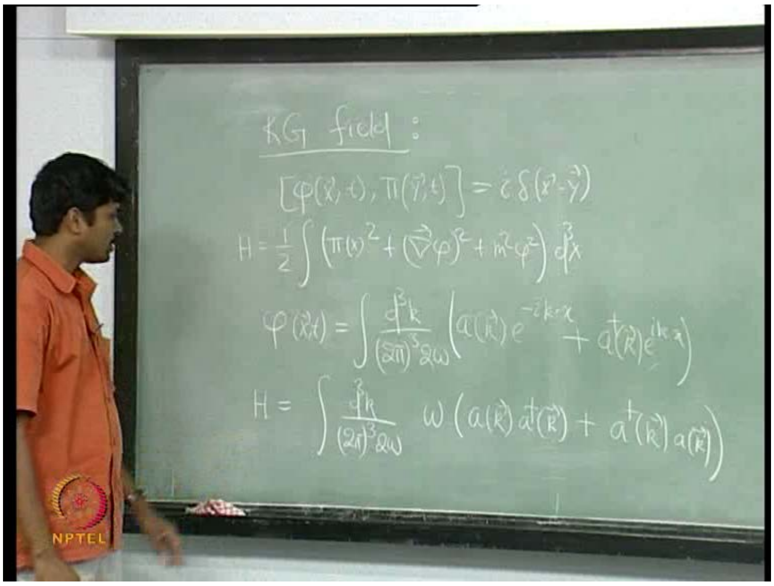
Quantum Field Theory
Prof. Dr. Prasanta Kumar Tripathy
Department of Physics
Indian Institute of Technology, Madras

Module - 1
Free Field Quantization Scalar Fields
Lecture - 5
Quantization of Real Scalar Field - III

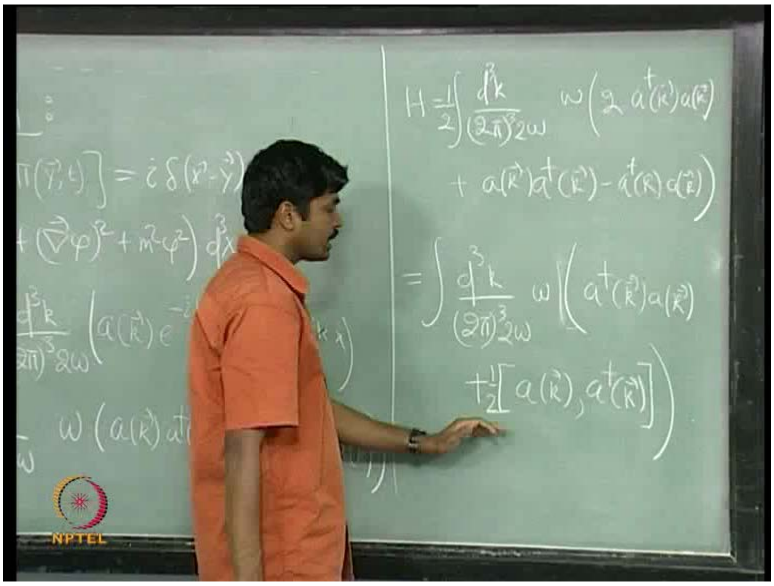
(Refer Slide Time: 00:20)



In today's lecture we will discuss a several important results. Let us briefly review what we have discuss so far we have been discussing quantization of a real scalar field or a Klein Gordon field. So, the fields are basically operator with the commutation relation pi of x t and pi of y t; commuter is given by i delta X minus Y. And using this commutation relation; we have seen that the Hamiltonian for the real Klein Gordon field which is given by half pi X square plus grade pi square plus m square pi square d cube X; this can be expressed in terms of a operators a k and a digger k which appears in the expensive on the field pi.

So, the field pi (X, t) is written as a d cube k over 2 pi cube 2 omega a k e to the power minus i k dot x plus a digger k e to the power i k dot x; we substitute this expression for pi and grade pi and pi in the Hamiltonian. And when I do that what we have seen in the last lecture is that the Hamiltonian can be written as integration d cube k over 2 pi cube 2 omega times omega times a k a digger k plus a digger k a; this is the Hamiltonian. So, we will start with this Hamiltonian and then we will find this spectrum for this system.

