Quantum Field Theory
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Module - 4
Quantum Electrodynamics
Lecture - 26
Feynman Rules in QED II

In the last lecture, we were discussing various physical processes in Quantum Electrodynamics.

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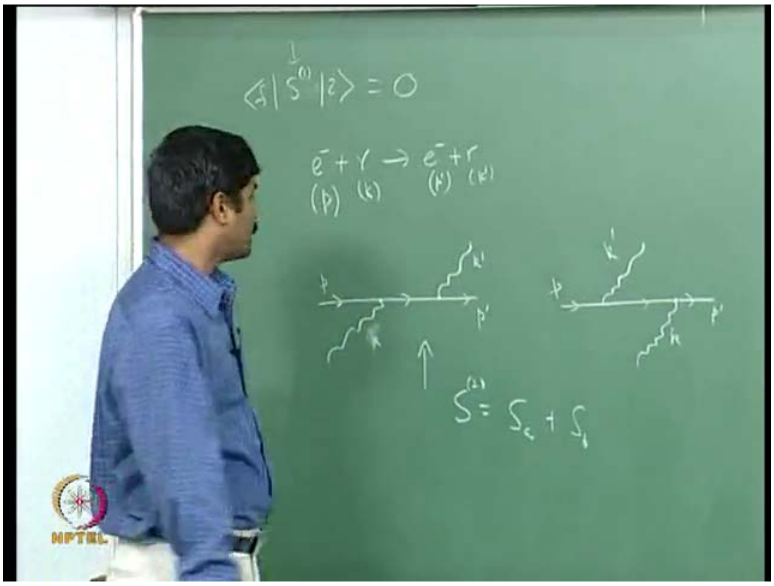

We have seen that the first order terms in the S matrix do not contribute at all to any physical process. So, as far as any physical process is concerned, the contribution to the amplitude from the first order terms is 0. Whereas, we have seen that, in the second order there are several physical processes, which gives non zero amplitude. As an example, we were considering the Compton's scattering, where e minus plus gamma going to e minus plus gamma, where the original electron head for momentum p before term head for momentum k and the outgoing electron head for momentum p prime and the outgoing photon head for momentum k prime.

There are two diagrams which contribute to this process, these diagrams are given by, you have an incoming electron of momentum p and then you have an incoming photon of momentum k. Then at the end, you have an outgoing electron of momentum p prime and then you have an outgoing photon of momentum k prime, there is a propagator from the X point, from X 2 to X 1. We would also seen that, you can have an incoming electron of momentum p emitting first and outgoing momentum of photon k prime.

And then the incoming photon of momentum k is observed and finally, you get an outgoing electron of momentum k prime. So, these are the two possibilities and then you have evaluated the amplitude per one of these processes. So, I denote S, the term in S matrix which gives non vanishing contribution for this process to be S a and for the second process to be S b.

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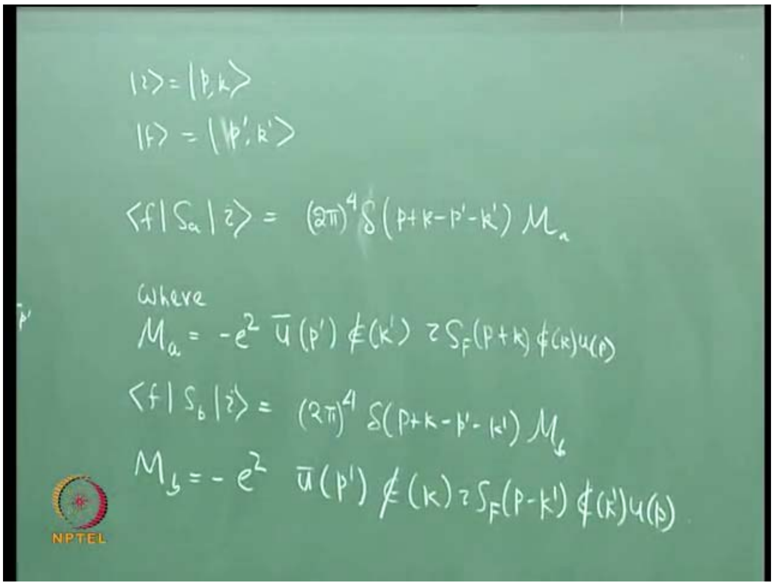

And what you have computed in the last lecture is, we considered this initial state i, which is an electron of momentum p and a photon of momentum k. And the final state consist of an electron state of momentum p prime and a photon state of momentum k prime. And we have compute this term S a and i and what it show is that, this quantity is equal to 2 pi to the power 4 delta p plus k minus p prime minus k prime, times M a, where the Feynman amplitude M a is given by minus e square times u bar of p prime epsilon slash of k prime i S F p plus k epsilon slash k u of p.

