**Quantum Field Theory**
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**Module - 01**
**Free Field Quantization - Scalar Fields**
**Lecture - 06**
**Quantization of Real Scalar Field – IV**

We have seen that, the field operator phi of X is given by the form d cube k over 2 pi cube 2 omega a k e to the power minus i k dot x plus a dagger k e to the power i k dot x.

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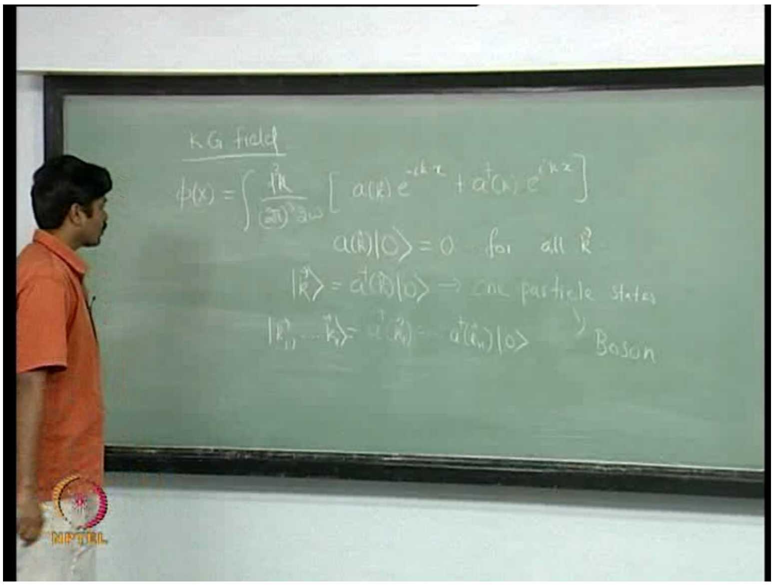


And, in the last lecture, we have seen that, these a k's and a daggers are respectively the annihilation and creation operator. We have also defined the vacuum, which is annihilated by all the a k's; that is equal to 0 for all k. And, the creation operators acting on the vacuum give us one particle states, which I have denoted as k. These one particle states are eigenstates of Hamiltonian as well as the momentum operator with a eigenvalues omega and k respectively. So, these are one particle states. And similarly, you can construct multi-particle states by acting number of a daggers on the vacuum – a dagger k n. This will give you k 1 to k n. And, we have also seen that, this particle like excitations actually obey Bose-Einstein statistics. So, this represents bosons. So, the excitations of the Klein-Gordon field are the bosons.

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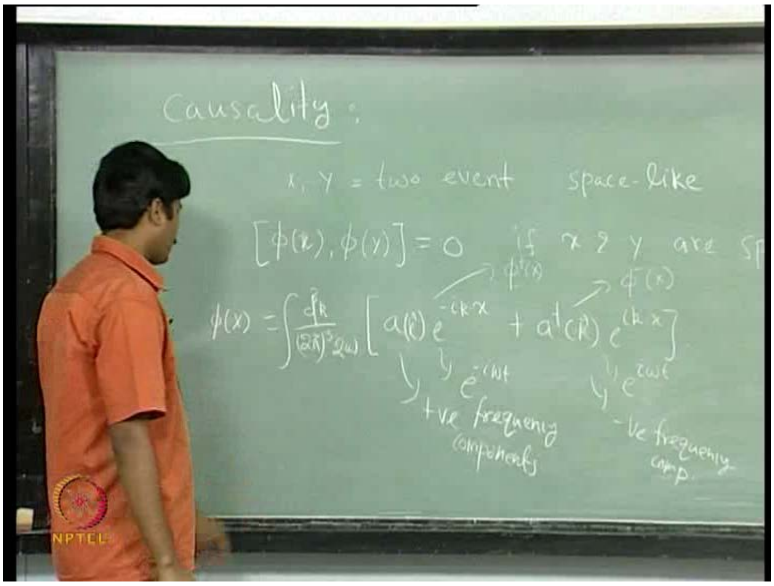


In today's lecture, we will discuss about the implications. One of them is causality. Causality basically says that, if you have two events, which I will say let us say one of them is x; another is y; if these are two events, which are separated by space-like distance from each other; then, if you consider an operator at this space-time point x and you consider another operator at this space-time point y; then, measurement of one of the operators will not affect the measurement of the other operator. So, quantum mechanically what should you do? The commutation of these two. So, you consider any operator at x – any other operator, any other measurable quantity, any other observable at y. If x and y are space-like separated; if x and y are space-like with respect to each other, then the commutation of phi of x – the operator – any operator at the space-time point x with respect with any other operator at space-time point y must ((Refer Slide Time: 04:15)) The simplest one is the observable phi of x. If x and y are space-like, then especially phi of x must commute with phi of y if x and y are space-like with respect to each other. So, let us see.