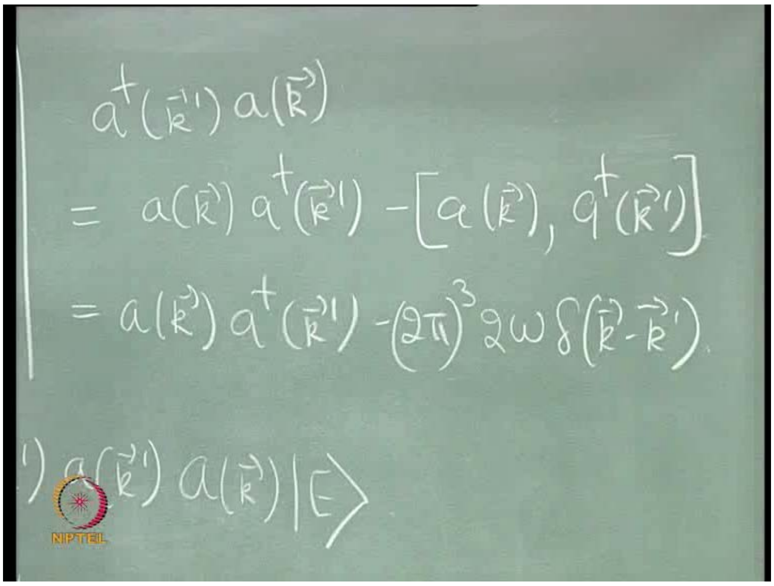
(Refer Slide Time: 03:22)



So, let us re write the Hamiltonian in the following way H is a this times; here what I do is a is that half here; I will write it as a twice a digger k a k. So, I have added I wrote this term as I have added one more time like that. So, I have to subtract that plus a k a digger k minus a digger a k. So, I am done something which is tritely I just ended and subtracted this pieces. So, this is what is the Hamiltonian? But now you consider in the second line I have the commutator of a k and a digger k. So, let us re write the Hamiltonian; this is d cube half here d cube k over 2 pi cube 2 omega times omega; this half will cancel this 2.

So, I have a digger k a k and then this one is the commutator of a k digger k; there is a half here that is all we get from the Hamiltonian. But now you see the second term which is equally a commutator is actually c number. So, we will discuss the detail about the second term in a moment but for the timing being we will just ignore this term; this is just a c number. So, it is enough to find this spectrum for first term here; if I find this spectrum for the first term I find the spectrum for the full Hamiltonian. So let us now focus on the first term.

(Refer Slide Time: 05:58)

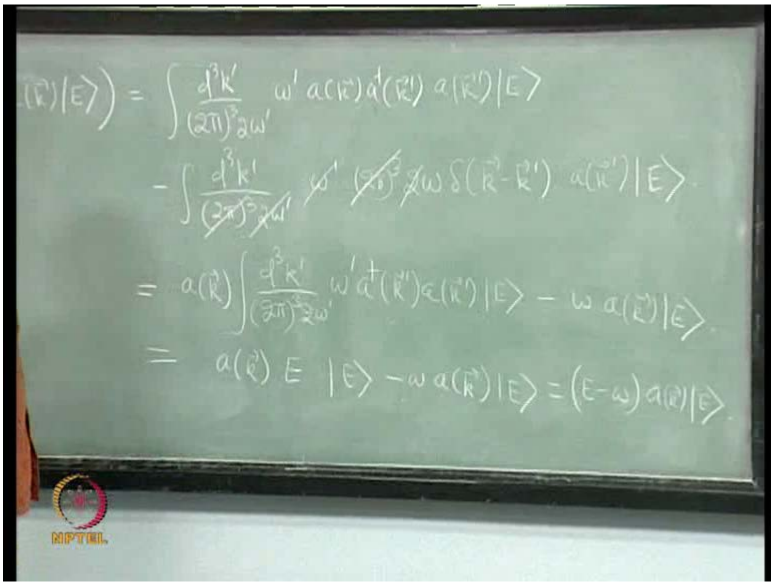


So, let us write the first term in the Hamiltonian which I will denoted H N. So, H N is d cube k over 2 pi cube 2 omega times omega a digger k a k. Let us assume that e is an Eigen state of this operator with Eigen value H of H N acting on e gives me e as an Eigen value; the question that I would like to ask is if I consider this state here which is a k acting on the Eigen state e. So, this is some state; what do I get by acting this operator H N on this state; this is what I am interested to find. So, let us act H N on this state; here because I am using this level k here I will use k prime as the integration variable.

So, this is integration d cube k prime over 2 pi cube 2 omega prime times omega prime a digger k prime a k prime; this acting on a k e a k; I have to find what I get from this expression. Now, look at this 2 terms a k and a k prime commute with this. So, I can just move it to the left and when I do that; what I will get is integration d cube k prime over 2 pi cube 2 omega prime times omega prime a digger k prime a k; a k prime acting on this state e.

