Quantum Field Theory
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Module - 04
Quantum Electrodynamics
Lecture - 31
Moeller Scattering II

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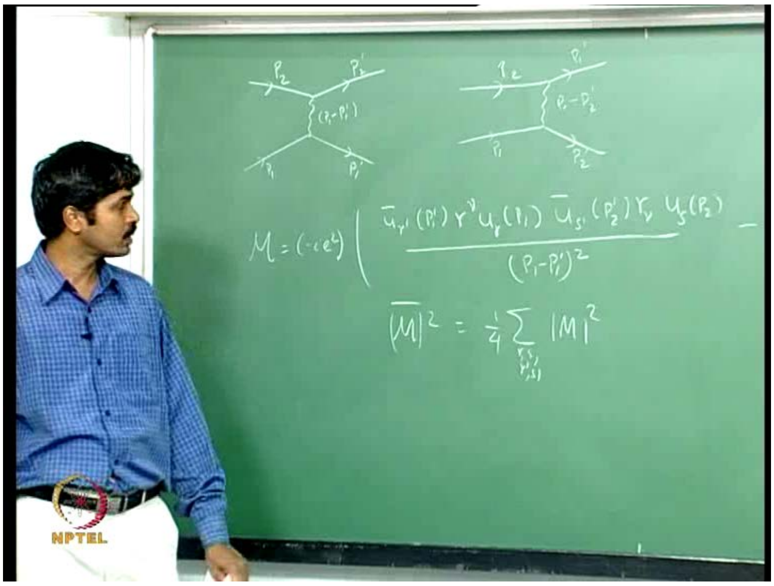

We are discussing the electron electronics scattering and their two diagrams which they contribute to the process. In the order, these two diagrams are given by the following P 2 P 2 prime P 1 going to P 1 prime and there is photon of momentum P 1 minus P 1 prime diagram is P 2 going to P 1 prime P 1 going to P 2 prime and the virtual photon carries a momentum is P 1 minus P 2 prime. We can easily write down the Feynman amplitude for the process and as we have seen in the last lecture, we defined amplitude is minus that is square mu bar r prime P 1 prime gamma mu u r P 1 mu bar prime P 2 prime gamma r mu u s P 2 divided by P 1 minus P 1 prime square minus u bar s prime P 2 gamma rho u r P 1 u bar r prime u, one prime gamma r o u s P 2 divided by P 1 minus s P 2 prime whole square and what comes in the a differential scattering across section is mode m square average which is one fourth of some over r s r prime s prime mode m square.

So, the mode m square contains 4 terms one of them is mode of square of this. The other one is mode square of this and there are cross terms. So, we have evaluated the mode square of this in the last lecture and we will now evaluate one of the cross terms. There are two terms obtained by exchanging P 1 P prime with P 2 prime.

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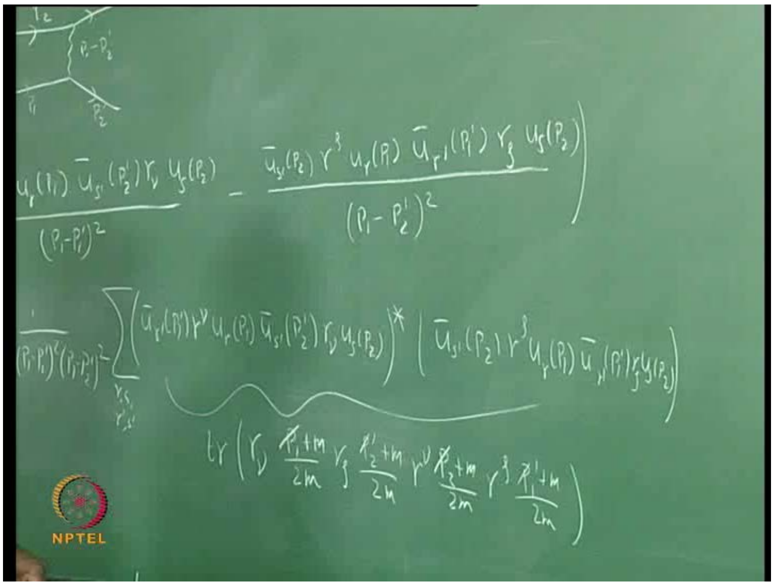

So, the cross term 1 of the cross term here is just given by the following quantity which is a cross term is just a minus one- forth and then 1 over P 1 minus P 1 prime whole square 1 minus P 2 prime whole square then conjugate of this quantity u bar r prime P 1 prime gamma mu u r P 1 u bar s prime P 2 prime gamma mu u s P 2 conjugate of this times the same this quantity which is mu bar s prime P 2 gamma rho u r P 1 u bar r prime P 1 prime gamma r o u bar s P 2, where r s r prime s prime are summed over all possible values. So, again you can write this quantum in terms of the components and then you use the identical u bar u or u bar is this P slash m over 2 m repeatedly. Then, it is a straight forward to see that here we will reduce to a trace and this will be given by trace of gamma mu P 1 slash plus m divided by 2 m gamma rho P 2 prime slash plus m divided by 2 m gamma mu P 2 slash plus m divided by 2 m gamma rho P 1 prime slash plus m divided by 2 m.