We have already got the expression for phi of x by solving the Klein-Gordon equation. We can put this expression for phi of x here and then derive this commutation relation; and then, we can see that, they in fact commute. So, let us consider the solution for phi of x, which is… Look at the time dependence here. Here the time dependence comes like e to the power minus i omega t; whereas, here it comes with a plus sign – e to the power i omega t. So, this part here involves the positive frequency modes; whereas, the second part in the expression for phi involves the negative frequency components. So, this involves the positive frequency components; whereas, this one negative frequency. We will denote this to be phi plus of x and this one to be phi minus of x, so that it will be convenience for us to do the computation. So, let us represent this to be phi plus x; whereas, this to be phi minus of x. phi plus x involves only the annihilation operators; whereas, phi minus x involves only the creation operators. So, phi plus x will commute with phi plus y; and also, phi minus x will commute with phi minus y; whereas, phi plus x may not commute with phi minus y and the vice versa. So, we can express… We can try to evaluate this commutator and express it in terms of phi plus x and phi minus x. When we do that, what will get is the following.

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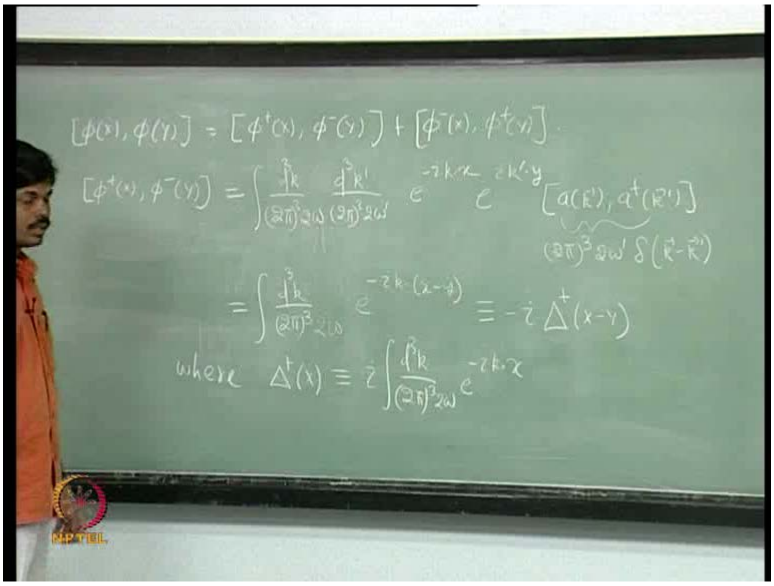


So, phi of x, phi of y is a phi plus x, phi minus y plus phi minus x phi plus y. If we compute the first term here, we can obviously get the second term just by interchanging x and y and a sign. So, what we will do is that, we will first evaluate this commutator of phi plus of x and phi minus of y. What is this commutator? For phi plus x, we will use the variable k. So, you have d cube k over 2 pi cube 2 omega. And, for phi minus y, we can use the variable k prime. So, I have d cube k prime over 2 pi cube 2 omega prime is the integration measure. Then, I have e to the power minus i k dot x from phi plus of x and e to the power plus i k prime dot y from phi minus y. And then, I have the commutator of a k, a dagger k prime.

We already know what this commentator is. So, we can put the value of this commutator here. This commutator I have already evaluated and then this is 2 pi cube 2 omega prime delta k minus k prime. So, you can substitute this part in the commutator and then we can see that, this 2 pi cube 2 omega prime will cancel with this 2 pi cube 2 omega prime. We can integrate all to one of these k primes variable. When we do that, what we get is d cube k over 2 pi cube 2 omega times e to the power minus i k dot x minus y. This is the standard notation. This is represented by something which is known as delta plus. This is equal to minus i delta plus of x minus y; where, the function delta plus of x is defined to be i d cube k over 2 pi cube 2 omega e to the power minus i k dot x. So, we have already evaluated the first term.

