Quantum Field Theory
Prof. Dr. Prasanta Kumar Tripathy
Department of Physics
Indian Institute of Technology, Madras

Module - l
Free Filed Quantization - Scalar Field
Lecture - 3
Quantization of Real Scalar Field – I

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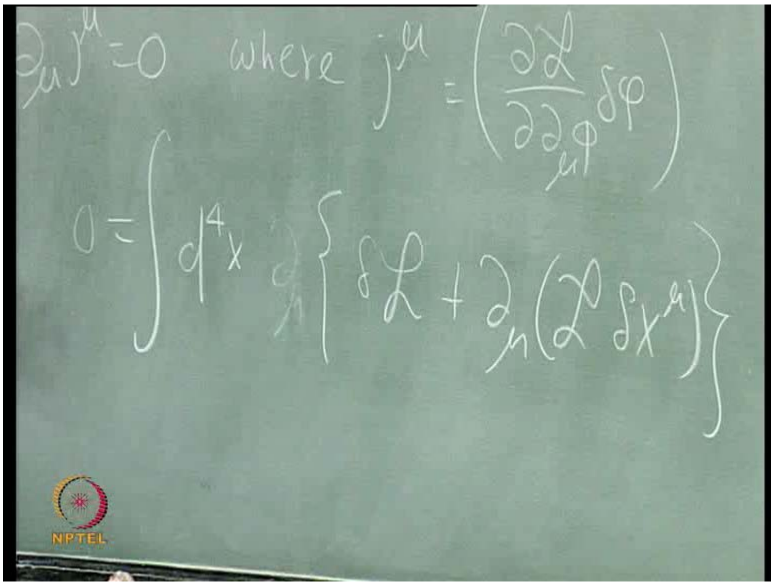

In the last lecture we were discussing in Noether theorem. It basically states that whenever we have the continuous symmetry under is the theory this is variant you have corresponding concern quantity, and as an example that it is theory is in variant under a local symmetry. So, 5 goes to 5 plus delta 5 then you see that delta L is equal to 0. And this implies Del mu j mu is equal to 0 were j mu is a Del L over del, del mu phi, delta phi right; this is what we have seen in the last lecture. If you have in an addition to this if you consider more general symmetry transformation were the field 5 of (x) goes to phi prime of the x prime.

And, the coordinate x goes to x prime which is x delta x; then you have to be bit more careful what you need to consider in this case is the in variants of action. And the invariants of the action under such a transformation basically implies that this quantities is equal to 0; delta L plus Del mu L del x mu. To see why the in variants of the action leads to the condition like this let us consider this one-dimensional case ok.

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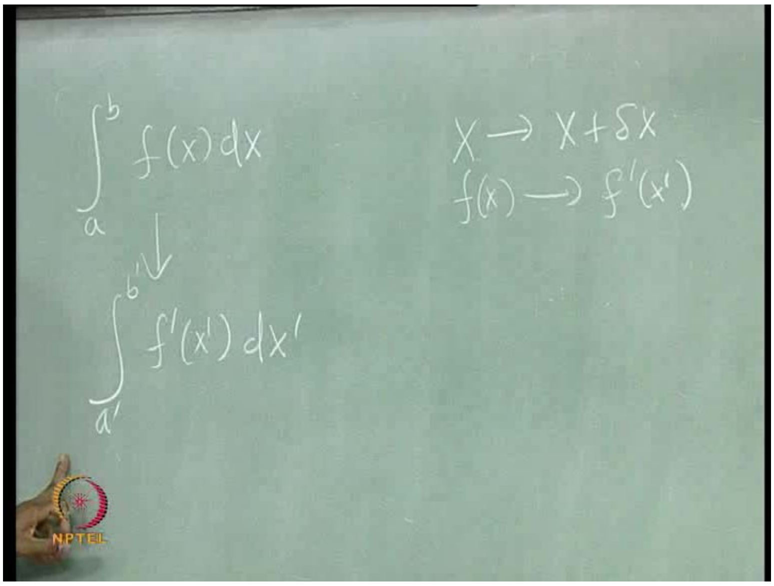

So, consider this integral in one-dimension from the interval a to b f of (x) d x; let us make a transformation x goes to x plus delta x. And let us assume that under such transformation the function f of (x) goes to x prime of (f prime) which is again f of (x) delta x so on. So, under such a transformation this integral will transfer to f prime of (x prime) d x prime. And the interval now acutely the integrate limits are now from a prime to b prime; if this quantity is invariant under such transformation then this difference is equal to 0.

