, we will do a few steps; then, I will leave it has an exercise for you.

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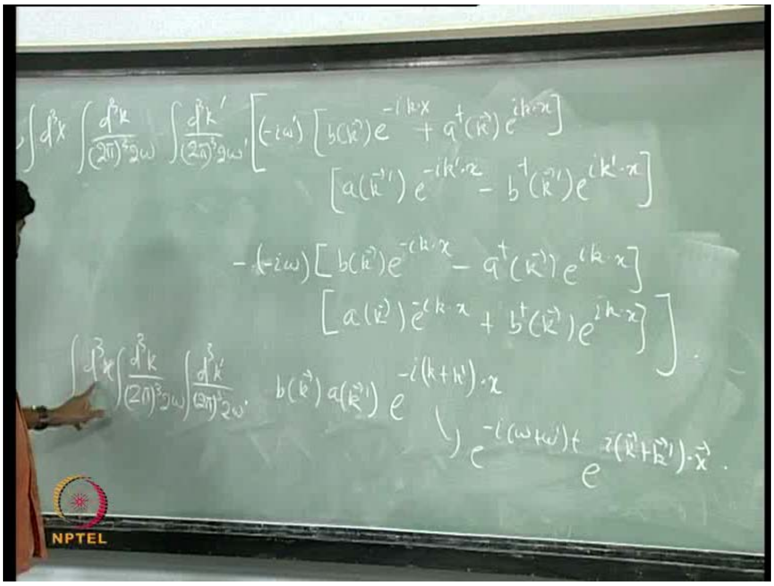

So, for the second one, I am using the level integration variable here as k prime; whereas, for the first operator, I will be using the integration variable as k in both these terms. When I do that, what I get is phi dagger del 0 phi. So, minus i omega times – minus i omega prime actually – b k e to the power minus i k dot x plus a dagger k e to the power i k dot x times del 0 phi with a minus i omega prime; and then, a k prime e to the power minus i k prime dot x minus b dagger k prime e to the power i k prime dot x. Then, the second term, which is minus of minus i omega; and, del 0 phi dagger, which is b k e to the power minus i k dot x minus a dagger k e to the power i k dot x. And then, phi, which is equal to a k e to the power minus i k dot x plus b dagger k e to the power i k dot x. So, what you can do now is you can multiply these terms. This one will give you four terms. Similarly, the second one will give you four terms. And then, you can evaluate the x integration.

When you evaluate the x integration, it will give you a delta function. And then, you can evaluate one of these k integrations by using this delta function. You can see the following. So, for example, you consider here the first term here. The first term will give you integration d cube x d cube k over 2 pi cube 2 omega d cube k prime over 2 pi q omega prime; and, b k a k prime; and then, e to the power minus i k plus k prime dot x. This is a product with a term you knew; here the metric is a term you knew. So, it will give you two terms: one is e to the power minus i omega plus omega prime t; and then, e to the power i k plus k prime dot x. So, this x integration – integration of d cube x times this will give you a delta k plus k prime. So, this delta k plus k prime – now, you can integrate out this k prime; you can carry out the k prime integration. Then, you get a b k a of minus k; and then, there is e to the power minus 2 i t. This term…

There will be an analogous term here; which will give you b k a k and again e to the power minus 2 i t. Because of this minus sign here and because of the fact that b and a commute, you can show that, this term here exactly cancels with this term. So, this way all the terms, which contains either two annihilation operators or two creation operators – they will cancel with each other. Only term which will remain in this integration is the one which is one creation and one annihilation operator. So, from here you get b k b dagger k prime term and a k a dagger k prime term. And, the same expression you will get from here. Instead of cancelling, they will add up. And finally, you have to remember that, you are using normal ordering. So, ultimately, you get after normal ordering, it is a dagger a minus b dagger b. a dagger a will come with a plus sign and b dagger b will come with a minus sign.