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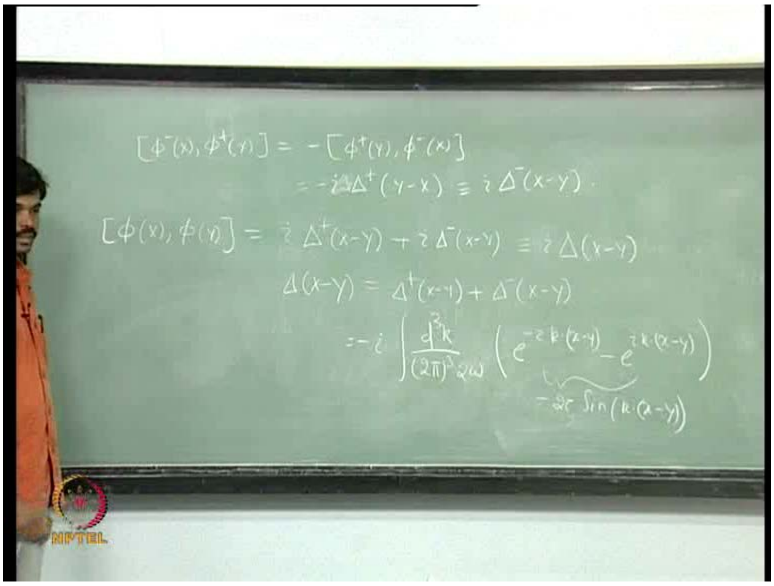


The second term will simply be phi minus of x phi plus of y is simply minus phi plus of y, phi minus of x. So, this is minus… The standard notation is such that there is an i here. So, there is a minus i in the definition of delta plus of x. So, this is minus i delta plus of y minus x; which is also denoted as i delta minus of x minus y. When I substitute that in this expression, what I get is the commutator of phi of x with phi of y is i delta plus of x minus y plus i delta minus of x minus of y; which is defined by i delta of x minus y; where, delta of x minus y is delta plus of x minus y plus delta minus of x minus y. When I substitute the expression for delta plus and delta minus, what I get here is i times d cube k over 2 pi cube 2 omega e to the power minus i k dot x minus y minus e to the power i k dot x minus y.

Student: Minus i…

There is a minus i here.

Student: Because delta plus is minus i times ((Refer Slide Time: 14:08))

Thank you. So, when I substitute it here, I will get this commutator here to be the one without a factor of i. Somehow I am missing a factor of i somewhere. I think it is fine. So, because of this minus sign, you can express this delta of x minus y as also a sign… You get a minus… So, when you get this, what you get is minus 2 i sign k dot x minus y. And hence, the delta plus x y; this i will go away; and then, you can see that this involves integration of d cube k over 2 pi cube 2 omega times 2 sine k dot x minus y. So, this delta of x minus y has several properties. First thing is that, it is an odd function of x minus y; it is an odd function of its argument, because sine is an odd function. Second thing is that, it is Lorentz invariant.

And finally, we can show that, this in fact satisfies the Klein-Gordon equation when the argument is not 0. You can apply the Klein-Gordon operator on this delta of x minus y. And then, you can say that, it in fact satisfies the Kelvin-Gordon equation when this is nonzero. However, we would like to show that, this commutator in fact… When he says when x and y are space-like with respect to each other. Or, in other words, you would like to show that, this delta function actually becomes 0 when x minus y is space-like. So, let us do that. If you consider a quantity, which is Lorentz invariant; and, if this quantity is 0 in any one inertial frame, then it will remain 0 in all inertial frames because of Lorentz invariance. So, we will exploit that property; then, we will go to a frame, where we will compute this delta of x minus y to be 0 if x minus y space-like. And then, this will tell us that, it will always remain 0 as long as x minus y is space-like.

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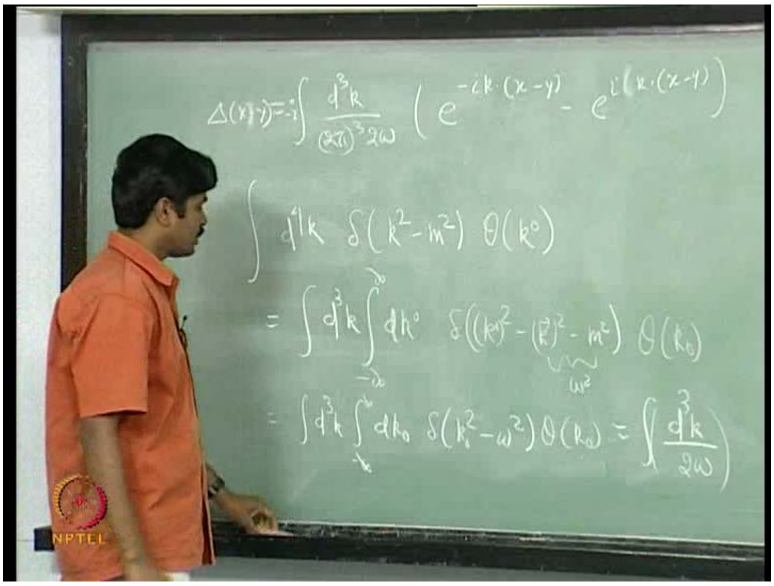