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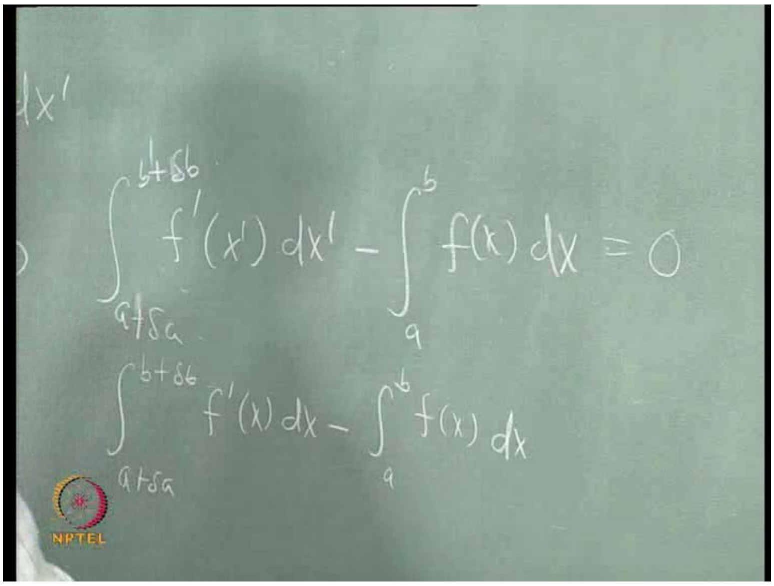

So, the invariants implies a prime to b prime f prime of (x prime) d x prime minus integration a to b f of (x) d x is equal to 0. Now, since we are considering infinity transformation here this a prime will differ from a by m infinity small amount. So, is the quantity b prime; so this integration is basically a plus delta a. And here again the integration limits goes from a plus delta a, to b plus delta b. Now, you can do even better in this integration this explains is just a dummy variable you can say this to x. So, what you get is a plus delta a, b plus delta b; f prime of (x) d x minus a to b f of (x) d x.

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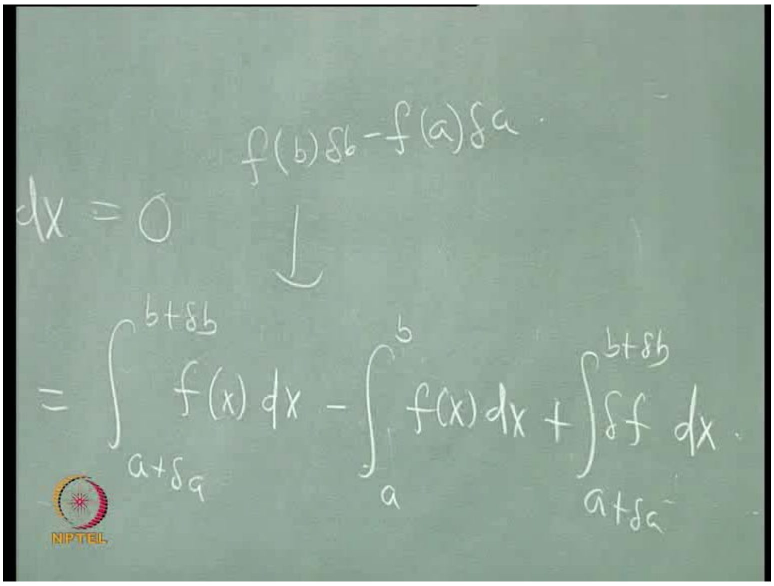

So, f prime of (x) is nothing but f of (x) plus delta L. So, you will substitute here then this is equal to a plus delta a, b plus delta b right. Now, what is this quantity, this difference here; you can see that this difference is nothing but f of (b) delta b minus f of (a) delta a.

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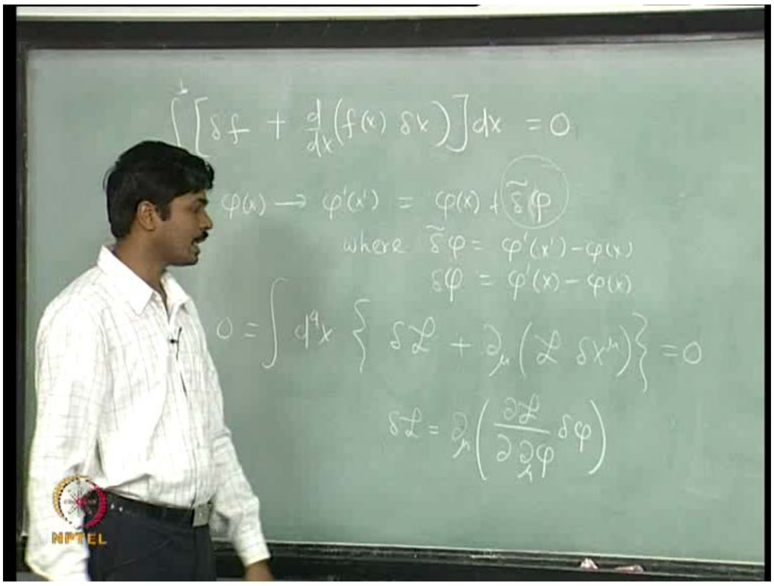

So, what you get is if you require this quantity to the invariant under such a transformation. Then you see that integration delta f plus d over d x, f of (x) delta x integrated from a to b is equal to 0; if you consider quantity up to first order in variation this is quantity which is equal to 0. So, you can do similar analyses in 4-dimension. And when the invariants of the action under this symmetry transformation 5 of (x) goes to pi prime of (x prime) which is equal to phi of (x) plus delta phi were delta phi is phi prime of (x prime) minus phi of (x); whereas, delta phi is equal to phi prime of (x) minus prime of (x) is given by this d 4 x delta L plus delta mu L, delta x mu is equal to 0.