Now, these 2 terms which is a digger k prime a k; this I will write as a commutator this I can write it as a k a digger k prime minus the commutator of this 2 terms minus commutator of a k a digger k prime. And the commutator is nothing but 2 pi cube 2 omega k times delta k minus k prime. So, this equal to a k a digger k prime minus 2 pi cube 2 omega delta k minus k prime. So, this is what I will substitute in this term here. So, when I do that I will get 2 terms.

(Refer Slide Time: 10:03)



So, let us write both this terms separately I have H N acting on a k e is equal to. And then omega prime then a k a digger k prime. Then a k prime acting e and the second term is minus integration d cube k prime over 2 pi cube 2 omega prime times omega prime; then 2 pi cube 2 omega delta k minus k prime a k prime acting on e. So, let us look at the first term first. Here, all this are numbers in this operator a k the argument is k whereas integration variable is k prime. So, I can take this a k to the extreme left outside the integration. So, when I do that what I get is a k time integration d cube k prime over 2 pi cube 2omega prime a digger k prime a k prime acting on e.

Then, I have the second term I can because there is a delta function I can carry out the integration; when I carry out the integration this 2 pi cube will cancel, this 2 omega prime will cancel here. And this will simply gives me omega minus omega times a k acting on e. Now, what is this term here? This is H N. So, I can now write this is a k H N acting on e minus omega a k e.

Student: ((Refer Time: 13:21)).

No, because I have carried out this integration.

Student: No.

Here, right.

Student: No.

Here, where is omega prime? There is omega prime here.

Student: ((Refer Time: 13:39)).

There is one more.

Student: ((Refer Time: 13:43)).

Now, this omega prime is cancel this; are you talking about the first line or second line?

Student: First line.

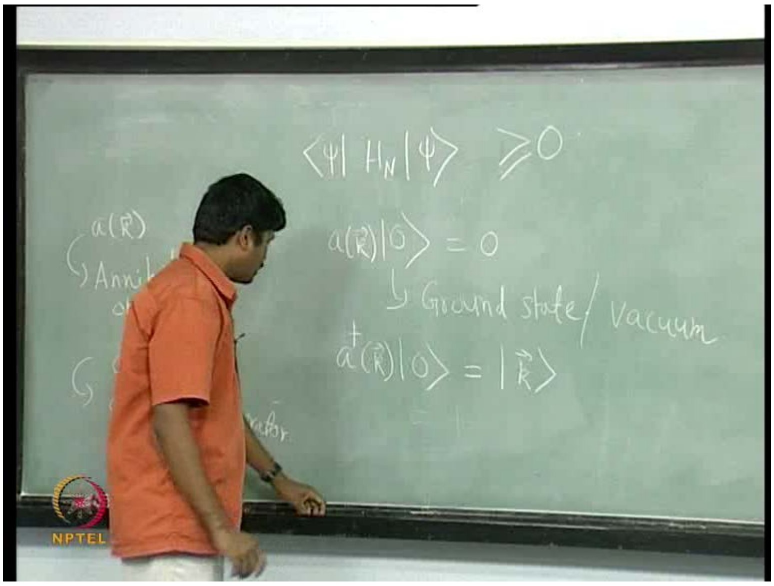
First line.

Student: ((Refer Time: 13:53)).

Here, there is a omega prime thank you. So, there is an omega prime. So, the entire thing is the a first term in the Hamiltonian which I do not as H N and the second term as it is. So, I forgot to write this omega prime here. Now, look at this term we have assume that is Eigen state of H N with Eigen value e. So, this term now is written as I can now I can re write it as e times this. So, the whole thing is equal to e minus omega a k acting on e. So, what we have learned from here? If e is an Eigen state of H N with Eigen value e then a k acting on e is also an Eigen state of H N with Eigen value e minus omega; this what we have learned from here.