Since Lorentz invariant is so crucial, let me prove that, this delta function as in fact Lorentz invariant. This is d cube k over 2 pi cube 2 omega times minus i e to the power minus i k dot x minus y minus e to the power i k dot x minus y. This is already Lorentz invariant; this is also Lorentz invariant. I have claimed earlier that, this integration measure here is Lorentz invariant. I have not proved it so far. So, let us prove that, this integration measure is Lorentz invariant. We know d 4 k; unlike d cube k, d 4 k is Lorentz invariant. When you make a Lorentz transformation, it will give you determinant lambda, which itself is invariant under proper ((Refer Slide Time: 18:43)) Lorentz transformation. So, this is Lorentz invariant. Delta of k square minus m square – you consider this quantity; where, k square is k mu, which is Lorentz invariant. Therefore, this is Lorentz invariant; theta of k 0 is also Lorentz invariant. So, you consider this quantity here, which is Lorentz invariant.

And, let us evaluate this integration. This is integration d cube k integration d k 0; where, k 0 runs from minus infinity to plus infinity; and, delta of k square minus m square is k 0 square minus k square minus m square theta of k 0. Here k 0 is a variable, which ranges from minus infinity to plus infinity. We have already seen that, this is omega square. So, let us rewrite this. This is d cube k minus infinity to infinity d k 0 and delta of k 0 square minus omega square. That is the argument of delta function and then theta of k 0. This theta of k 0 restricts k 0 to take all positive values. So, you can already see that, if you write this k 0 square minus omega square, is k 0 plus omega times k 0 minus omega;

and, you use the proprietary delta of a x is 1 over a delta of x. So, then you will see that, you can evaluate this k 0 integration here. And, this will turn out to be d cube k over 2 omega. So, you have started with quantity, which is Lorentz invariant. And, when we have carried out the k 0 integration; and, at the end, we got this integration over d cube k over 2 omega. Therefore, this here is this factor 2 pi cube is for convenience. So, this is actually Lorentz invariant. So, we have proved that, delta of x is in fact Lorentz invariant.

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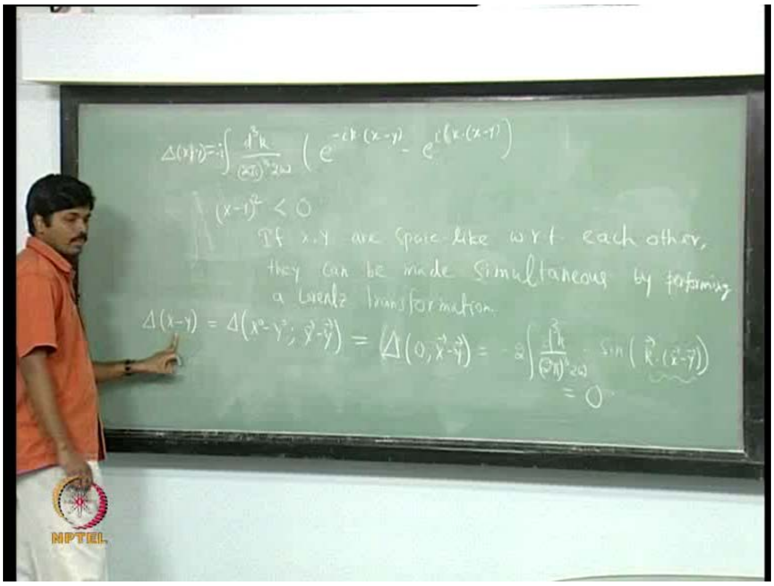


Now, it is very easy to show that, this is in fact 0 when x minus y square is negative. If two events: x and y are space-like with respect to each other; then, you can perform a Lorentz transformation; and, make them simultaneous by performing a Lorentz transformation. So, I can set the 0-th component of x to be equal to the 0-th component of y. If that is the case, then… So, I start with delta of x minus y. It is delta of x 0 minus y 0; x minus y. And, I can make a Lorentz transformation – says that x 0 minus y 0 is 0. So, this is also equal to delta of… Since delta is Lorentz invariant, this will be equal to delta of 0, x minus y. What is delta of 0 x minus y? This is equal to minus 2 integration d cube k over 2 pi cube 2 omega times sign k dot x minus y. There is a minus, which will from sine; and, there are 2 i's, which will give minus. So, as a result, it will be plus.

Now, you can see that, this in fact is 0, because this is an odd function of k, which is integrated throughout the entire range of k. Therefore, you can change k to minus k; it will give a minus sign. And hence, this will become 0. This is 0 because of the fact that, the integrand here is an odd function of k. This is not the case when the argument here is not 0. If you cannot make this to be 0, then you have a four vector here in the argument of the sine; and, it is not an odd function of k usually. So, this does not vanish when you cannot make this argument here to 0; whereas, if you can make it to 0 by a Lorentz transformation, then this vanishes identically. So, what we have seen is if x and y has space-like with respect to each other, then you can make them simultaneous. And, when they are simultaneous; these two events are simultaneous; then, the delta function vanishes. So, the delta of x becomes 0; delta of x minus y becomes 0 if you make the argument simultaneous. And, they are simultaneous if they are space-like with respect to each other.

