# Quantum Field Theory

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**Module - 4**
**Quantum Electrodynamics**
**Lecture - 25**
**Feynman Rules in QED l**

We have considered the S matrix for QED and then we locked into all the terms in the first order as well as second order in e, and then we looked at various physical process and then we discussed which term in the S matrix will contribute to what physical process and so on. We also have seen that, that can be more than one term, which can contribute to a given physical process and each of these terms can actually infect the represented in terms of this local Feynman diagrams.

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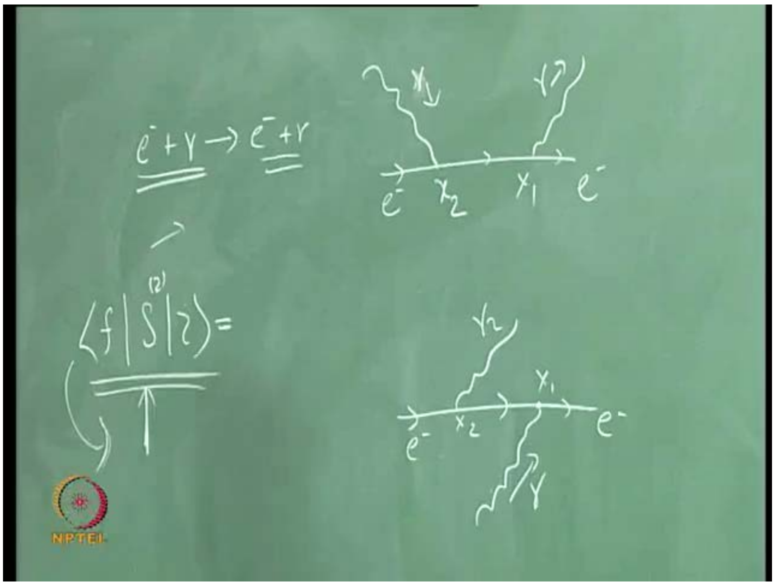

So, for example, suppose you consider a process like Compton's scattering let us say e plus plus gamma e minus plus gamma going to e plus gamma then there are two terms in the S matrix, which gives non zero contribution to this process. It is second order and these two terms are represented by the following to Feynman diagram. So, you have an incoming electron and you have a incoming photon and then you have an outgoing electron and you have an outgoing photon at X 1 or you too can also have an electron e minus, which emits a photon at X 2.

And there is an Fermion propagator between X 1 and X 2, and incoming photon is observed at this specific point x 1, finally there is an outgoing electron. So, these are the two possibilities and correspondingly there are two terms in the S matrix, so it gives non zero contribution for this physical process. And these are the two Feynman diagrams for the two terms in the S matrix, what you need to do is, we actually need to consider the S matrix, the relevant terms in the s matrix for this process and then we have to compute the amplitude for this process.

So, you have some initial state given by these incoming electron and photon, and you have outgoing state represented by some outgoing electron and outgoing photon. And you know what are the terms in the S matrix per second order, you need to compute this amplitude in order to know, what is the cross x or anything you want to do in the quantum field theory, you need to evaluate this amplitude. What we will do now is, we will formulate a set of rules to get this amplitude so that, you do not have to evaluate again and again every time, you want to compute this amplitude.

So, we will get a simplified form for this amplitude from what is known as the Feynman rules. We will derive the Feynman rules for QED and then you can write down this amplitude using Feynman rules, instead of working out it is and every time whenever you need it. So, this is what we will be doing in a today's lecture, let us first summarize the use full expressions that will be needed in this lecture.

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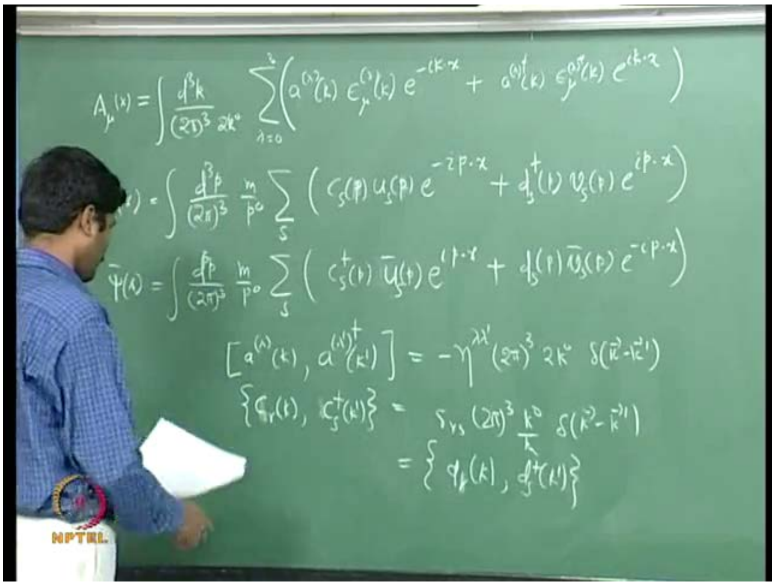