So, I leave it as a home work for you to evaluate. So, that this end is equal to trace of this line notice that this contains 1 2 3 4 there are 8 gamma matrices. So, you need to evaluate these tricks; however, there will be a considerable simplification, if you note that this gamma nu here is contracted with this gamma nu. This index nu contracted with this index nu and this can make a lot of simplification.

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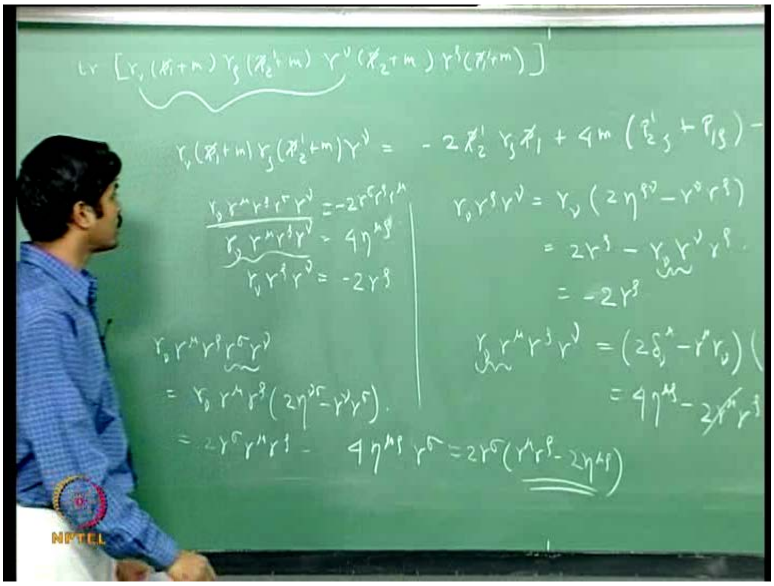

So, our goal is to evaluate this trace or what we need to do is we need to evaluate trace of gamma nu P 1 slash plus m gamma rho P 2 primes plus m gamma nu P 2 slash plus.

M gamma rho P 1 prime slash plus M1, this is what we need to evaluate and we will use various identities involving the gamma matrices to evaluate this trace. So, far this note that, first of all we can consider this, what I will do is that I will show gamma nu P 1 slash plus m gamma rho P 2 prime slash plus m gamma nu this quantity is equal to minus 2 P 2 prime slash gamma rho P 1 slash plus 4 m P 2 prime rho plus P 1 rho minus twice m square gamma rho. So, I will show this quantity is equal to this and then I will substitute here and then it will be quite simpler to evaluate this trace. So, let us first consider this and then prove this identity to prove this. We will note several identities.

This involves terms it is like gamma nu gamma mu gamma rho gamma sigma gamma nu. This also involves gamma nu gamma mu gamma rho gamma nu and also term like gamma nu gamma rho gamma mu. The co- efficient of M square is this and the term is this linear in M contains something like this where, the other remaining term contain some expression like this. So, all this things we can simplify directly. So, let us consider the last and first gamma nu gamma rho gamma nu is nothing, but gamma nu times. I will use the identity gamma rho gamma mu is twice eta rho nu minus gamma mu gamma rho.

Now, the first quantity is simply twice gamma rho and minus gamma nu gamma nu gamma rho. What is gamma nu gamma mu?