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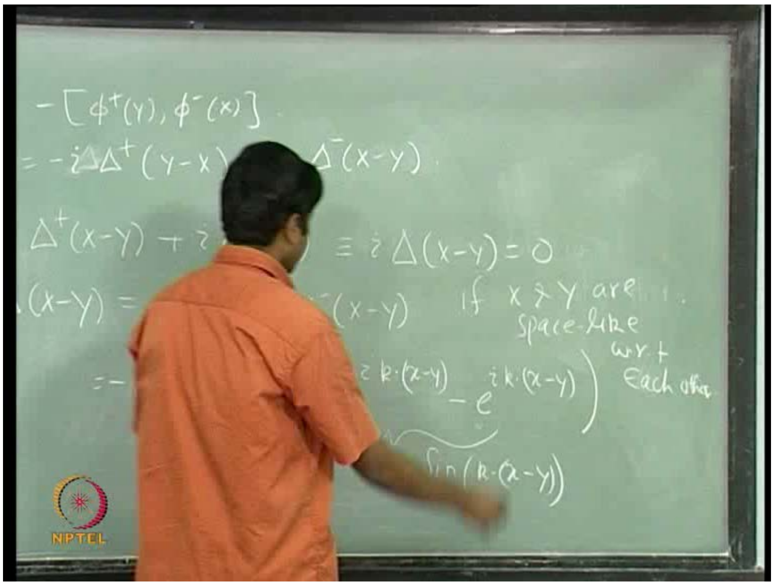

So, the conclusion is that, this delta here vanishes if x and y are space-like with respect to each other. Since this delta of x minus y is up to a factor of i equal to the commuter of phi of x and phi of y; therefore, phi of x and phi of y commute when x and y are space-like with respect to each other. This is what causality tells us. And hence, we have seen that, at least the Klein-Gordon theory is causal; in the sense if two events are space-like with respect to each other, then the measurement of an observable at one of these events does not affect the measurement of any other observable at the ((Refer Slide Time: 27:29)) other space-time event.

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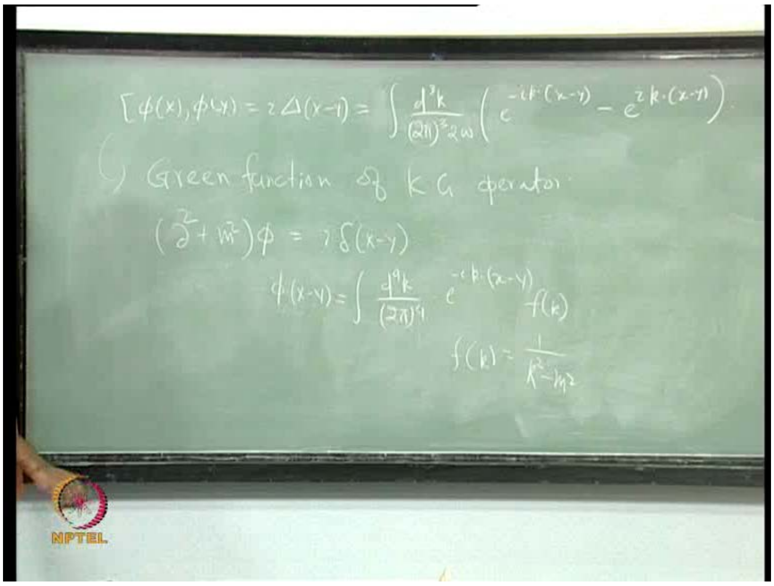


Now, we will start with this commute… And, we can show that, this actually is a Green function of the Klein-Gordon operator. How can you do that? The Green's function for the Klein-Gordon operator can actually be obtained by solving this equation del square plus m square phi equal to i delta of x minus y. And, the solution here – if you write phi of x minus y to be d 4 k over 2 pi 4 e to the power minus i k dot x minus y f of k, you can substitute it here. And, when you substitute it here, you can see that, f of k is in fact 1 over k square minus m square. It is very straightforward. This delta… You can have the integral representation for this 4-dimensional delta, which is same as this without this vector f of k here. So, you can see that, this in fact… When you operate del square over m square here, you will get k square minus m square in the left-hand side. So, f of k times k square minus m square is 1; and hence, f of k is k square minus m square.