Delta L is what we have derived in the last lecture. And what we saw is delta L is equal to Del mu of Del L over L del mu phi delta phi. However, we want to express this quantity inside the squarely bracket, in terms of the quantity transformed quantity terms of the delta phi and delta x. So, let us express everything in terms of delta tilde phi.

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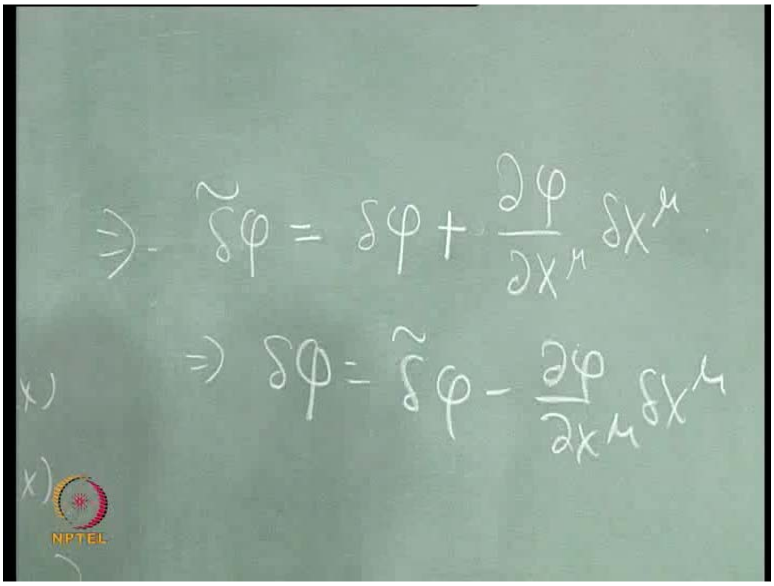

We have already derive the line delta phi is equal to delta pi minus Del phi over Del mu, delta x mu. So, we will substitute and delta plus; we will substitute first delta phi here delta phi is delta tilde phi minus Del x mu, delta x mu you will substitute here. And we will put it in this expression then we will get a conservation law.

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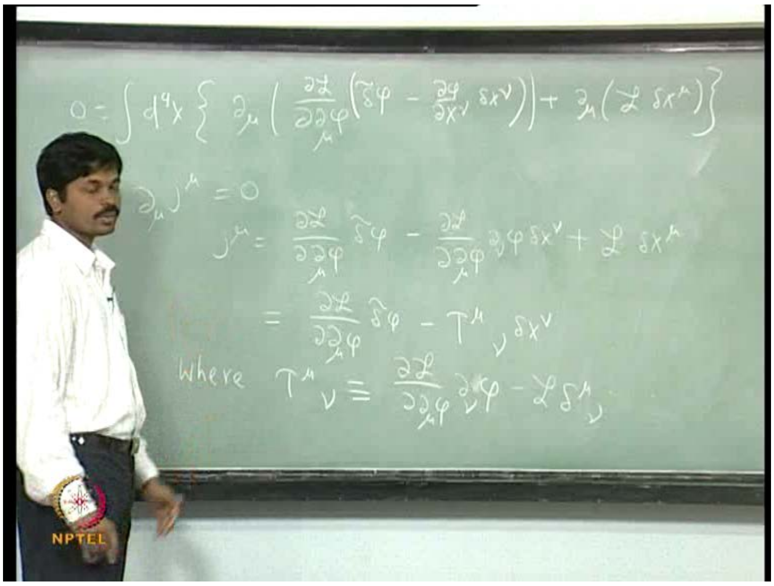

So, let us do that when you do that we get d for x del mu of del L over del, del mu phi, delta tilde phi minus del phi over del x mu, delta x mu plus del mu L delta x mu. Now, since this variations are arbitrary variation the integrants equal to 0. And hence what we get is we get again the conservation law which is del mu z mu equal to 0; were the conserve quantity is mu given by del L over del, del mu 5 delta tilde phi minus del L over L del mu phi delta x mu plus L delta x mu this is the most general conserve current for the transformation under consideration.

We can rewrite this quantity as follows this is this can be rewritten as del L over del, del mu phi delta tilde phi minus T mu, mu delta x mu were the quantity T mu, mu is define to be del L over del L del mu phi, del mu phi minus L delta mu; we will see what is the physical interpretation of this in a movement for that let us consider the transformation for this delta tilde phi equal to 0. And delta x mu is simply some transformation a mu. So, we will consider the symmetry transformation and translation invariants; if the theory is translation invariants.