So, this is 4 2 minus 4 which is minus 2 gamma rho. So, this quantity here is nothing, but minus 2 gamma rho, now we consider this one. So, what is this is gamma nu gamma mu gamma rho gamma nu now what I will do that I will click use the gamma matrices here and as well as here. So, the first two gamma matrices can be written is twice delta nu mu minus gamma mu gamma nu and the second last two terms are twice eta rho nu minus gamma nu gamma rho. Now, I will expand this. The first term is simple 4 times delta mu nu dealt rho nu. It arrow nu which is eta mu rho then this term here will give minus 2 gamma mu gamma rho and is multiplied again. This gives me nu mu minus twice prime minus gamma mu gamma rho. Finally the last term is gamma mu gamma nu gamma mu gamma rho this is 4. So, this will cancel with these two terms. So, what is left is 4 eta mu rho. So, this quantity here is nothing, but 4 eta mu rho now we can easily evaluate the this the first term here. So, what is the first term you consider gamma nu gamma mu gamma rho gamma sigma gamma nu now you fit the last two gamma matrices that will give you gamma nu gamma mu gamma rho times two twice eta nu sigma minus gamma nu gamma sigma all right now what is the first term this eat nu sigma can be constructed with this give twice gamma sigma gamma mu gamma rho the first term and the second term is minus gamma nu gamma mu gamma rho gamma nu gamma sigma now what you can do you can use this identity here they will give me this quantity is simply 4 times eta mu rho this is mu rho right here. So, minus 4 eta mu rho gamma sigma. So, twice gamma sigma if you take gamma two gamma sigma then what is left is gamma mu gamma rho minus twice eta mu rho what is this quantity minus gamma rho gamma mu.

So, what you get here is minus 2 gamma sigma gamma rho gamma mu. So, this order here is click and you get a minus 2 factor. So, so we will use this identity repeatedly and when.

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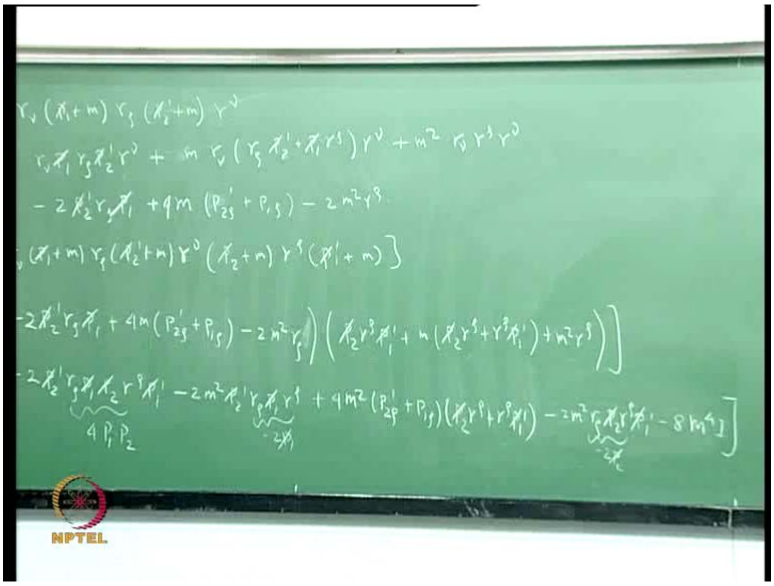

So, so let us consider the left hand side here gamma rho gamma nu P 1 slash plus m gamma nu slash half plus m gamma mu. I will expand this, so this will give me gamma nu P1 slash gamma rho P 2 prime slash gamma mu that is the first term the second term plus gamma plus m times gamma nu gamma rho P 2 prime plus slash plus P 1 slash gamma rho gamma mu and finally, plus m square gamma nu gamma rho gamma mu. So, we know this identity. So, first identity will make this very simple.

So, this will simply tell me that this is nothing, but minus 2 P 2 prime slash gamma rho P 1 slash and this term here will tell me that this is m times gamma 4 m P 2 rho prime plus P 1 nu and this 1 minus 2 m square gamma rho, this is obvious. Now, if I just expand the proof obvious once I used these three identities. So, now what I will do is that, I will insert this here and expand all this terms to evaluate this term. So, what we need to do is we need to look in to trace of gamma mu P 1 slash plus m gamma rho P 2 prime slash plus m gamma nu and then P 2 slash plus m gamma rho P 1 prime slash plus m.

What I will do is that for this, I will substitute this one and then here I will expand this and a term that will give me trace of minus 2 P 2 prime slash gamma rho P 1 slash plus 4 m P 2 rho prime plus P 1 rho minus twice m square gamma rho times P 2 slash gamma rho P 1 prime slash and m P 2 slash gamma rho plus gamma rho P 1 prime slash and then there is m square gamma rho, this is what I will get by straight forward multiplication. Now, what I will do is that I will multiply this term with this term and then I will keep in mind that trace of odd number of gamma matrices vanishes. So, if I do that what I will get this one. So, I will do that and whenever we have odd number of gamma matrices we will new that out. So, so what I get here is trace of minus 2 p 2 prime slash gamma rho slash p 2 slash gamma rho p 1 prime slash then minus. So, this contains three gamma matrices, this contains two gamma matrices. So, this vanishes finally, minus 2 m square P 2 prime slash gamma rho P 1 slash gamma rho and then I will consider this prime multiply by this when I take this trace will be 0. There are three gamma matrices. There are no gamma matrices here, where this will survive. What I have is 4 m square P 2 P prime plus P 1 rho then P 2 slash gamma rho plus gamma rho P 1 prime slash, then this multiplied this will get me 0.