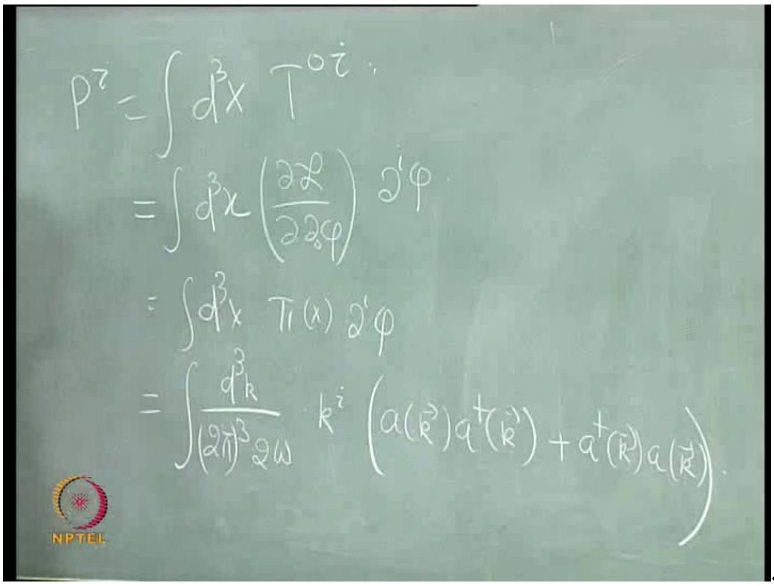
Therefore, this a k is an annihilation operator; it annihilates a quantum of energy omega in this state here. Similarly, if you consider instead of a k acting on e if you consider a digger k acting on e; you will see that again it will be Eigen state of H N with Eigen value e plus omega. Therefore, a digger creates a quantum of energy omega. So, a digger is a creation operator; this is annihilation operator and this is creation operator.

(Refer Slide Time: 16:19)



And, if you look at the expectation value of H N in some states psi; since H N involves a digger a k a k; this quantity is positive definite is greater than or equal to 0; whereas this a k lowest the energy by omega. So, there must exist a state such that a k acting on this state gives me 0 this annihilates the annihilation operators annihilates this state; this I will call as the ground state of the system or I will call it as the vacuum. Now, once I have this vacuum; I can act on this the creation operator on the vacuum. And this will give me some state which I will denote as k; whose H N Eigen value is omega a digger k will create a quantum of energy omega from the vacuum.

(Refer Slide Time: 18:01)

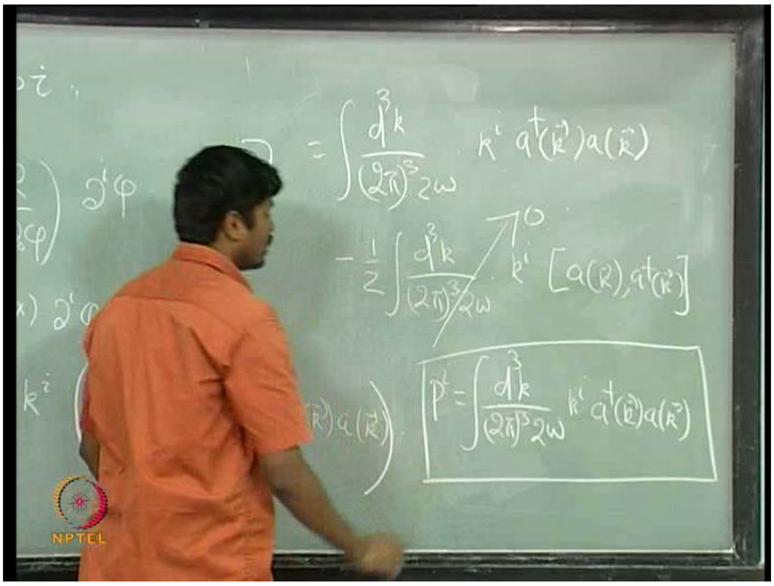


Now, we can look at the momentum operator you know the momentum operator is integration d cube x t 0 i.

Student: ((Refer Time: 18:16)).

Here. Right now I am just using this as a level. And this state is such that H N acting on this state is omega acting on omega times k. Now, we will see why we have to use this level k here; that is why I am considering this operator P here; if you substitute for the annihilation momentum what you will get that is d cube x del L over del 0 pi times del i pi. And then you substitute for the fields pi or this is equal to d cube x pi of x Del i pi; you substitute for pi and Del i pi. And carry out the integration then what you will get I will not work it out I will leave it as an exercise for you; what you show is that this momentum operator P i will give me d cube k over 2 pi cube 2 omega times k i a k a digger k plus a digger k a k. So, this is an exercise for you this is what you will get. Now, again I can write this as twice a digger k a k plus commutator of a k and a digger k where I do that there is a half here I think.