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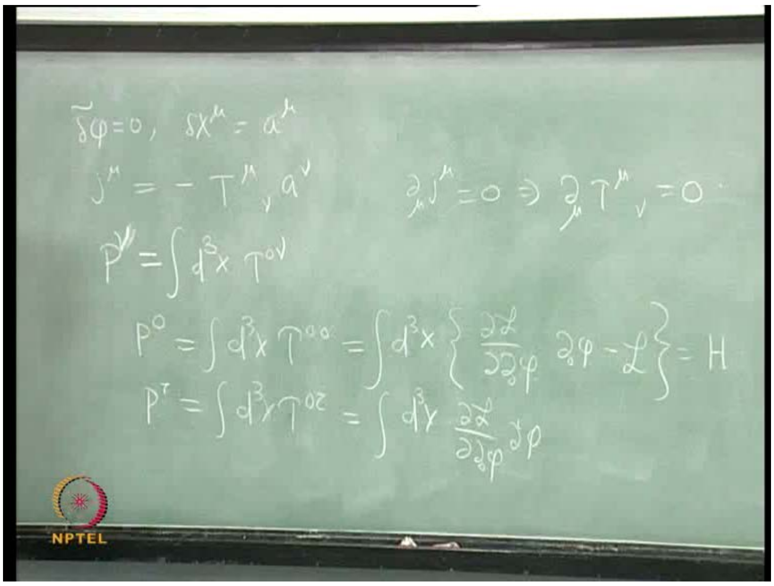

If you have translation invariants then we have delta tilde phi equal to 0. And delta x mu it is some infinity parameter which I have called a mu for this transformation our concerned current a mu is equal to minus T mu nu a mu or the conservation law is simply del mu z mu is equal to 0; simply implies del mu, mu nu equal to 0 this is what we get. So, the conserve instead of getting single conserve star. Because you have free index mu we get set of 4 the conserve stars here. The conserve stars are d cube x T 0 mu this I will define to the d mu.

Let us see P 0 the component of the conserve stars P 0 is basically d cube x T 0, 0 what is t 0, 0? If you look at the expression for T mu nu then this is nothing but d cube x del L over del 0 phi minus L what is this quantity? This is the Hamiltonian density and this is integrated over the entire space. So, P 0 is the Hamiltonian H and P i; the Pi the components are basically d cube x p 0 i. So, this is nothing but d cube x Del L over Del 0 phi, Del i phi. So, this is the momentum contained this is total momentum contained in the field configuration and this is the total energy.

So, this is the reasons this quantity T mu nu is known as the energy momentum denser. And what translation invariants imply is the conservation of the energy momentum denser. So, the conservation of energy momentum denser is actually is the consequence of translation invariance.

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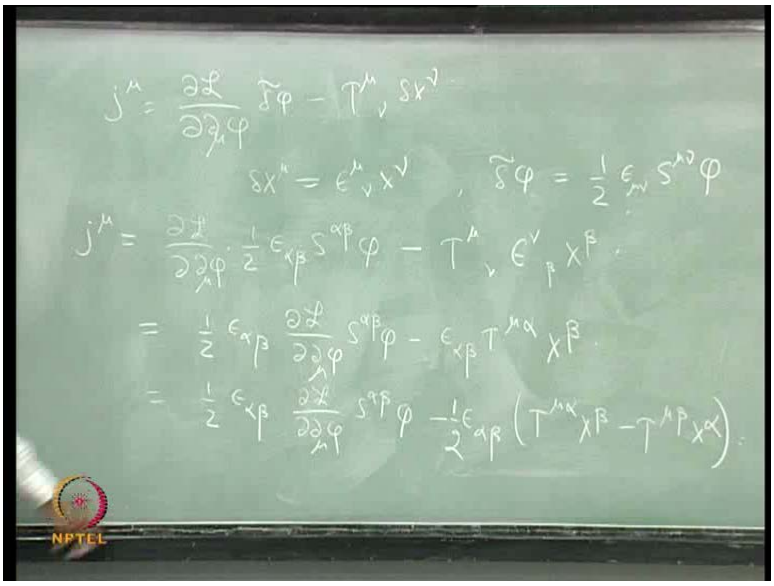

Now, we can consider Laurence transformation in variants under transformation. So, let us again consider the conserve current a mu which is delta L over de L del mu delta L phi minus T mu, mu delta x mu; when you make the Laurence transformation delta x mu basically become epsilon mu nu x mu correct. And the fields usually nontrivially under Laurence transformation depending upon what is what kind of field we have considering we have some transformation. So, the field transformation delta tilde phi I will write it is half epsilon mu nu S mu nu phi; in general this phi will also carry some index. If it is the vector it will carry some index mu, if it is spinner it is carry spinner indices, if it is a carries different index and so on.

And, accordingly we have some transformation law which I will, I am writing in a compact way like this. So, this is how the field transforms and this is the transformation of the coordinate; we will substitute this transformation here. And then what we get is j mu is equal to Del L over Del mu phi half epsilon alpha beta S alpha beta phi minus T mu, nu epsilon mu beta x beta I can raise this index nu here and L over the index mu in this place.