Now, this will survive. So, that you will give the minus twice m square gamma rho P 2 slash gamma rho P 1 prime slash this will give me 0 and finally this multiplied by this will give me minus twice m forth gamma rho gamma rho, but gamma rho gamma rho is 4. So, this will give me minus 8 m 4 times the identity matrix, this is what I got now. Again, I will repeatedly use these identities in the first one. So, the last two identities are useful now. Look at this term is gamma rho P 1 slash P 2 slash gamma rho, what will this be 4 P under P 2. So, 4 P under P 2 and here this one is minus 2 gamma rho.

So, this is minus 2 P 1 slash minus 2 P 1 slash and here of course, this you have to evaluate and here minus 2 P 2 slash. Right! So, now I will use when we substitute that I will take a simple form.

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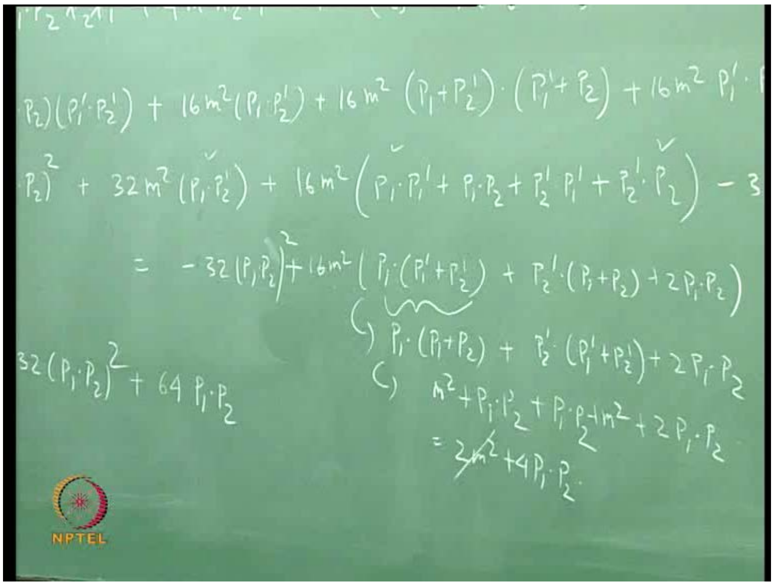

So, what I get is trace of minus 8 P 1 0 P 2 P 2 prime slash P 1 prime slash plus 4 m square P 2 prime slash P 1 slash plus 4 m square P 2 prime rho plus P 1 rho P 2 slash gamma rho plus gamma rho P 1 slash, then plus 4 m square P slash plus P 1 prime slash minus 8 m 4 times the identity. This is the trace that we need to evaluate.

But, now it is very easy to evaluate the trace first term. What is trace of this? So, this is 4 P 1 prime 0 P 2. So, this is minus 32 P 1 0 P 2 P 1 prime dot P 2 prime. This 16 m square P 1 0 P 2 prime and here you can see that this will give me 4. So, this is 16 m square P 1 plus P 2 prime 0 P 1 prime plus P 2 and here this is 16 m square P 1 prime 0 P 2 minus 32 m forth is right . So, now what we will use the annihilation momentum of conservation relation repeatedly that is P 1 plus P 2 is P 1 prime plus P 2 prime that will tell me that P 1 0 P 2 prime is P 1 0 P 2. So, this is minus 32 P 1 0 P 2 whole square and here I will leave it as it is. I can write this and this because P 1 dot P to prime is P 1 dot P 2 prime.

So, this is 32 m square P 1 0 P 2 prime and then 16 m square. I can write it as P 1 0 P prime plus P 1 0 P 2 plus P 2 prime 0 P 1 prime plus P 2 prime 0 plus P 2 and minus 32 m forth. Now, look at this you can consider one of the terms here and then you can combine it with P 1 0 P 2 prime. There is P 1 dot P 1 prime. So, that this will give me half of this with this term will give me 16 m square P 1 0 P 1 prime plus P 2 prime and then again the remaining term here P 1 0 P 2 prime. I will combine it with this one last term.