(Refer Slide Time: 20:29)

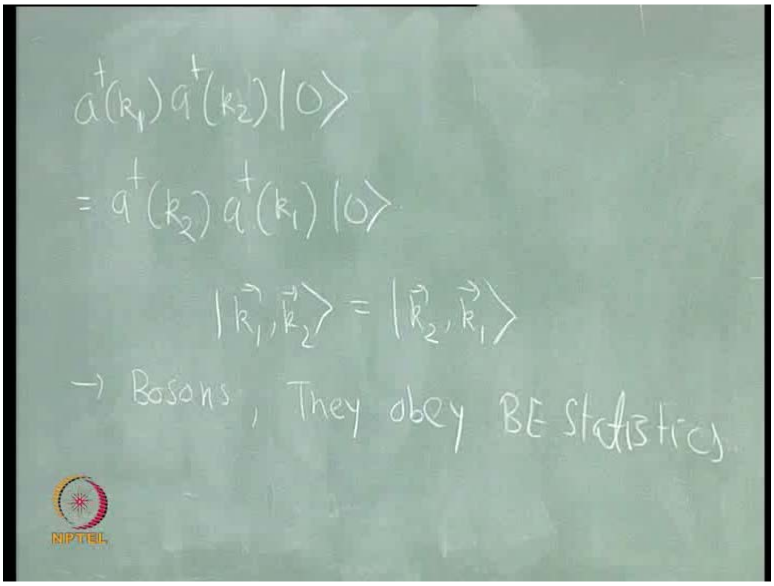


When I do that what I will get is d cube k over 2 pi cube 2 omega k i a digger k a k minus half d cube k over 2 pi cube 2 omega times k i commutator of a k a digger k. In the second term this is a c number; which is even when k goes to minus k. Because this involves some delta function which is the even function of k; where as there is a k i which is odd. So, when you carry out this integration it will since sine. So, this quantity will be equal to minus of itself because of the appearance of k i here; hence this integration is 0.

So, the second term when it is identically therefore what you get for the momentum P i is equal to d cube k over 2 pi cube 2 omega k i a digger k a k. Now, you take this momentum operator and you yet on this state k here. Then what you will get is not only that it H N Eigen value omega; it is also an Eigen state of the momentum operator if you yet P i on this then you will get k i. So, this state k is an Eigen state of H N as well as P i with Eigen values omega and k i respectively.

And, that is the reason I level this s k here because omega as you know I am not explicitly writing. But omega is actually of k which is square root of k square plus m square. So, it is sufficient to level it by the momentum of the state. So, what this operator a digger k does is it creates a quantum of energy omega and momentum k of a. Therefore, this state k here I will interpret it as a one particle state with an energy omega k and momentum k. So, these are one particle states.

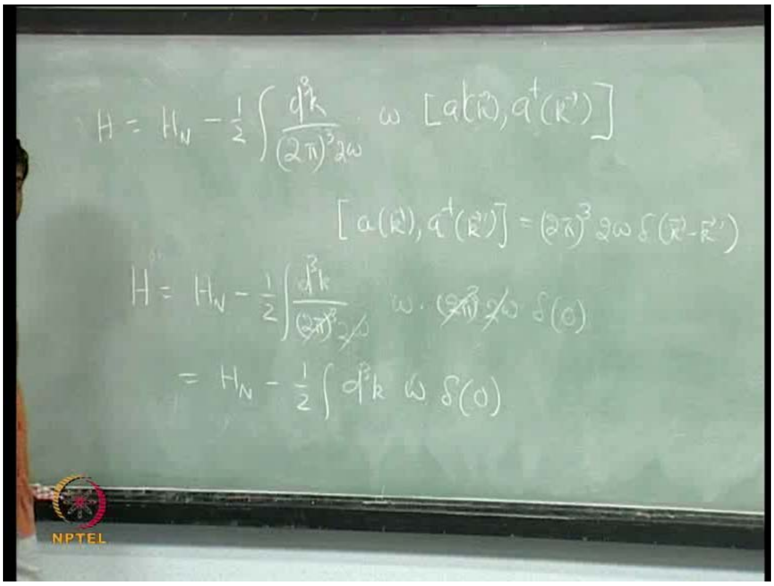
(Refer Slide Time: 24:12)



Similarly, you can write 2 such creation operators, you can start with the ground state and at a digger k 1 a digger k 2 on this; this will create 2 quanta’s of energy or of H N value to omega 1 plus omega 2 and momentum a k 1 plus k 2. So, therefore this I will interpret as a 2 particle state, similarly if you at n number of creation operator on the ground state then you will get n particle states. So, this is the way you can construct the entire spectrum and you know what are its Eigen value? What are its energy and momentum of Eigen values? The other important thing here if you consider the 2 particle states.

Let us say a digger k 1 a digger k 2 acting on this because a digger k 1 commutes with a digger k 2. Therefore, this is also equal to a digger k 2 a digger k 1 acting on the aground state. So, this state k 1 k 2 is identical this states k 2 k 1. So, these particle like states if you interchange to particles you get the same state. Therefore, these are actually bosons they obey both statistic. So, this particle like excitation in Klein garden are actually bosons; they obey both science and statistics. Let us now go back to the full Hamiltonian; I told you earlier that I will come back to this Hamiltonian and discuss the second term, which I have not discussed in any detail earlier.