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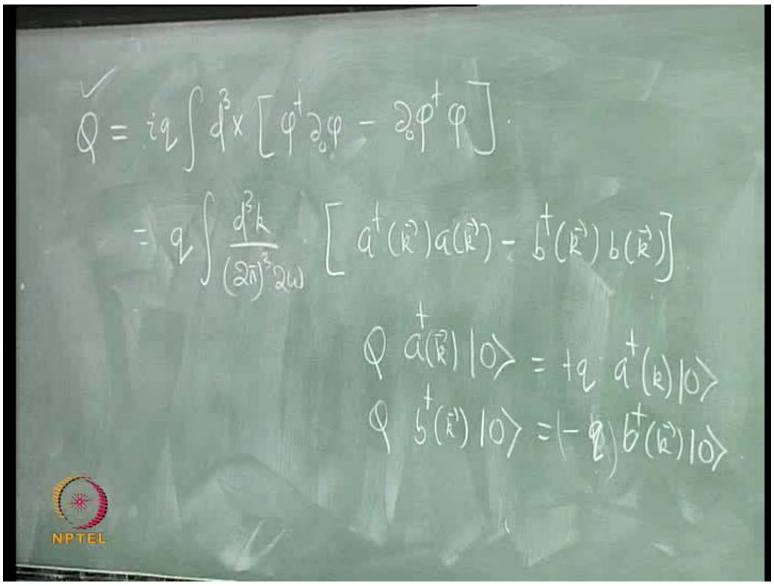

So, when you carry out the integration, what you will get for Q here is the following. At the end of the day, you will find that, this is q times integration d cube k over 2 pi cube 2 omega times a dagger k a k minus b dagger k b k. You understand the origin of this minus sign here? This minus sign here is very important. And, you can see how this minus sign comes here once again. This minus sign comes here because there is a minus here in this term or equivalently there is a minus here. And, this minus here comes because you have del 0 phi… So, you have phi dagger and del 0. And then, this time derivative here brings you a minus sign here. And, because of this minus sign, in the entire term, when you evaluate this product here, you get a dagger a with a plus sign; whereas, you get b b dagger with a minus sign. And, with the normal ordering prescription, this b b dagger becomes b dagger b. So, that is how you get a minus here.

But, now, what you can see; you can now consider let us say one particle state of a particle of type a. So, one particle state of a particle of type a is given by this. Now, you look at what you get by acting Q on it; Q on this will give you a charge, which is plus Q times a dagger k acting on this. So, this is an eigenstate of the charge operator with a charge plus q. This plus here comes because of this plus sign here. Whereas, if you now consider a particle of type b and you try to find what is the eigenvalue of this operator q here for such state; so, b dagger k 0 acting on this will give you an eigenvalue, which is minus q b dagger k 0. So, this one particle states – all are eigenstates of this charge operator Q. But, the particles of type a have charge plus q; whereas, the particles of type b are having charge minus q. Therefore, the particles of type a are nothing but positively charged particles; whereas, the particles of type b are negatively charged particles. This is what which was not there in the real Klein-Gordon case. And, we have this in the complex Klein-Gordon case.

So, a complex Klein-Gordon field can describe particles, where there are two types of charges. For example, a system of pi plus and pi minus. Here the pions – this is a positively charged pion; this is a negatively charged pion. This can be described by a complex Klein-Gordon field. However, this need not be electric charge. You can for example, consider hyper charge. You can use this Q to denote hyper charge. And, you have a system of k 0, k 0 bar with hyper charges plus and minus 1; which can also be described by a complex Klein-Gordon field. So, to summarize, we have quantized real as well as complex Klein-Gordon field. And, we have seen that they can find the entire spectrum. These particles are bosons, because the obey Bose-Einstein statistics. And also, we have seen the importance of normal ordering. If you consider blindly the Hamiltonian, it gives you answer. So, it is nonsensical. And hence, you need time ordering to get a sensible answer. And then, we have also introduced… You need normal ordering to get a sensible answer. Then also, we have introduced concept of time ordering; and then, we have found integral representation for the vacuum expectation value of the time-ordered product, which is the Feynmann propagator. We have also discussed causality in this case; and then, we have seen that, the operators commute if they are at a space-like separation. So, next class, we will introduce an interaction and then we will see how particles interact with each other.