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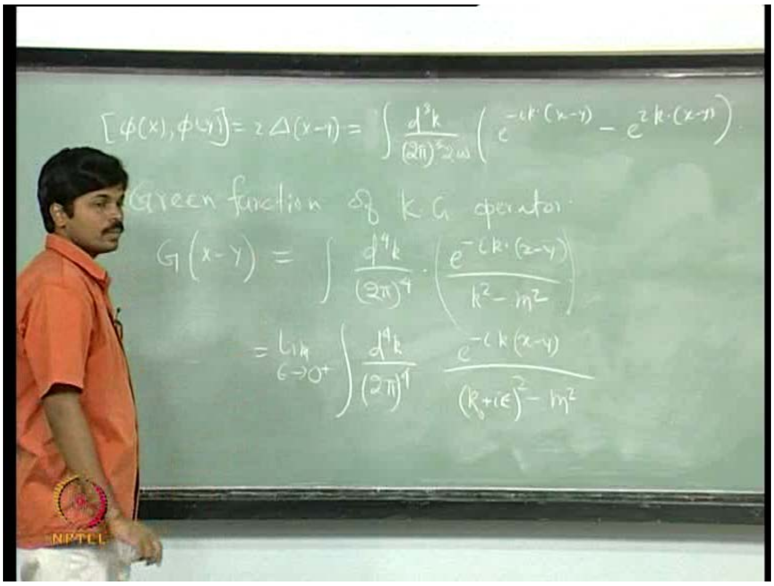


So, the Green function for the Klein-Gordon operator is… I will denote this as G of x minus y is d 4 k over 2 pi 4 e to the power minus i k dot x minus y divided by k square minus m square. Again this k square minus m square has a pole when you consider the range of k 0. k square minus m square is nothing but k 0 square minus omega square. And hence, it becomes 0 when k 0 is equal to plus or minus omega. Therefore, the integrand diverges at these two points. So, you need to define what you mean by this integration. You can evaluate it in four different ways.

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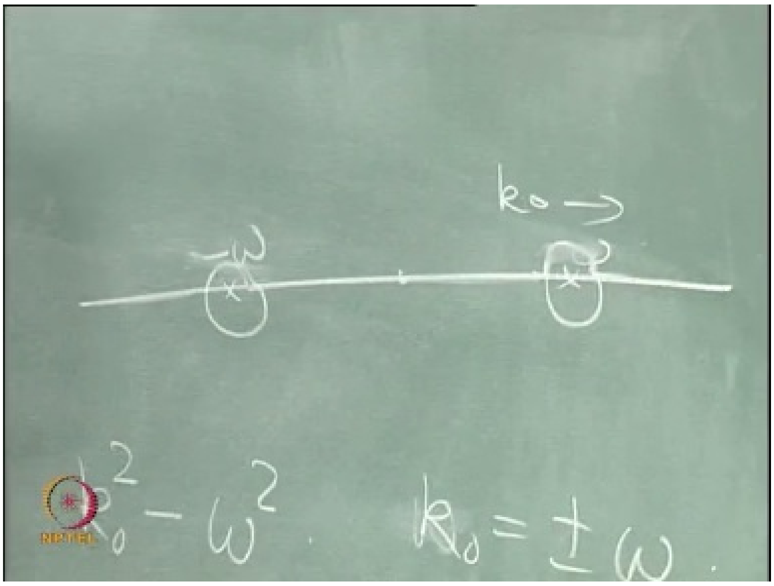

Therefore, if you consider the integration on the k 0 line, this represents k 0 here. This is minus omega and this is plus omega here. So, this integrand diverges. So, you can use, you can evaluate this integration by the means of Cauchy's residual theorem. You need to define what you mean by this integration. You can do this integration by removing these two poles in four different ways. One of them is you can in fact consider contour to be like this or you can consider the contour downwards here or you can consider one of them upwards, one of them downwards and so on. So, there are four different ways you can evaluate this integration. And, accordingly, you will get four different answers. So, depending on the boundary condition, one of these integrations will be the correct integration. Or, what you can do is you can adopt this epsilon prescription and then you can consider this integration to be the limit epsilon tends to 0 plus integration d 4 k over 2 pi 4 times e to the power minus i k dot x minus y divided by k 0 plus i epsilon square minus m square. You can consider k 0 minus i epsilon also. That will tell you that you are choosing a different contour and so on. So, this is one particular way of choosing a contour. What we can do is that, we can show that we can evaluate this integration here. And then, this will be equal to the commutator of phi of x and phi of y under certain condition. So, let us see that.