(Refer Slide Time: 27:17)

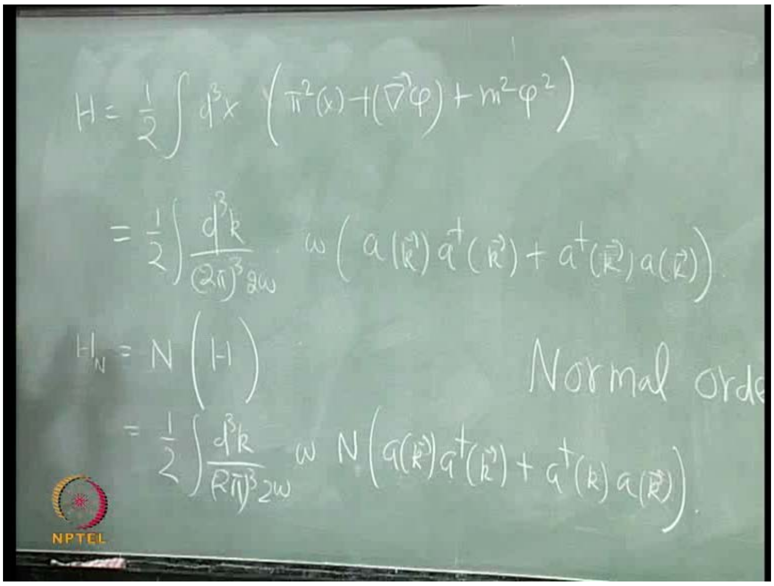


So, the full Hamiltonian is H N minus half d cube k over 2 pi cube 2 omega times omega times commutator of a k a digger k right. So, let us look at the commutator here; we know a k and a digger k prime commutation is given by 2 pi cube 2 omega delta k minus k prime; but here the argument is k in both this operators. So, what you will get in this commutator is a delta 0. So, the Hamiltonian H is H N minus half d cube k over 2 pi cube 2omega times omega 2 pi cube 2omega delta 0; when we consider this term for the momentum operator it vanished.

Because there is k i here which was a odd function of k whereas here you have omega instead of k i this is H N minus half d cube k this will cancel here 2 pi cube 2 omega. So, half d cube k omega times delta 0. So, there is no way this term vanishes; this term is still there and its divergent it gives you infinity. So, this of course does not make any sense; you do not want ah the particle states to have a finite energy. However, what really matters when you make measure is the relative energies not some absolute value energy. But the relates energy between various states. So, when you do that this term of course does not give any contribution.

So, you just forget about the second term. And you consider this to be the Hamiltonian; the other thing a classical you know there is a well defined way a on what you call to be the Hamiltonian. But when you quantize this system; you do not have the fields are not commuting a functions they are operators who are nontrivial commutation and relations. So, you do not know what is the operator ordering to start with; so there is an ambiguity on operator ordering. And you can reserve that ambiguity by saying that you order the operator in such a way that this term is not there.

(Refer Slide Time: 30:57)



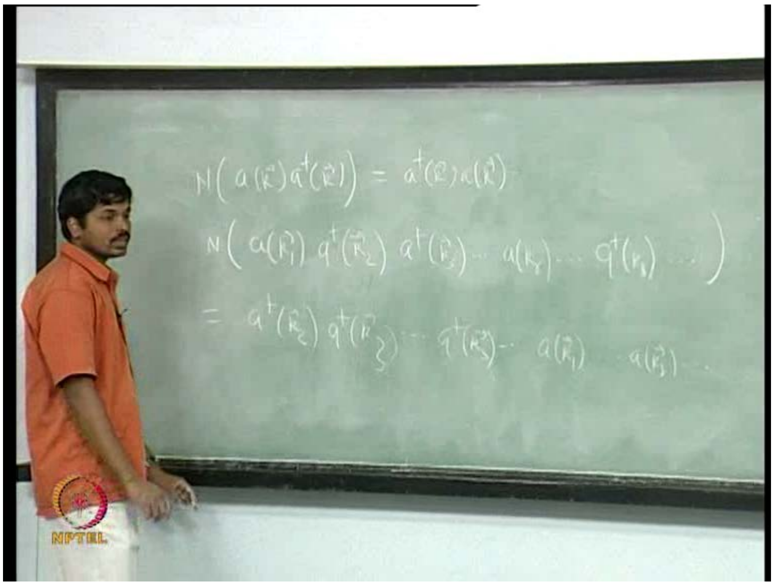
To the more specific what we did is we blindly started with the Hamiltonian which is half d cube x pi square plus grade pi square plus n square pi square; which is perfectly fine. If you study classical field theory but when you quantize this operators and nobody tells you have this operator should be ordered; when you write this operators in term of a case and a digger case; what you get is half d cube k over 2 pi cube omega times omega a k a digger k plus a digger k a k. And when I rewrote this term what a saw is that this is equal to H N minus half integration d cube k omega delta 0; this operator H N admit finite Eigen states; Eigen sates with finite Eigen values. But there is an additional term which gives you infinite answers. But this additional term is there because you started with Hamiltonian which is this. But nobody tells you that the operator should be ordered like that.