We will require the mode expansion for the photon field, which is give by d cube k over 2 pi cube 2 k 0, sum over the polarization lambda is going to 1 to 3 a lambda k epsilon lambda of k e to the power minus i k dot x plus a lambda dagger of k epsilon mu lambda star of k e to the power i k dot x. Then if the Dirac psi of x which is given by d cube p over 2 pi cube m over p 0 is the normalization I have chosen. And sum over this spins c s p u s p e to the power minus i p dot x then d s dagger of 9 v s p e to the power i p dot x.

Correspondingly, we have psi bar of x, this is given by d cube p over 2 phi cube m over p 0 sum over s, c s dagger p u s bar of p e to the power i p dot x plus d s p v s bar of p e to the power minus i p dot x. And these operator is a’s and c’s and d’s, they obey the following computation and anti computation relations, a lambda of k, a lambda prime dagger of k, prime commutator is given by minus eta lambda lambda prime 2 pi cube two k 0 delta k minus k prime.

Whereas, c r k c s dagger k prime anti commutator is given by delta r s 2 pi cube k 0 over n delta k minus k prime, which is also same as the commutator of d s d r k, d s dagger k prime and all other commutators and anti commutators relating to the Feynman and prosodic operators from this. So, these are the non vanishing commutation and anti commutation relations. So, what we will now do is, we will lock at the first order term in the S matrix that I will call as S.

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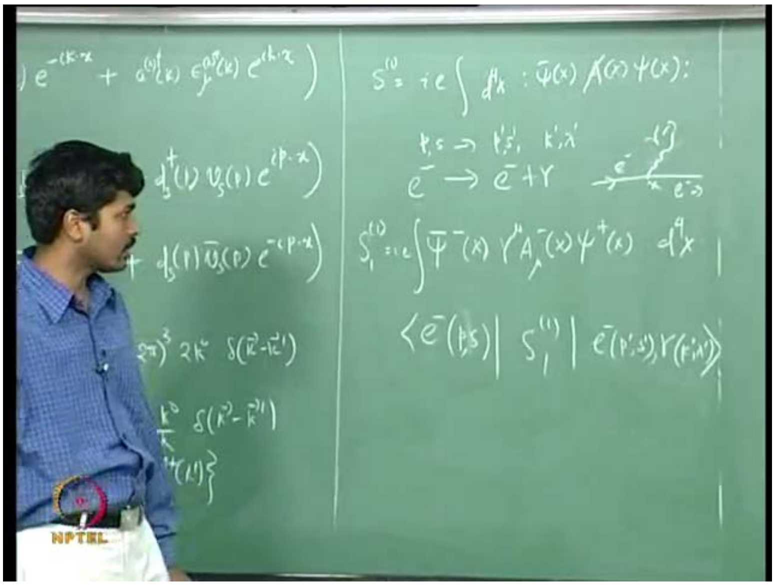

I have denoted as S 1, which is i e times d 4 X normal under that are psi bar of X A slash of X psi of X. And we look at a physical process, for which this may, this might give a non zero amplitude although will see later that, there is no physical process, to which this one gives a non zero contribution and then we will evaluate the amplitude for this process in detailed. So, let us consider this process of e minus going to e minus plus gamma, this is one possibility, because you can have this term.

For example, can contain one term, it is a psi bar minus of X gamma mu a mu minus of x psi plus of X, one of the terms in this normal under ring given by this and this psi plus represents any lesser than electron at X. Whereas, m mu minus m psi bar minus represents emission of or creation of m emitton at it is phase time point X. If this is X then an electron is an elected here and an outgoing photon and an outgoing electron are created at x. So, this process this is the term in S matrix, so it can contribute to this process.

So, we will consider this process and then we will evaluate, so S 1 1 I will call, i e times to this d 4 X, so we will take this e minus incoming electron with some momentum p and some polarization s, S 1 1 e minus e prime s prime gamma of k prime lambda prime. So, the point is, suppose you have this incoming electron with momentum p and the polarization of s, going to some outgoing electron of momentum p prime polarization s prime and some outgoing photon of momentum k prime and polarization lambda prime.

Then, what is the amplitude for this process, that is what is the question that we would like to ask. So, naturally, we need to evaluate, what happens when this psi plus acts on these states and a mu minus acts on this states and so on. So, this is what we need to know.