Then, what I will get is P 2 prime 0 P 1 plus P 2, now we can see why I wanted to combine a trace function because with us here I can use the annihilation momentum of conservation relation and then I can write it as P 1 plus P 2, similarly here I can write as P 1 prime plus P 2 prime. So, this will take care of the spectrum of this term and these two terms. What is left to this two here. So, they are equal. So, this is twice P 1 and P 2 that is what I get when I get this term. So, this is equal to minus 32 P 1 dot P 2 square plus this minus 32 m 4, that is what I get now. As I said I will use the annihilation momentum of conservation term here is P 1 0 P 1 plus P 2 and here this term is P 2 prime 0 P 1 prime plus P 2 prime and here two P 1 0 P 2.

Now, what is this is P 1 square which is m square plus P 1 0 P 2, here I will get P 2 prime 0 P 1 prime which is again P 1 0 P 2 and P 2 prime square which is m square plus

2 P 1 0 P 2. So, this simply twice m square plus 4 P 1 0 P 2, now twice m square multiply with by 16 m square. So, that is that will give me plus 32 m forth that will cancel with minus 32 m forth and so the end of the day what I get is minus 32 P 1 0 P 2 square and then plus 16 in to 4 which is 64 P 1 0 P 2. There is nothing else left. So, this is what I get. So, what I did is I evaluated this place here and at the end of the day this is simply equal to minus 32 into P 1 0 P 2 square minus twice P 1 0 P 2, this is what I get for this tricks.

So, there is m square here 2 m square this is 64 P 1 0 P 2 m square. So, what we will do is that will substitute all these things in the expression for m square average remember

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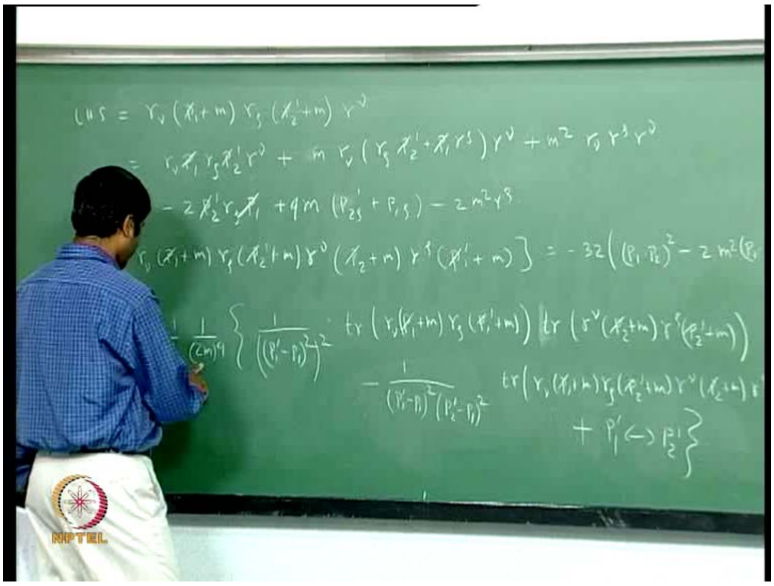

When we computed m square leverage there was a factor of 1 over 4 squares because this average over spins of incoming electron and then P slash plus 2 m over to m there are 4 things. So, that will give me 1 over m to the power forth and finally, we have this trace. So, one of the diagram at the photon propagator which is P 1 prime minus P 1 square in the Feynman amplitude.

We take mode m square, its square of this and I have the trace which is trace of gamma nu P 1 slash plus m gamma rho P 1 slash plus m times trace of gamma nu P slash plus m gamma rho P 2 prime slash plus m. This is the first term, the second term is evaluated today is minus 1 over P 1 prime minus P 1 square for the photon propagator from the first diagram P 2 prime minus P 1 square for the photon propagator in the second diagram. Then this place here trace of gamma nu P 1 slash plus m gamma rho P 2 prime slash plus m gamma nu P 2 slash plus m gamma rho P 1 prime slash plus m this are the first two terms and then the remaining two terms are obtained from these two by exchanging P 1 primer with P 2 prime.

So, this is the expression for mode m square average. What I will do is that will substitute here for the value of this and this was evaluated in the last lecture and this we have seen is equal to minus 32 P 1 0 P 2 square minus P m square P 1 dot P 2. So, we will directly substitute this here, if we write this quantity here is nothing, but 32 times some factor. So, there is an overall factor of 32 here which will cancel 8 into 4 here. So, what will be left is 1 over 2 m forth. So, two to the power 3 and 4 that will cancel 32 and that will be two left m forth.