And, then this is the dummy variable I can since do that mu alpha when I do that what I get is half epsilon alpha beta Del L over del, del mu phi if alpha beta phi; and here minus epsilon alpha beta T of mu alpha x beta. The second term here I can rewrite this is half epsilon alpha beta Del L over Del mu phi S alpha beta phi minus epsilon alpha beta half d mu alpha x beta minus T mu beta x alpha.

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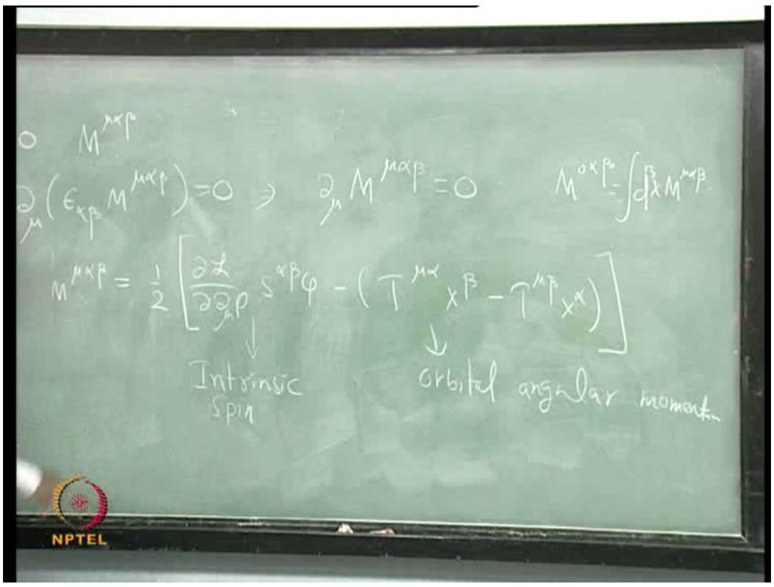

Now, since epsilon alpha beta are obituary infinity quantities. So, what you get is actually a conserve quantity which I can denote this I can denote is M mu alpha beta. And Del mu j mu is equal to 0; implies del mu of epsilon alpha beta, M mu alpha beta equal to 0 since epsilon alpha beta are obituary infinity quantities. So, this simply implies Del mu M mu alpha beta equal to 0. So, what we get is actually a set of 6 conserve quantities which I denoted by M 0 alpha beta d to x M mu alpha beta these are the quantity which are conserve in a consequence of Florence in variants; what is M mu alpha beta?

M mu alpha beta is given by half del L over del, del mu pi S alpha beta minus T mu alpha x beta minus d beta, x alpha. Now, what is the physical interpretation for this quantity? If you look at M 0 alpha beta, look at the second term of M 0 alpha beta, the second term basically T 0 alpha x beta minus T 0 beta x alpha. If you consider i z i z components of M 0 alpha beta then you get T 0 i, x g minus T 0 g x i which is basically the i z component of the orbital angular momentum. So, the second term here is the orbital angular momentum. And hence this first term here must be the intrinsic angular momentum of the field.

So, this is the intrinsic spin and this is the orbital angular momentum. So, as the consequence of the Laurence invariants the sum of the angular momentum and the angular momentum remains conserved. And you can see the spin actually spin of a field is determined by how is the field transform under Laurence transformation. If the field does not transform under Laurence transformation; if you consider a field scalar under Laurence transformation then; obviously, the fast term here it is 0. And hence it is no intrinsic field that is why this scalar quantity is the quantity which is scalar under Laurence transformation has been 0.

And, whether you have spin 1 or spin half etcetera are determined by how the field transforms under Laurence transformation; this is the term which basically defines what the spin of the field age. And the some of the spin orbital part is concerned. So, what do we have seen, what we have discussed so far is discuss classical field theory; we derived Lagrange equation. And then we consider theory which are invariant under continuous symmetry and there consequences.

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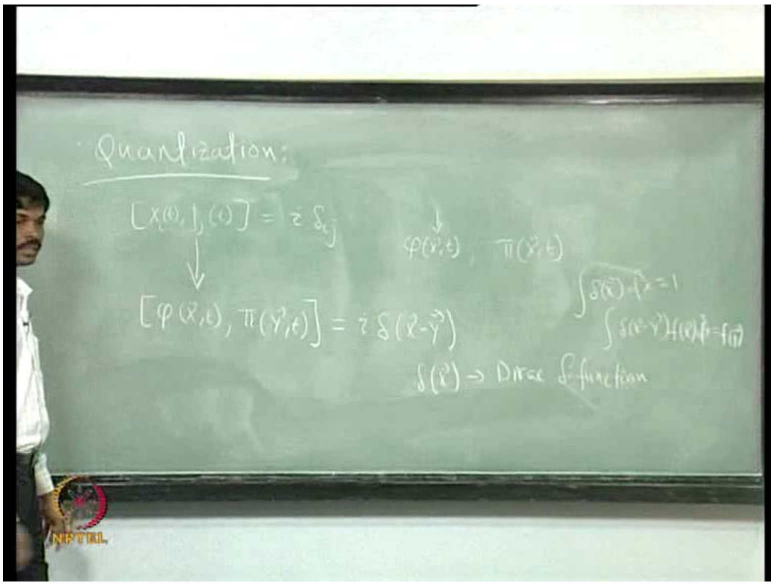