So, this is what we have derived in the last lecture, we can do a similar computation and then we can compute this quantity S b i. And this will be equal to 2 pi to the power 4 delta p plus k minus p prime minus k prime times M b, where the amplitude M b is given by minus e square times u bar p prime epsilon slash k i S F p minus k prime epsilon slash of k prime u of p.

So, I will leave it for you to derive this result, you can show in a very similar way that, if you compute this amplitude then at the end of the day, you will get this to be equal to 2 pi to the power 4 delta p plus k minus p prime minus k prime times M b, where M b has this expression.

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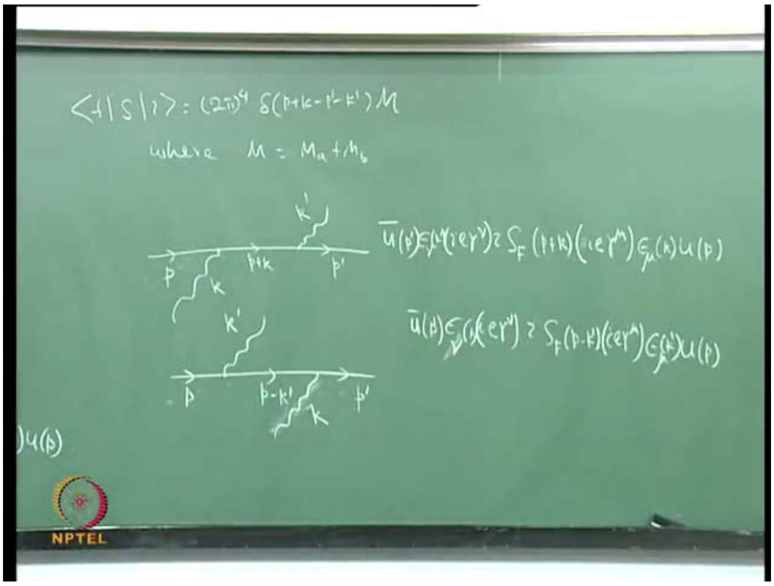

Now, things look very simple. So, finally, this amplitude here is simply given by 2 pi 4 delta p plus k minus p prime minus k prime times M, where M is given by M a plus M b. The question that I would like to ask is, is there are simpler way to compute M a and M b, given these diagrams. And the answer is yes, you can simply write M a and M b by just looking at these two diagrams and these are provided, you have set of very simple rule to compute these amplitudes, so these rules are known as the Feynman rules.

And then now you know, at least you can guess what can these rules be, especially this term here, represents the incoming electron. This term here are not directly, but it schematically represents the incoming photon, this term represents the propagator and this term here represents the outgoing photon and this term represents the outgoing electron. So however, there are this interaction vertices, that is why I said, it is not exactly representing the incoming photon, it represent the incoming photon and the interaction vertex.

So, let us look at this term again, let us look at this diagram, you have an incoming electron of momentum p and you have an incoming photon of momentum k, the interact and then finally, you have a Fermion propagator of momentum p plus k. And finally, you have and then you have an outgoing photon of momentum k prime and outgoing electron of momentum p prime. Let us write u of p for the incoming electron, write epsilon mu of k for the incoming photon, the interact.

And for the interaction vertex, you write minus i e gamma mu for the interaction vertex or according to our notation, it is plus i e gamma mu then there is a Fermion propagator of momentum p plus k. So, I have i S F p plus k for a Fermion propagator and then you have the interaction vertex, which is i e gamma nu then you have an outgoing photon of momentum k prime. For that, I will write epsilon nu k prime and you have an outgoing electron of momentum p prime, I will write it is u bar p prime.

You can see that, this is exactly what your M a is and on the other hand, you have the second diagram here, where you have an incoming electron of momentum p, which emits a photon of momentum k prime here and then you have a propagator with momentum p minus k prime. Finally, the incoming photon is observed, photon of momentum k is observed and you have an outgoing electron of momentum p prime.

So, let us follow the same set of rules, this will help us that, you have a u of p for the incoming electron, you have epsilon mu of k prime for the outgoing photon here and then there is this interaction vertex. For this, I will write i e gamma mu then you have a Fermion propagator of momentum p minus k prime. For that, I will write i S F p minus k prime and finally, the incoming photon is observed here, there is an interaction vertex, which I will write as i e gamma nu. For the incoming photon of momentum k, I will write epsilon nu of k and there is an outgoing electron, for which I will write u bar of k prime. You can see that, the amplitude computed by following this set of rules is exactly this amplitude here.