So, what you do is you define your Hamiltonian to be the one; where the operator are order in such a way that the creation operators are to the left of the annihilation operators. If you do that then they both the terms are identical and this infinite term is not there. On the other hand this Hamiltonian H N is nothing but the Hamiltonian hold and the Hamiltonian H; where you look at all the terms and you shuffle this operator. And then you put all the creation operator to the right and the creation operators to the left and all the annihilation operators to be right; this process is known as normal ordering.

So, you take this Hamiltonian H and you do normal ordering you will get the Hamiltonian H N that is why I have donated this with subscript N; N stands for normal ordering. And what you get is H N is I will denote as normal ordering of the Hamiltonian H; by normal ordering what I mean is you take d cube k over 2 pi cube 2 omega times omega times normal ordering of a k a digger k plus a digger k a k; this term when you do normal ordering this will give you a digger k a k. Because here the annihilation of operator is the left and the creation operator is towards the right. So, normal order or ordering of these is a digger k a k which is identical to this; therefore you get H N as a result, so N of a k a digger k simply a digger k a k.

So, to state it again; if you start with this Hamiltonian you get a spectrum where the Eigen value of this Hamiltonian are all infinite. But if you subtract one term for from the Hamiltonian; then you get a Hamiltonian which I have a new Hamiltonian which I denoted as H N. And this new Hamiltonian is finite Eigen values the spectrum is such that all the Eigen states a finite Eigen values. So, we will start with this, we will take this to be definition of Hamiltonian here. And this is obtained from the old Hamiltonian by what I call is normal ordering in the Hamiltonian that I obtained normal ordering is known as the normal order Hamiltonian; what do I mean by normal ordering; by normal ordering what I mean is I put all the creation operators towards the left of the annihilation operators.

(Refer Slide Time: 36:37)



So, you take for example N such operators a k 1, a digger k 2, a digger k 3, a k r a digger k s and so on; when you do normal ordering what it does is a digger k 2, a digger k 3 and a digger k s all this things are there; to the left of a k 1, a k r and so on this is what I mean by normal ordering. And when I do normal ordering you get a Hamiltonian which says finite Eigen values. So, we will use this ambiguity by all the physical observable are by normal order whether you consider the Hamiltonian or momentum operator or any other operator; are all depend to be by normal ordering.

Student: What is the normal ordering gets the infinity?

It gets rids of the infinity because the other term is not there; when you do normal ordering you get H N we have already shown you that this H N is s finite Eigen values.

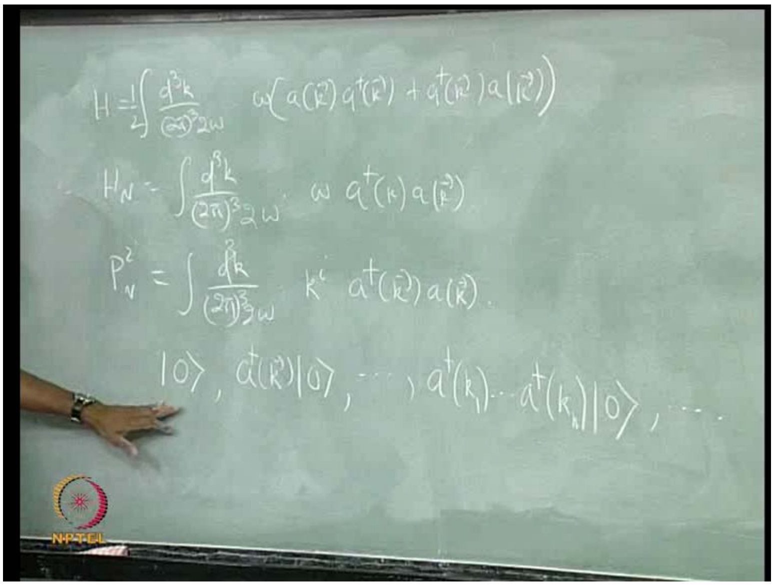
Student: ((Refer Time: 38:19)).