Now, we will discuss how to quantize a given classical theory. So, how we carry out quantization of a classical field; we will consider the usual non relativistic quantum mechanics you have x of (t) and x i and P (t) these are operator in the haze burgee picture. And then all of us know the fundamental relation that the commutations of x i and t j is i delta j; we will take this fundamental relation in non relativistic quantum mechanics. And then we will discuss the generalize 2 quantum.

And, obvious guess to generalize is that this x the analyses of this x is field phi of (x, t). And the conjugate momentum is the conjugate variable phi of(x, t) only thing is that there are descript level i and j. But here you have continues parameter in this field phi of (x) a set of three continuous parameters. So, what we do is we start with the competition relation which looks like phi of (x, t), phi of (y, t) since x and y are continuous variables. So, instead of conical delta you have the dirac delta which is given by i delta of (x) minus y were delta of (x) minus y is the dirac delta function; 3-dimensional dirac function which as the property that integration delta of (x) d cube x is equal to 1 delta x minus y f of (x) b is f of (y) all of are familiar with the dirac delta function.

So, we will assume the field phi of (x, t) and phi(y, t) to be operator. And then we will start with this formal relation and then we will quantization here; what is the commutation relation between phi S and the field phi of (x, t) and phi of (y, t). And the other commutation relation phi of (x, t) phi of (y, t) commutator of (x, t) phi of (y, t) is equal to 0 just as in quantum mechanics x i term x i, x j is equal to 0; analytical calculation theory is 2 phi commute and so on pi of (x, t) phi of (y, t) is equal to 0.

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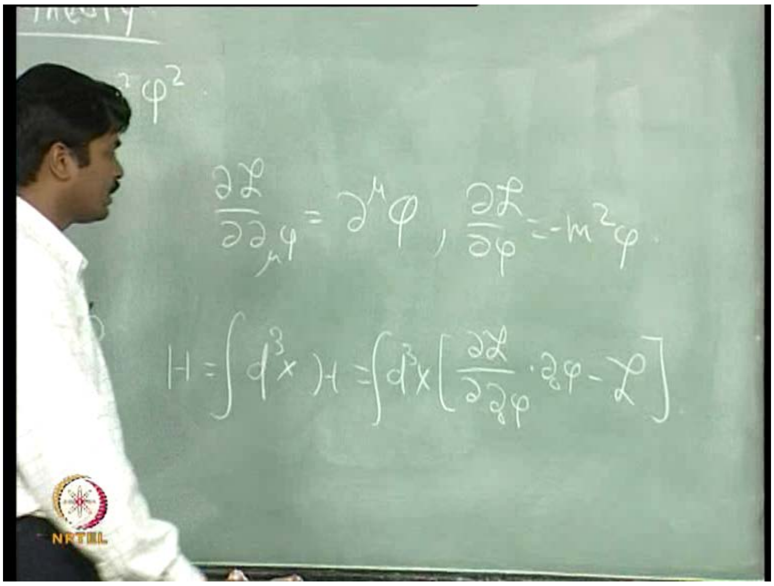

So, let us start with this basic commutation relation and then see what you get. So, we will consider a very simple example; an example of a Klein Gordon field and quantize it. So, Lagrangian for a Klein Gordon field which is a real scalar field of the mass M it is given by half del mu phi minus half m square phi square; we will start the system and then we will quantized. So, let us look at the equation of motion; the Euler-Lagrangian equation minus Del L over Del pi minus Del L over Del, Del mu phi is equal to 0. The second term here del L over del mu phi basically del mu phi.

And, del L over del pi is minus m square phi when I substitute this what I get is del cube phi plus m square phi is equal to 0; what is the Hamiltonian of the system we have integration d 2 x time the Hamiltonian H which is . So, the Hamiltonian del L over Del, Del 0 phi time phi dot minus l.

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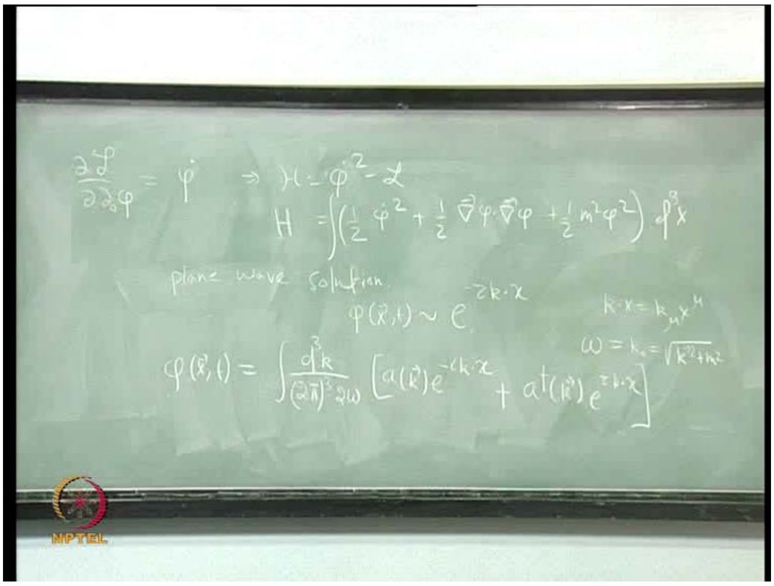