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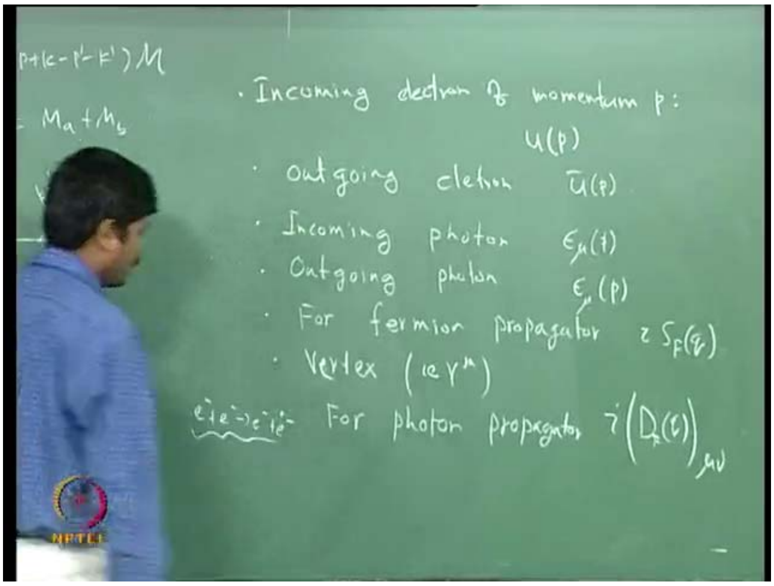

So, if this immediately suggest a set of rules which say that, if you have an incoming electron line of momentum p you write u of p. For an outgoing electron you write u bar of p, over it is for an incoming photon you write epsilon mu of p. For an outgoing photon, again if you are choosing a rear polarizations for photon, you again will have epsilon mu of p, otherwise you write epsilon m u star p. For Fermion propagator, you write i S F of q, where q is the propagator momentum, for each vertex you write i e gamma m u.

So, interaction vertex, so if you follow this set of rules then you can recover the amplitudes just by looking at this Feynman diagrams. Vertex left is, suppose you have approaches, where instead of a Fermion propagator, you have a photon propagator. So then you simply replace this i S F q by I will represent the photon propagator as D F. So, for photon propagator, you have minus eta mu nu or I will write as D F q mu nu, the mu nu th component of the photon propagator with effective of i.

Now, suppose you have identical particle set rating just like in the case of e minus plus e minus going to e minus plus e minus then you have to take care of the minus sign, because of the exchange and interactions. So, when you compute the amplitudes, finally you have to add them with a relative minus sign, if in case you have identical particles scatter with each of that, so you have to keep that in mind.

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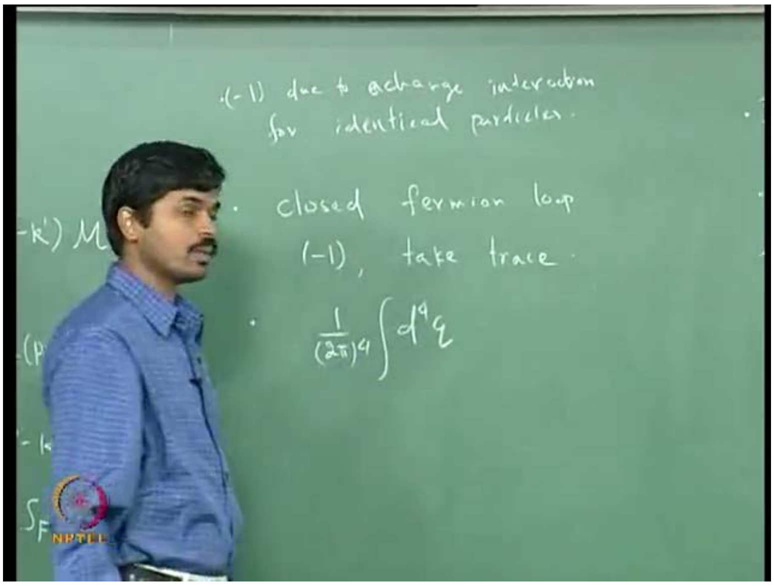

So, in addition to all these things, you have to keep the exchange interaction minus 1 due to identical. And then so you have follow these set of rules and then you write this in a sequence, just you wrote sometime back and then you add the amplitudes appropriately by taking care of minus signs, wherever it is required. And then you add all these amplitudes together, you will get the total Feynman amplitude for any physical process.