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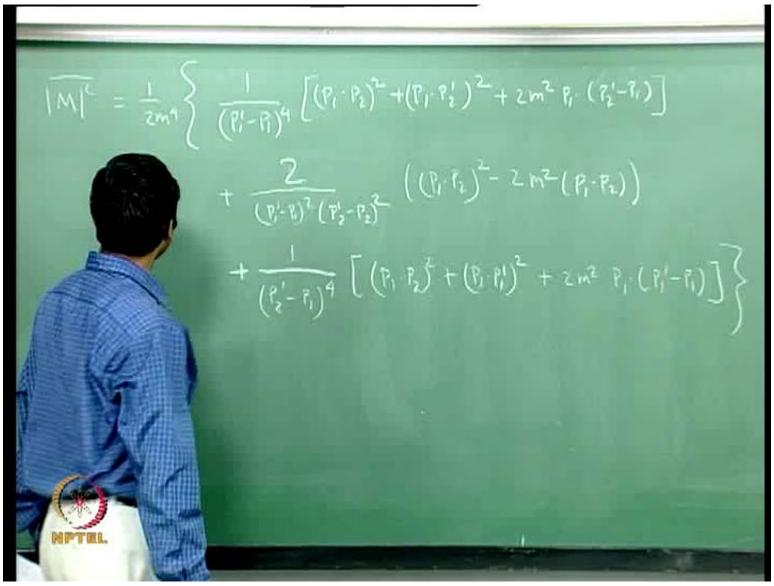

So, that will give me mode m square bar is a 21 over twice m plus, then this one over P 1 prime minus P 1 whole forth times, if you substitute 1 over 32 times, then what we get is P 1 0 P 2 square plus P 1 0 P 2 prime square plus 2 m square P 1 0 P 2 prime minus 0 P 1 and this term here will give me plus as you can see there is a minus which will give me plus this 32. We have cancelled it here with a 4 into 2 to the power 3. So, this is plus this times this.

So, plus 1 over P 1 prime minus P 1 square P 2 prime minus P 2 square times P 1 0 P 2 whole square minus 2 m square P 1 0 P 2, these are the first two terms and then remaining two terms will be obtained from these two by expanding P 1 prime and P 2 prime. So, from the first term I get 1 over P 2 prime minus P 1 to the power 4 times P 1 0 P 2 square plus P 1 0 P 1 prime square and plus 2 m square P 1 0 P 1 prime minus P 1 that is from here and you can see that this term is in variant under exchange of P 1 prime and P 2 prime. So, I will just get a factor of 2.

So, you have got this now. We will use the fact that we are doing the calculations in the centre of mass frame. So, in the center of mass frame we will evaluate, what is the value of P 1 0 P 2 P 1 0 P 1 prime and so on and finally we will substitute those and at the end we will substitute this in the expression for the differential scattering cross section.

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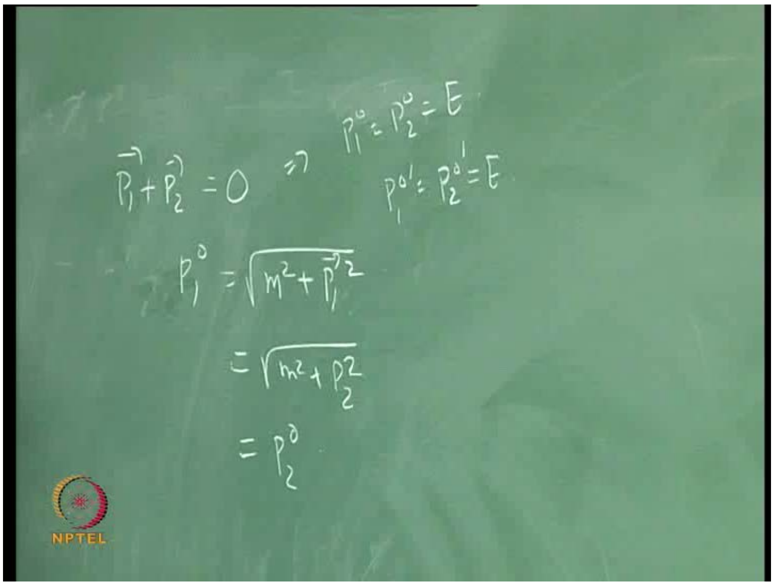

So, let us go to the centre of mass frame you have the incoming electron with momentum P 1 and momentum P 2 and then scatter and then out going electron with momentum of P 1 prime P 2 prime. What are the energies of these electrons? Let us consider this P 1 plus P 2 in the centre of mass frame this equal to 0, now if you consider P 1 0 which is a just s a square root of m square minus P 1 square. Right now because of this mode P 1 is equal to mode P 2.