We can evaluate this it is very straight forward; you can see that Del L over Del, Del 0 phi is basically phi dot. So, the quantity inside the bracket is phi dot square the Hamiltonian density H is phi dot square minus L. And this will give us half phi dot square plus half phi plus half m square phi square. And the Hamiltonian of the system is just a H is integration this now for the space. Now, we have to take this first all this quantities inside the bracket to the operators the question given such a system how can you find the spectrum what are the Eigen value, Eigen state and so on the way usually we find is if we can write this Hamiltonian to be a sum of normal modes then we know how to find the spectrum first at the system.

So, we will do that to do that you know the equation of motion actually admits plain wave solution which are of this form pie of (x, t) goes like e to the power minus z k dot x were k dot x is k mu, x mu. And this becomes a solution of the equation motion provided omega which is defined to the k 0 is equal to square root k square plus m square; if k 0 is equal to this then satisfy the equation of motion. So, this is the plain wave solution of Klein Gordon equation a general solution most general solution will be super position of solutions like this.

So, more generally phi of (x, t) will be given by d cube k over 2 phi cube, 2 omega times a (k) e to the power minus i k dot x. Since, the field pie is real we have to also add the complex conjugate of this quantity here; a digger (k) e to the power i k dot x. Here, note that this vector 2 omega is just for convenient; we will see later that if you take this then the integration major Lagrangian invariant.

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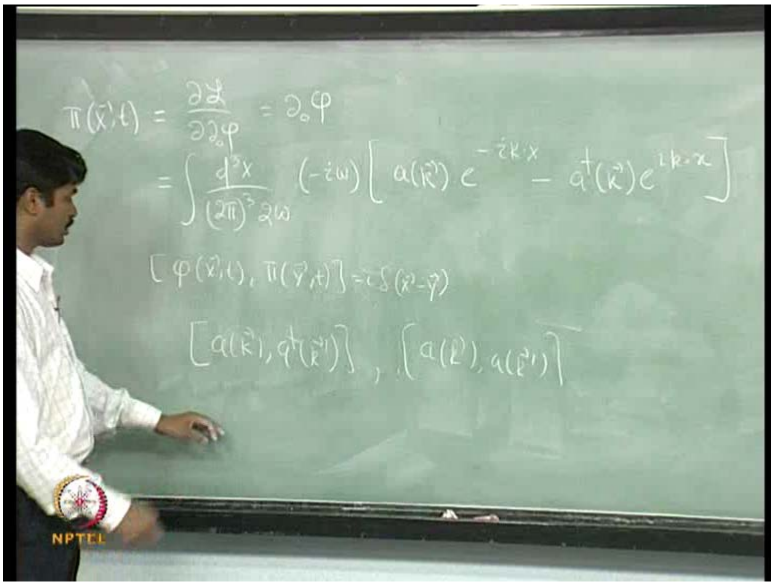

So, if this is phi what is the conjugate momentum phi of (x, t) is basically del L of a del L del 0 phi which is del 0 phi. And from this expression we get this to be d cube x over 2 phi cube, 2 omega times minus i omega a (k) e to the power minus i k dot x minus a digger (k) e to the power i a (k) dot x; what we will do now is we will take the fundamental commutation relation which is phi of (x, t) phi of (y, t) equal to i delta of (x) minus y.

And, then we will see if this is the commutation relation between phi and it is conjugate momentum then what is the commutation relation between these operators a (k) and there conjugate a digger (k). So, derive this commutation relation what you need to do is we can invert this relation and then we can write is a (k) and a digger (k) in terms of the fields pie of (x, t) and phi of (x, t). So, we will invert this relation and then we will derive commutation relation a of (k) a digger (k) and what is commutation between a (k), a (k) prime and so on all right. So, this is what we will do.