There are a couple of things that remains, one is that suppose you have a closed Fermion loop then you have to write a minus 1 for every closed Fermion loop and then you have to take trace. So, we will see, why you need minus 1 and then why you need to take a trace in a moment and finally, suppose the momentum of an integral line is not fixed by any of this. So, the point is that, at every vertex, the energy and momentum is to be conserved, that is why I wrote here for the Fermion propagator, the momentum p plus k and here, I wrote p minus k for the Fermion propagator.

That is because at every interaction vertex, the energy momentum is conserved, in case the energy momentum conservation does not fix the momentum for any internal line then you need to integrate over that particular momentum. For example, suppose you consider a process like this, you have a photon of momentum k. All it says is that, if the momentum here is q for this Fermion then for this Fermion the momentum is k minus q, it does not say anything other than that.

So therefore, in this process, the momentum q for the internal Fermion line is not fixed, therefore in those cases, you need to integrate over all possible values of momentum. So, you need to write 1 over 2 pi to the power fourth integration over the 4 q, whenever the four momentum for the internal Fermion line is not fixed at the interaction vertices. So, you follow this set of rules then you will get Fermion amplitudes for any physical process that you consider.

So, let us now look, why we need to have a minus 1 factor for a closed Fermion loop and then why we need to take it trace. So now, let us ask the following question, when do we get a closed Fermion loop, let us consider this example of physical processes, which contributes at the second order. So, which term in the S matrix at second order contribute to any physical process, which has a closed Fermion loop.

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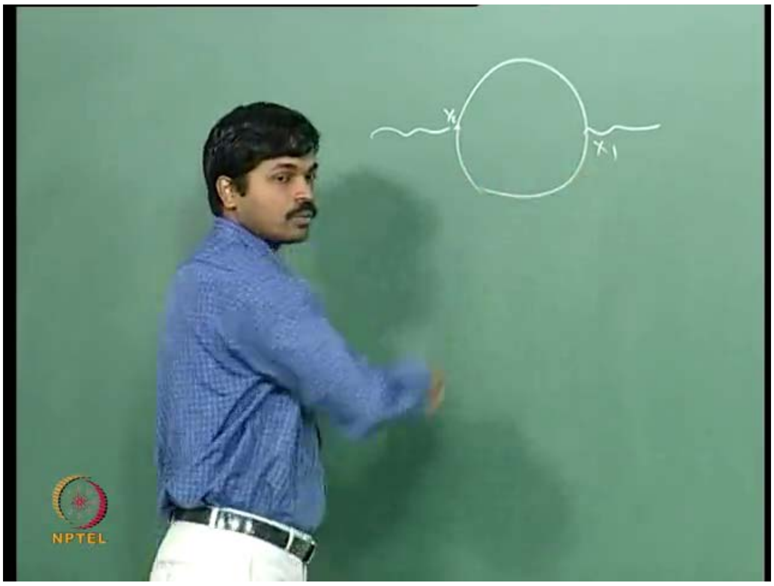

Closed Fermion loop means, you can have some interaction vertices and this loop has to be closed. So, if this is X 2, this is X 1, but simply means that, if there is a Fermion line, which is contracted from X 2 to X 1.

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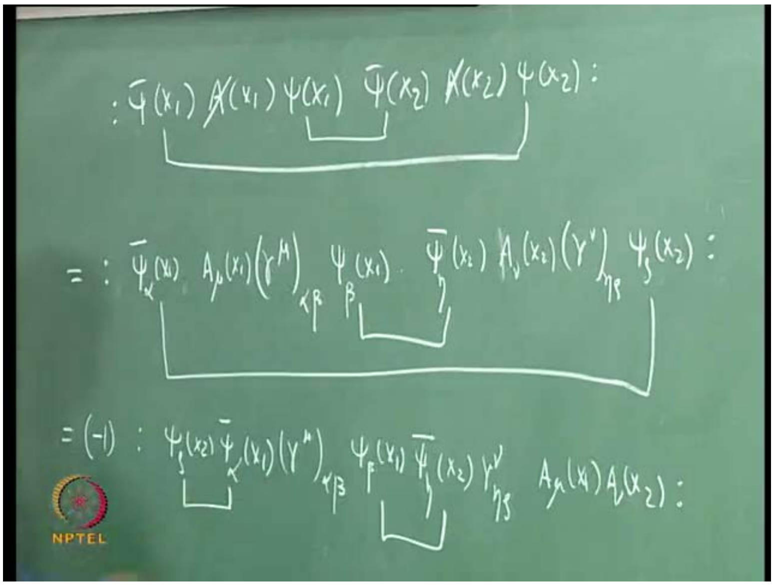

Then, you again have a Fermion, which is contracted from X 1 to X 2 or to state it more clearly, what do we have for the S matrix is, this term normal order product of psi bar X 1 A slash X 1 psi X 1 psi bar X 2 A slash X 2 psi X 2. The time order product of this quantity is what, which comes in the S matrix and then you do the weak expansions. If the weak expansion, you contract various term and then whenever you have this contracted be this, you have an internal Fermion line. Now, if to get a closed Fermion loop, not only this has to be contracted with this, this also is to be contracted with this, this is the case when you will get a closed Fermion loop.