So, which is m square minus P 2 square which gives you P 2 plus P 1 square? P 1 square m square equal to p. So, this is nothing but P 2 0. So, this simply says that P 1 0 is equal to P 2 0, which I will denote as e the total energy is conserved and you can use the same argument which will say that P 1 0 prime is equal to P 2 0 prime therefore, this is also equal to e. So, this is what the conclusion that we got now.

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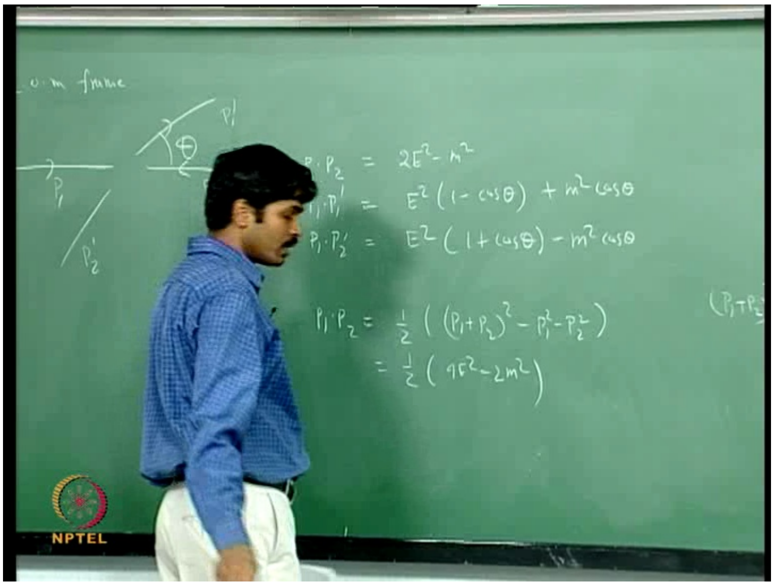

So, we have what you need to do is we need to value P 1 0 P 2 P 1 0 P 1 prime P 1 0 P 2 prime. What I claim is that this is equal to twice E square minus m square this equal to E square. 1 minus cos theta is the single plus m square cos theta, then this equal to E square 1 plus cos theta minus m square cos theta. Lets first look at this relation here, P 1 0 P 2 is nothing, but P 1 plus P 2 square minus P 1 square minus P 2 square, but P 1 plus P 2 square P 1 0 plus P 2 0 square minus P 1 0 P 1 plus P 2 0 P 1 plus P 2, but P 1 plus P 2 is 0 in the centre of mass frame and this is twice E.

So, therefore, I got this to be half 4 E square minus 2 m square this is what is given here. Now, I will evaluate P 1 0 P 1 prime and then you can evaluate P 1 0 P 2 prime in a similar way.

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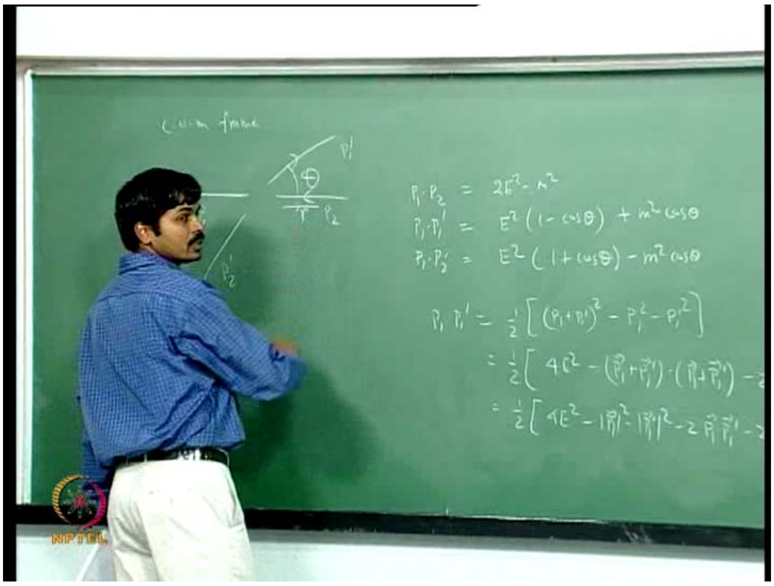

So, P 1 0 P 1 prime is again half P 1 plus P 1 prime square minus P 1 square minus P 1 prime square right half. So, the 0 component will give me 4 E squares minus P 1 plus P 1 prime 0 P 1 plus P 1 prime. This is a minus twice m square then what you get from here half 4 E square minus mode P 1 square minus mode P 1 prime square minus twice P 1 0 P 1 prime minus twice m square. What is P 1 square mode P 1 square e square minus m square and mode P 1 square is also E square minus m square? So, this will give me half 4 E square minus twice E square minus m square and what is this P 1 0 P 1 prime minus cos theta, this is the direction of P 1, right is cos theta.