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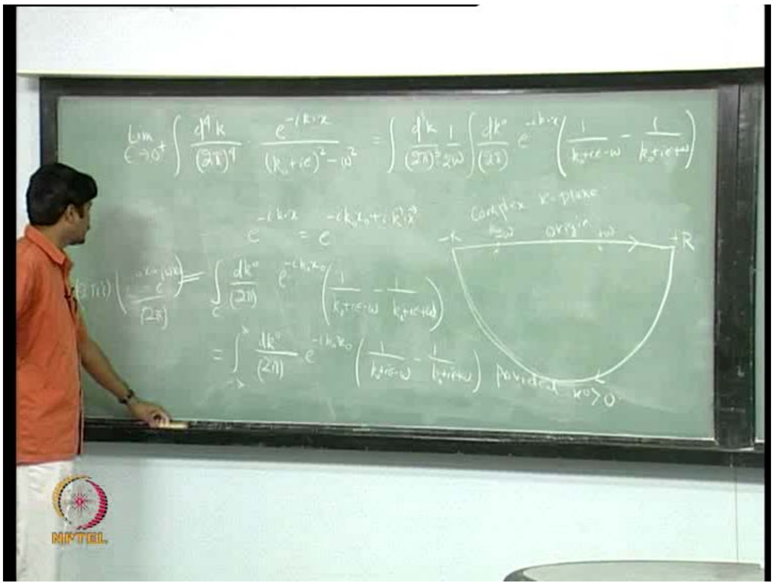

So, let us consider this integration limit epsilon tends to 0 plus d 4 k over 2 pi to the power 4 e to the power of minus i k dot x divided by k 0 plus i epsilon square minus…

Student: Omega square.

Omega square; thank you. Omega square. So, this here I can write it as 1 over 2 omega times 1 over k 0 plus i epsilon minus omega minus 1 over k 0 plus i epsilon plus omega. All right? And, I will substitute it here. When I substitute it here, what I get is… This d 4 integration – I can write it as integration over d cube and d k 0. So, this is d cube k over 2 pi cube and 1 over 2 omega; then, integration d k 0 over 2 pi. Then, this quantity here – times e to the power – e to the power minus i k dot x times 1 over k 0 plus i epsilon minus omega minus 1 over k 0 plus i epsilon plus omega.

Now, the value of this integration depends on whether x 0 is positive or negative. If x 0 is positive, then you can in fact show that you can… So, what is the argument here? e to the power minus i k dot x, which is e to the power minus i k 0 x 0 plus i k dot. So, now, what you can do is that, you can consider the integration here on the complex k 0 plane; and, you can choose this contour of this integration to be this one. This is a semi circle of radius r; and, this is the origin; and, this runs… This here k 0 – this is complex k 0 plane. And, in this plane, this is the origin. This point here is minus omega; this point here is plus omega; and, this is minus R and this is plus R. You choose this contour. You consider this integration on this contour – integration d k 0 over 2 pi times e to the power minus i k 0 x 0 1 over k 0 plus i epsilon minus omega minus 1 over k 0 minus k 0 plus i epsilon plus omega. You choose this contour integration; where, you have contour c is this one. And, you take the limit of R goes to infinity.

And, you can see that, this contour here – if x 0 is positive, then this contour will in fact be equal to the integration minus infinity to infinity the same integration; where, you take instead of choosing this integration along the contour, you take it from minus infinity to plus infinity. That is because if x 0 is positive, then the integration along this semi circle actually becomes 0; it becomes exponentially small. And, in the limit R goes to infinity, this in fact vanishes. On the other hand, if x 0 is negative, then the contribution from this semicircle does not vanish. So, this is not equal to this when x 0 is negative. This integration along this contour is equal to this integration, where k 0 varies from minus infinity to plus infinity only when x 0 is positive, because only then, you can see that you can take k 0 to be some i R. And then, this will give you some e to the power minus R x 0. And, in the limit R goes to infinity, this contribution from the semicircle in fact vanishes.

On the other hand, if x 0 is negative, you will plus mode of this. So, this diverges. Therefore, this equality does not hold. In fact, if x 0 is negative, you can choose the contour the other way; instead of closing if from the down, you can close this semicircle from up. And then, the other integration here from the semicircle on the upper half plane vanishes. So, now, you can see that, this is in fact equal to d k 0 over 2 pi e to the power minus i k 0 x 0 1 over k 0 plus i epsilon minus omega k 0 plus i epsilon plus omega provided x 0 is greater than 0.

Now, the reason I am considering this contour here is I can evaluate this integration using Cauchy's residue theorem here. Therefore, I have evaluated this integration when x 0 is positive. So, what is the value of this integration when I use Cauchy's residue theorem? This in fact is 2 pi i times… There are 2 poles; and then, you can just at the residues at both these poles, e to the power minus i omega x 0 minus e to the power i omega x 0. This is what you get here – divided by 2 pi. So, this I considered without this factor of i k dot x here. When I have i k dot x, I will have an additional e to the power i k dot x. So, I can substitute this in this integration here.