So, let us consider this term and then write it in a simpler way, so let us write, so you have normal order product and then I can expand it in terms of it is components here. So, I will call this as psi bar alpha of X 1 and A slash is nothing but A mu of X 1 gamma mu of alpha beta. And then you have psi of X 1, which is psi beta X 1, that is for this term here then you have psi bar X 2 psi bar. Let us call it as eta X 2, A slash is again A mu of X 2 gamma nu of eta rho, psi rho of X 2 with this normal ordering.

Now, this is contracted with this, so you have a contraction here and this is contracted with this. Now, in this contraction, I can just pull it to this place, but you have to keep this in mind that, in this process, it crosses two, so if you pull it to the front, let us say you want to pull it to the full front, the net crosses 3 Fermion, 1 2 3, therefore it will gain a minus sign.

So, this is equivalent to writing minus 1 normal order product of psi rho of X 2 psi bar alpha X 1 and gamma mu alpha beta psi beta X 1 psi bar eta X 2 gamma mu eta rho and then A mu X 1 A mu X 2 normal ordering. Now, this is contracted with this and this is contracted with this.

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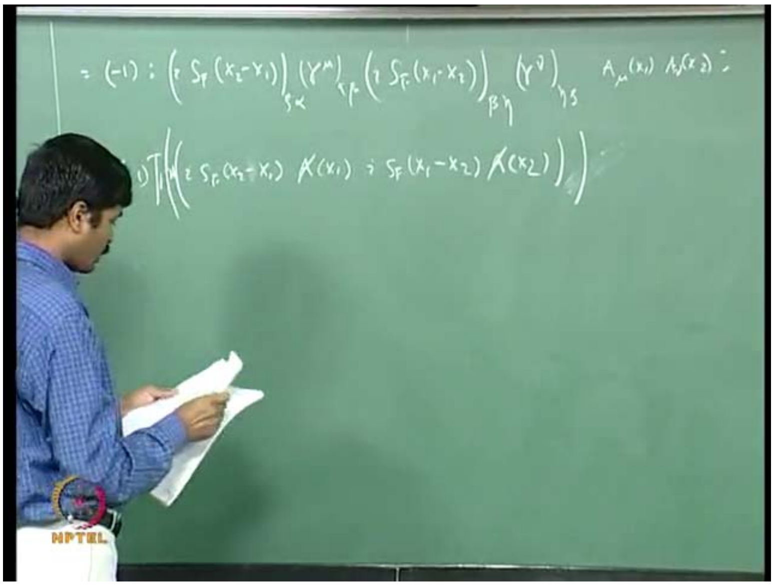

So, what you get here is now, this term is equal to minus 1 normal order product of, this is just i S F X 2 minus X 1 rho alpha and then gamma mu alpha beta then psi beta X 1. This contraction will give me i S F X 1 minus X 2, the beta eta competent of that and you have gamma mu of eta rho then A mu X 1 A mu X 2 normal order product. Or this is nothing but minus 1 i S F X 2 minus X 1 A slash X 1 i S F X 1 minus X 2 A slash X 2, this whole thing rho rho th component, sum over and normal order product.

This simply means that, you take this quantity and then you take the trace of this, so this is nothing but trace of normal order product of the whole thing. So, now you know why there is a trace and how did we get a minus 1 factor, so this result will hold whenever you have a closed Fermion loop. So, for every closed Fermion loop, you have to write a factor of minus 1 and then you have to take the trace appropriately, around the close Fermion loop. So now, you have the set of rules, what we will do is that, we will consider a couple of more physical processes and then compute the respective Feynman amplitudes for those processes, any question after this.