So, the first of all you this is what you defined to be your Hamiltonian. And it what you have seen here I mean in this discussion is that it admits is finite Eigen states with finite Eigen values whether this correspond to a physical situation, whether you consider a physical system. And whether such as is the Hamiltonian explains any physical system or not is a question that we can discuss. So, there are system and all this system you know quantum field theory are explained by such normal ordering. And then you can compute interaction and so on by taking the normal ordering Hamiltonian. And then you can compare whatever you get here with the answer that you do in real experiment.

And, if you find if your answer is agrees with experiment then of course this is correct Hamiltonian that you this is correct question. So, of course it is not so simple; you cannot just through the infinite term; the reason for that in all though in theory which does not contain a gravity adding a constant term to the Hamiltonian really does not have a any consequences, but if you consider a theory of gravity then of course the term that you through from this Hamiltonian H finite energy.

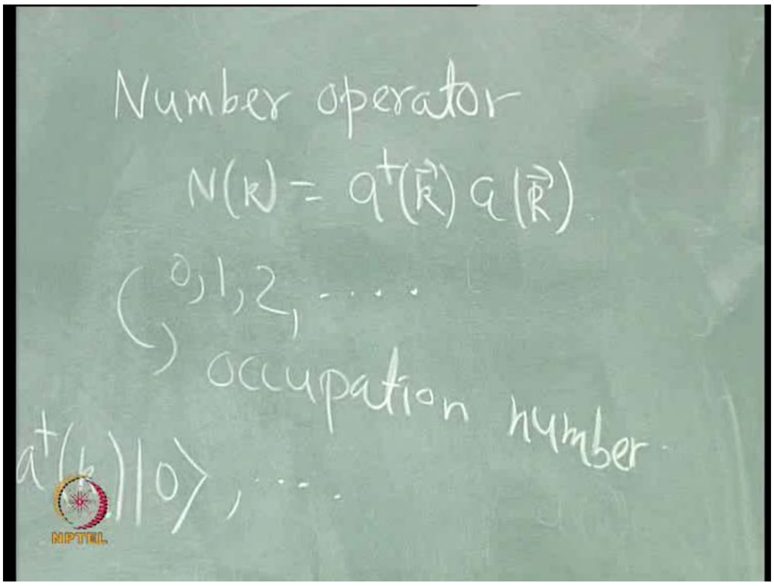
So, the energy of vacuum itself is not 0; it has infinite energy. And anything which has energy any particle can interact with any particle which has energy it can interact gravitationally. So, therefore it is actually not correct to add or subtract constant term Hamiltonian in a theory of gravity. But if you ignore that fact, if you consider a theory where the gravitational interaction is negligible compare to all of the interaction; you do not worry about this is issues. And then you start with Hamiltonian which is normal order. So, just to summaries what we did is we start with Klein Gordon field and we quantize it.

(Refer Slide Time: 40:59)



And, then we have seen that the Hamiltonian can be expressed in terms of the creation and annihilation operators; this is d cube k over 2 pi cube 2 omega times omega a k a digger k plus a digger k a k. And to get finite Eigen value to get normal ordering and Hamiltonian; as a result we got the normal order Hamiltonian to be d cube k over 2 pi cube 2 omega times omega a digger k a k; the momentum operator P i normal order momentum operator is one; where we have d cube k over 2 pi cube 2 omega times k i a digger k a k. And we have constructed this spectrum the spectrum of states are the ground states; which does not have any particle. And hence the ground is the vacuum and then one particle states are obtained by acting a digger k on the vacuum; multi particle sates are obtained by acting of several of this a diggers a digger k 1 up to a digger k n on the ground states; these are multi particle states all these are Eigen sates of H N and P I. I can introduced something which I call as number operator.

(Refer Slide Time: 42:58)



This I will calls n of k as a digger k a k. And the number operator is such that this states with an Eigen values which are 0; this states is Eigen value is 0, this sates will Eigen value 1 and so on. So, all this states in this spectrum will have Eigen value 0, 1, 2 and so on and these are known as the occupation numbers. So, the Eigen states of a number operator is the occupation number. And the Hamiltonian and the momentum operator it can be expressed in term of this number operator is H N is this times of k and P i is this times n of k all right. So, we will close today's discussion here; in the next lecture we will discuss a quality and a time ordering and a propagator and all this thing.