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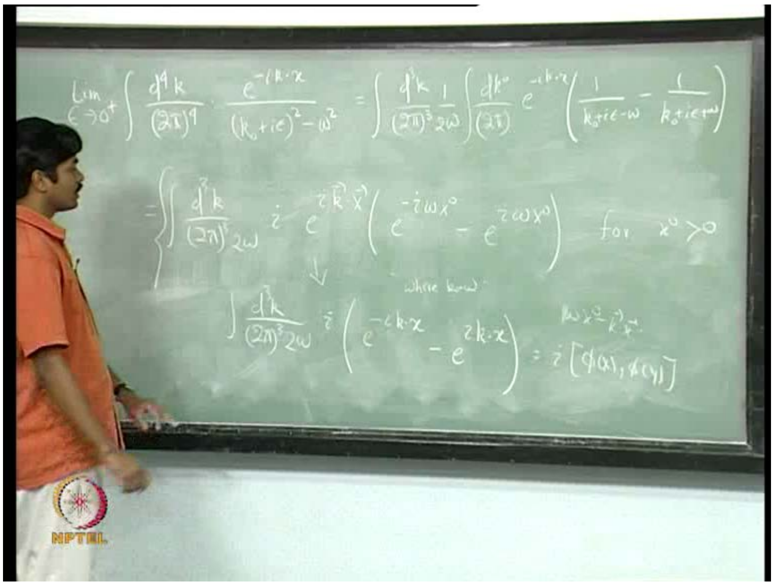

When I do that, what I get is… That is an i because there is a 2 pi i in the residue theorem; and then, e to the power i k dot x times e to the power minus i omega x 0 minus e to the power i omega x 0 for x 0 greater than 0. What is the value of this integration for x 0 less than 0? We can again use the residue theorem. We can choose this contour instead of the contour, where the semicircle is in the lower half plane. And, the contour – this – the integration on this contour will be equal to this integration if x 0 is negative. But, this contour – the interior of this region does not include any pole here. So, the contour integration here is 0. Therefore, this integration here is 0 for x 0 less than 0.

Now, this one you can rewrite it here. You can do a simple rewriting and then you can show that, this in fact is equal to the commutator of i of x with i of y. I will show this in a moment. So, this – you can consider this integration here. This is d cube k over 2 pi cube 2 omega i. Let us consider this for x 0 greater than 0. What you can do is you can take this inside here and then this simply becomes e to the power minus i k dot x here; where, k 0 is omega. Whereas, in the second part, you can change this integration variable here from k to minus k. Nothing will change here, because these limits are such that this will remain invariant. And, here what you will get is omega x 0 minus k dot x; which you can write as e to the power i k dot x; all right? So, this entire thing here you can write it in a Lorentz invariant fashion; and, you can see that this is nothing but i times the commutator of phi of x and phi of y. So, the Green function what we have seen is that, the Green function of the Klein-Gordon operator is equal to the commutator of phi of x times phi of y provided x 0 minus y 0 is 0. And, this is 0 if x 0 minus y 0 is less than 0.

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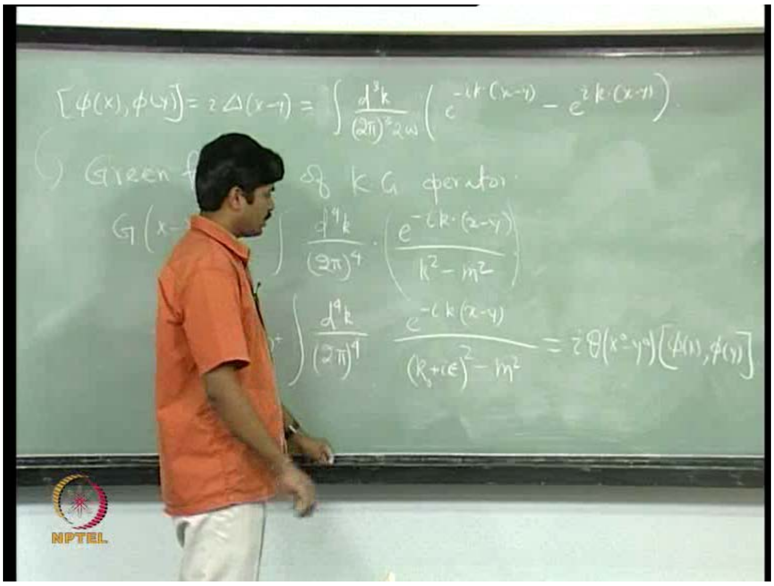

So, just to summarize, what we did is we have evaluated this and then we have shown that, this is equal to i times theta of x 0 minus y 0 times the commutator of phi of x and phi of y; all right?