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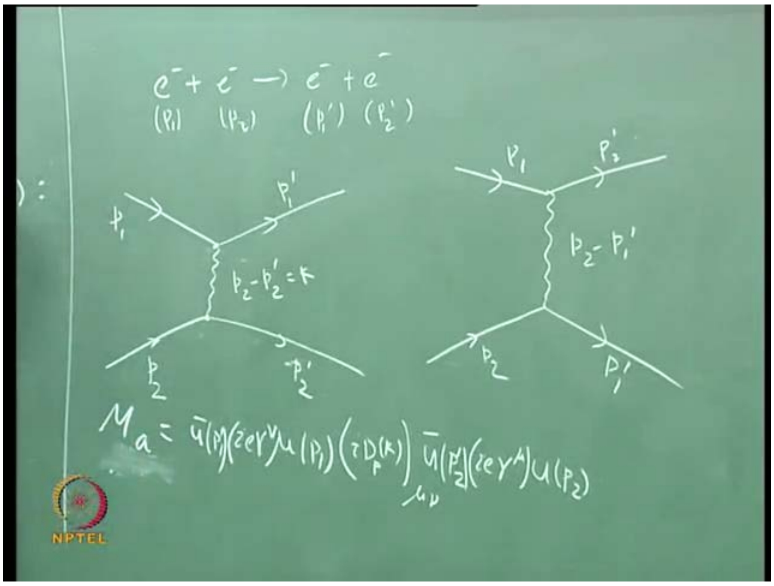

So, let us say, we consider this e minus plus e minus going to e minus plus e minus of momentum p 1, p 2, here the momentum of outgoing electrons are p 1 prime and p 2 prime. We have already seen, there are two diagrams, which contribute to this process, for one of them, you have incoming electron of momentum p 1, outgoing electron of momentum p 1 prime, incoming electron of momentum p 2, outgoing electron of momentum p 2 prime and then you have a photon propagator of momentum p 2 minus p 2 prime, let us say.

The other process that gives a contribution, the other diagram which gives a non zero contribution to this physical process is again the same, except that the p 1 and p 2 primes are interchange. So, you have p 2, p 1 prime, the p 2 minus p 1 prime, p 1, p 2 prime, so how will you write the amplitude for these two processes, this I will call as M a, the Feynman amplitudes for this one is M a. So, you start lesser from here, you have an incoming electron of momentum p 2, I will write this as u of p 2 and there is an interacting vertex here.

For this, I will write i gamma mu i e gamma mu then you have this outgoing electron of momentum p 2 prime, I will write u bar p 2 prime for this. Then you have a photon propagator, for which I will write as i D F, let us say this is k, i D F of k mu nu th component. Then here you have an incoming electron of momentum p 1, so you write u p 1, you write i e gamma nu for the interaction vertex, i e gamma nu and then you have u bar p 1 prime, so our M a is given by this.

What about M b, M b here, I can write it as, first of all, because these are identical particles, I have to write a minus sign. The phase vector rising, because of the exchange interaction and then I will follow the set of rules that we have here and then write down the amplitude.

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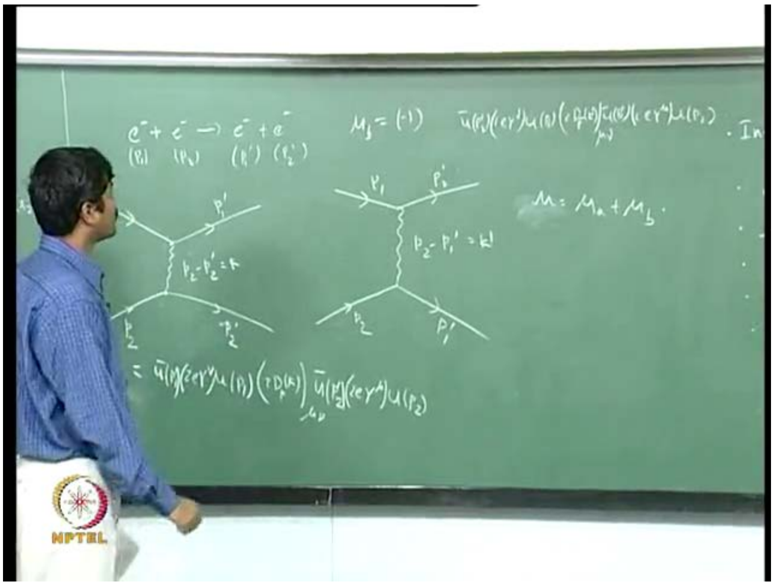

So, this is p 2, u p 2 i e gamma mu and then u bar p 1 prime then you have a propagator i D F k, which is p 2 minus p 1 prime here, that is call it as k prime, this is k and this is k prime. So, i D F k prime mu nu then you have an incoming electron of momentum p u p 1 i e gamma nu u bar p 2 prime. So, this is the amplitude for this, this is the contribution from the second diagram and then the total contribution here is M is M a plus M b.