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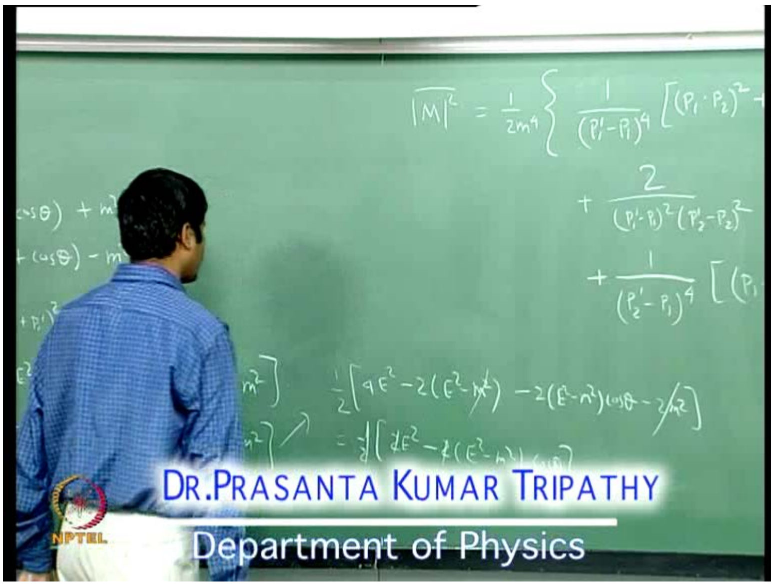

So, twice e square minus m square cos theta and then minus to m square all right. So, as you can see this m square here it will be cancel with this m square and what you have is half prime this will give me twice E square minus twice E square minus m square cos theta this factor of two will cancel here.

So, what you are left is E square into minus cos theta plus m square cos theta. So, you can similarly evaluate P 1 0 P 2 prime and then you can show that this equal to this finally at the end you substitute all this results here because this involves all this E square which are m square and then P 1 0 P 1 prime or P 1 0 P 2 prime nothing else. So, you substitute this expression for P 1 dot P 2 P 1 0 P 1 prime P 1 0 P 2 prime in mode m square and then you simplify it very straight forward you will see that the differential scattering cross section at the end of the day.

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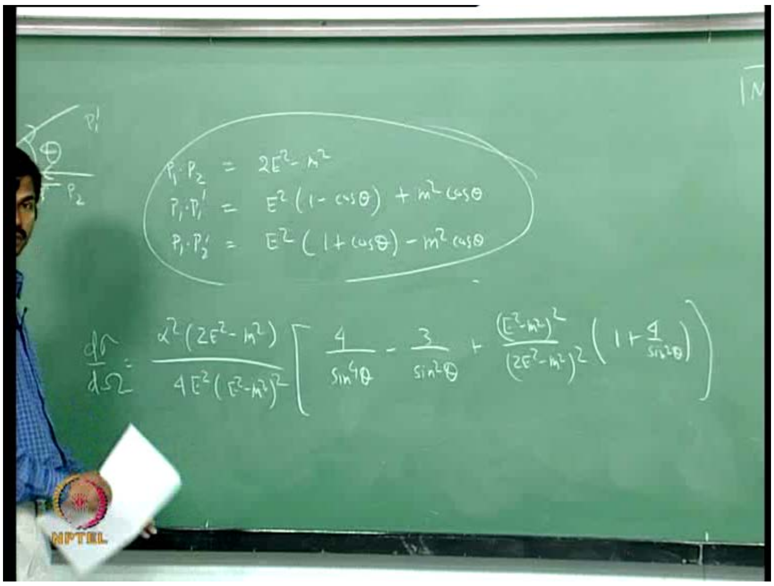

Will look like the sigma over d omega is equal to alpha square times twice E square minus m square. Now, we can see the origin of all these terms 4 E square E square minus m square whole square times 4 over sin cos theta minus three over sin square theta plus e square minus m square whole square m square whole square 1 plus 4 over sin square theta. So, this is the expression for the differential scattering cross section for electron scattering.