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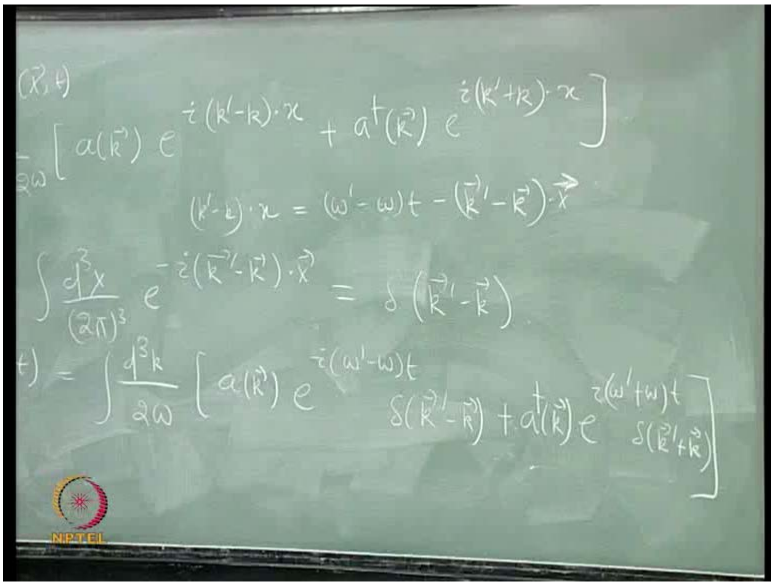

So, let us look at the expression for phi and from this relation; what we can do is we can consider integration d cube x, e to the power i k prime dot x times pi of (x, t). So, this is going to be integration d cube x when we use this solution here then integration cube k over 2 phi cube 2 omega times a (k) e to the power i k to the power minus k dot x. And here plus a digger (k) to the power i k prime plus k dot x. Now, look at this here k minus k prime dot x plus 2 terms which is omega time minus the 3 vector k prime minus k dot x.

So, this will get 2 terms when I integration over d cube x you will get a delta function you will use this relation integration d cube x divided by 2 pi. So, e to the power minus i k prime minus k dot x this is the delta function delta of k prime as k. So, first term here and second term here we will give you another delta function; in addition there will be the factor omega prime minus omega e to the power i omega minus omega e here and to the power i omega prime plus omega t. So, when I use that use this relation here what I get is d cube x e to the power i dot x phi of (x, t) is equal to integration d to k divided by 2 omega.

And, the first term a of (k) e to the power i omega minus omega t; delta of k prime minus k and the second term a digger (k) e to the power i omega prime plus omega t delta of (k) prime plus k. Now, you carry out the k integration and when you carry out the k integration you see omega prime is k prime is equal to k; then omega prime is equal to omega. So, this exponential factor will become 1 were as in the second term you will get e to the power to i omega t. So, when you carry out this integration.

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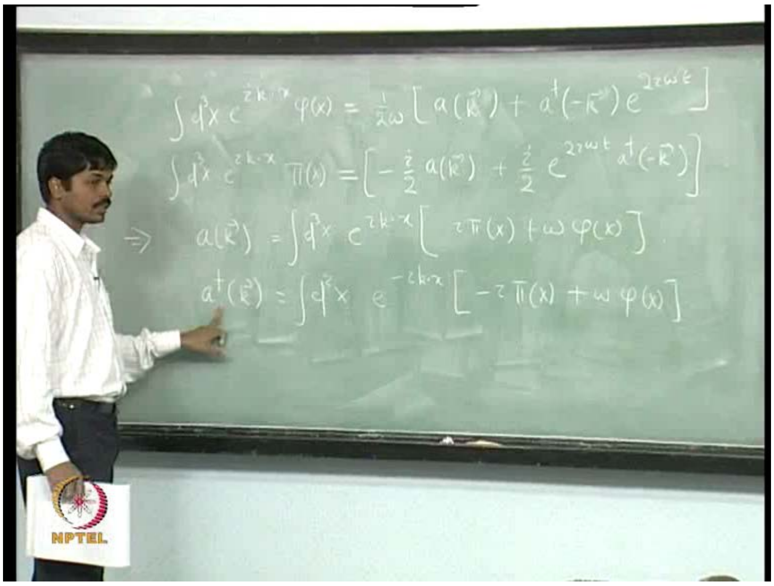

So, when you carry out this integration what you get is integration d cube x e to the power i a (k) prime dot x pi of (x) is equal to 1 over 2 omega a of (k prime) plus a digger of minus (k prime); because as you can see here you have delta of k prime plus k. So, when you carry out the integration over k here you have to substitute k prime by minus k which is what we have done here and then e to the power 2 i omega time k. Now, there is no reason to keep prime here. So, what I will do is I will denote a prime, I will delete it.

So, no prime all right; you do similar analyses and you can consider d cube x, e to the power i dot x phi of (x). And I will leave it has homework for you what you will get by doing such exercise is minus i by 2 a (k) plus i by 2 e to the power 2 i omega t a digger of minus (k). Now, it is very easy to write a and it is Hamiltonian conjugate a digger in terms of the field and phi of (x). So, this basically implies a of (k) to be integration d cube x e to the power of i k dot x times i pie x plus omega phi x. And a digger k is equal to d cube x e to the power minus i k dot x minus i plus omega phi of (x). So, what we will do in the next lecture; we will consider this two expression were a and for it is Hamiltonian conjugate. And then I will find what are the commutation relation using the fundamental commutation relation between phi and pi.