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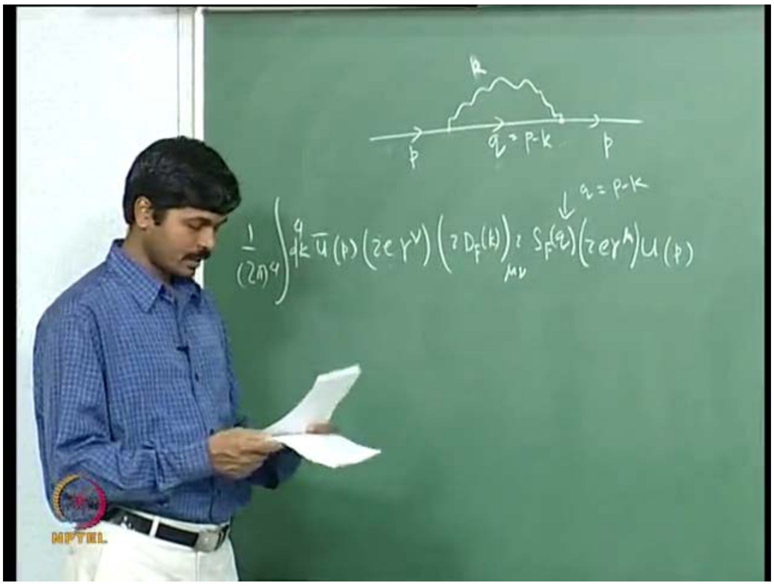

So, what about something with a closed loop, let us consider the electron self energy diagram. You have an incoming electron of momentum p and you have a Fermion of momentum q, a photon propagator momentum k, so q is just p minus k. Before momentum is conserved at each vertex, therefore you again will have an outgoing electron of momentum p. So, what is the amplitude for this process, you write again u p for this then you have an interaction vertex i e gamma mu.

You have a Fermion propagator, this I will write as i S F q then you have a photon propagator, which I will write as i DF k mu nu. Then you have this interaction vertex, for which I will write i e gamma nu and you have an outgoing electron, I will write here u bar of p for the outgoing electron, is this all, that is not all, because the k here is actually not fixed by this process. Therefore, you have to integrate over it, therefore you just consider 1 over 2 pi to the power 4 integration d 4 k, when you should to keep in mind that, this q here is p minus k, so this gives the amplitude for this process here. So, that way you can consider any physical process, you write down all the Feynman diagrams for this process. And then from the Feynman diagrams, you can just derive this amplitudes by following the Feynman rules, which we have stated. So, what we will do now is, we will consider, now we know how to compute the Feynman amplitude.

So, the next question that we will would like to ask is, how to compute the cross sections for a given process. So, to compute cross sections, we have a, let us say suppose, we will do this by considering example.

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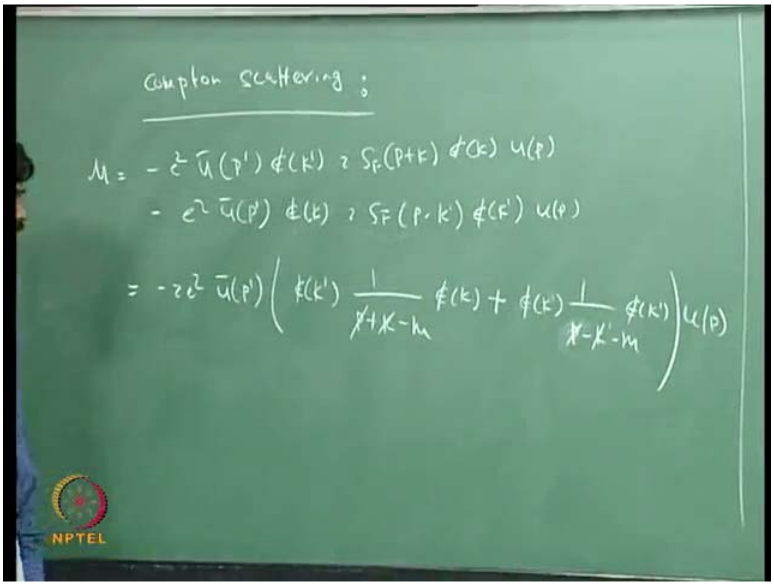

So, first what I would like to do is, we will consider the Compton's scattering again, which we have studied in great detail and then we will compute the cross section for the Compton's scattering. Then we will follow this process, electron electron scattering and then I will compute the amplitude at lowest order for this process as well and then I will say how to do that.

So, let us look at the amplitude M here, it is a given by minus e square u bar p prime epsilon slash k prime i S F p plus k epsilon slash of k u p minus i e square u bar p prime epsilon slash k i S F p minus k prime epsilon slash k prime u of p. So, I can write it in in simple terms, I can substitute for the Fermion propagator, this is minus i e square u bar p prime, it has epsilon slash k prime. For the Fermion propagator, I have 1 over p slash plus k slash minus M, here epsilon slash k and then from the second term, I get epsilon slash k prime 1 over p slash minus k prime slash minus M for this propagator here. And finally, this is k epsilon slash k prime u of p, that is all here. Then the cross section, we have derived a formula for the cross section in one of these earlier lectures.

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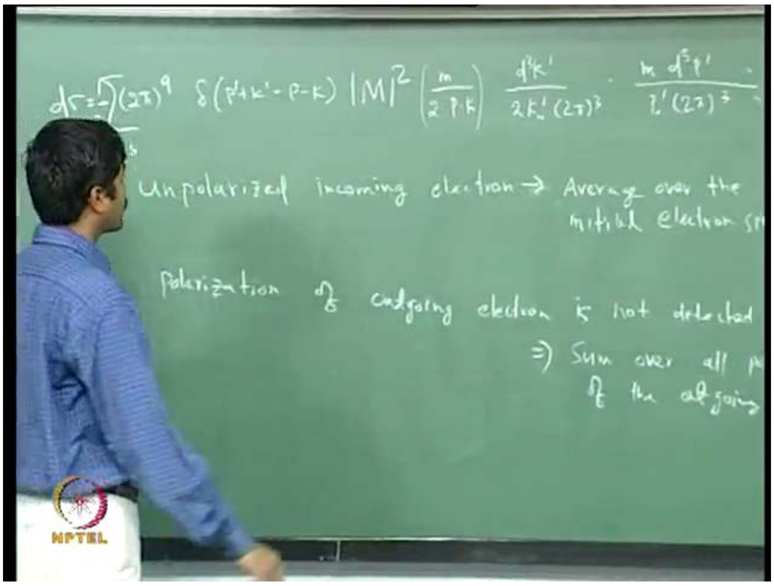

So, let me write down the differential scattering cross section for this process, this is given by 2 pi to the power 4 delta p prime plus k prime minus p minus k mod M square. Then M divided by 2 p dot k d q k prime over 2 k 0 prime 2 pi cube then M d cube p prime over p 0 prime 2 pi cube, you have an outgoing bosonic particle and one outgoing Fermionic particle. So, accordingly, your normalization section differ here and then you have this appropriate factor, which we have derived in one of these earlier lectures, this is the differential scattering cross section.

What you will assume in addition is, we will consider the initial electron, the incoming electron to be unpolarized. So, when it is unpolarized electron, initially what you need to do is, you need to average over it is polarization. So, unpolarized, so both spins are equally probable, therefore you just average over the initial electron spin. Then we will also assume that, the polarization of the outgoing electron is not detected. So therefore, what you need in the cross section is that, you have to sum over all the outgoing spins, all possible spins.

Polarization of outgoing electron is not detected, this simply suggest that, you need to sum on possible polarizations of the outgoing. So, this simply says that, you need to sum over all polarizations or entire I will say, because of this averaging, you need to have a factor of half. This is the differential scattering cross section then what we will further do is, we will integrate over this outgoing electron here, outgoing electron momentum.

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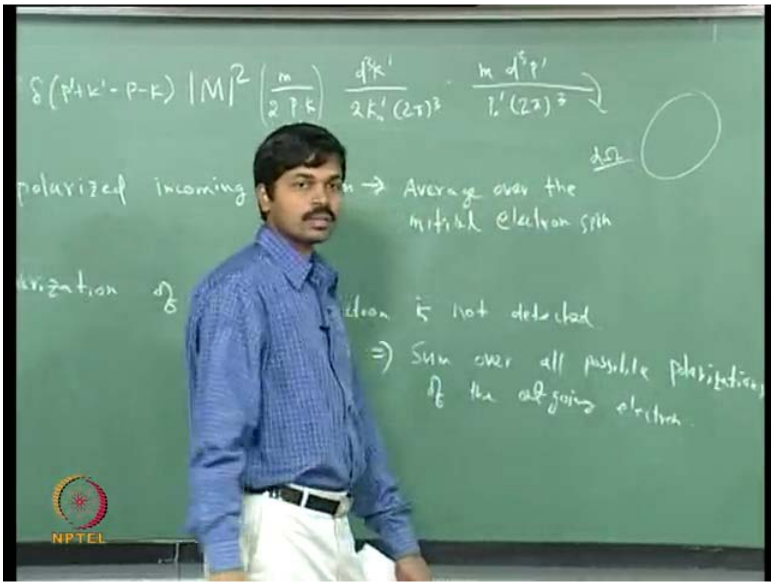

And then finally, what we will see is that, what is the differential scattering cross section when the outgoing photon momentum lies in some solid angle d omega, that is what we are interested to compute, so which we will do in the next lecture.