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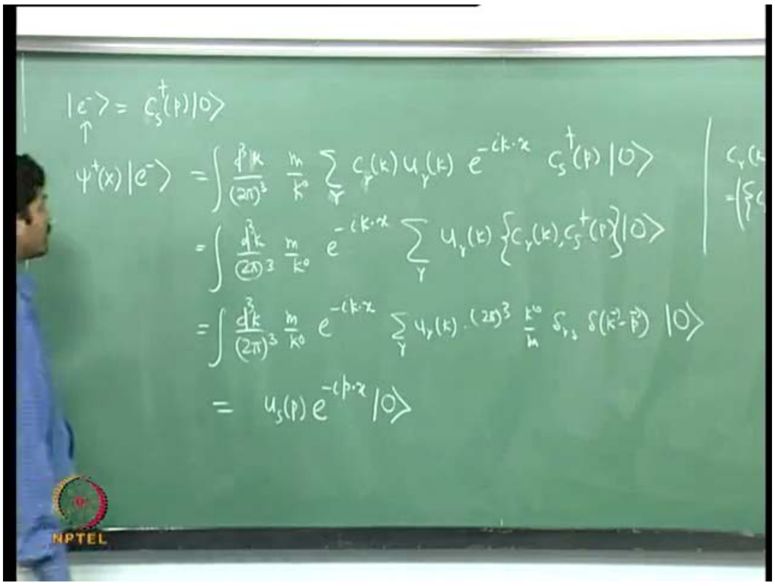


So, to do that, let us look at first the electron state, let us say this represents the state with a momentum p and spin s, therefore this state for electron is given by c s dagger of the acting on the vacuum. Now, we would like to know, what happens when I consider psi plus of X acting on this tact. So, what is psi plus of X, you can look at this expression here, psi plus of X is given by this term, so d cube p over 2 pi cube and let us use the integration variable to be k, m over k 0 sum over s c s k.

Let us denote this dummy variable by r, c r k u r k e to the power minus i k dot x, this one acts on this state, which is given by c s dagger of p acting on vacuum. So, what do we get, I can rewrite this term as d cube k over 2 pi cube m over k 0, sum over r or this time e to the minus i k dot x then you have sum over r u r k c r k c s dagger of p acting on the ground state. I just rearrange the terms and this expression.

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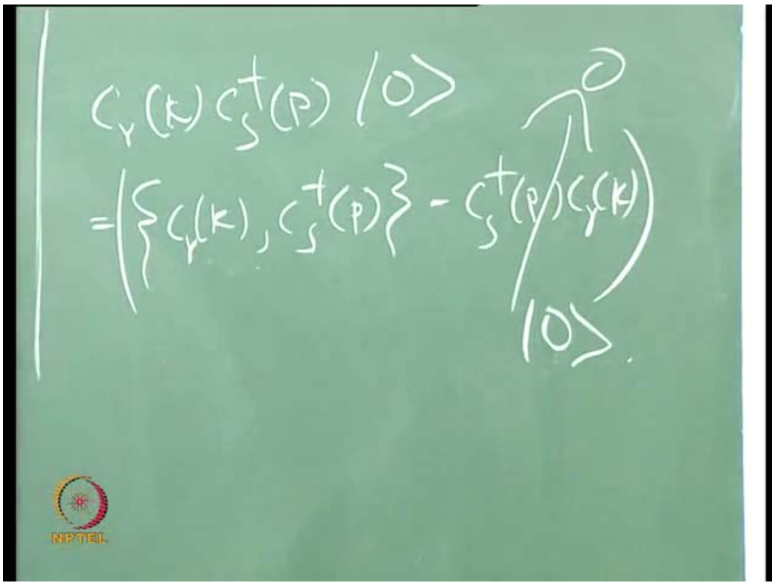


Now, what I will do is, I will take this term here, c r k c s dagger of p, I can rewrite it as the anti commutator of c r k c s dagger p minus c s dagger p c r k. Now, when this whole thing acts on the ground state, this simply means that, this and this acting on the ground state. But, as you can see, the second term acting on ground state, there is an annihilation operator. So, when c r k acts on 0 on the vacuum, it gives you zero contribution, so only thing that is left is the anti commutator of c r k c s dagger p.

But, the anti commutator we know what it is, this is nothing delta r s, upto some normalization this is delta r s and delta k minus p. So, you can just substitute that, we can say that, this is equal to the anti commutator of these two operators acting on the ground state. And we will put the known value for the anti commutator which is nothing d cube k over 2 pi cube m over k 0 e to the minus i k dot x sum over r u r k then 2 pi cube k 0 over m delta r s delta k minus p, this acting on the ground state. So, the sum over r, I can evaluate and also I can evaluate the integration over d cube k, that will give me simply this 2 phi cube and k 0 over m, they will cancel each other. So, what is left is, u s p e to the power minus i p dot x acting on the ground state.

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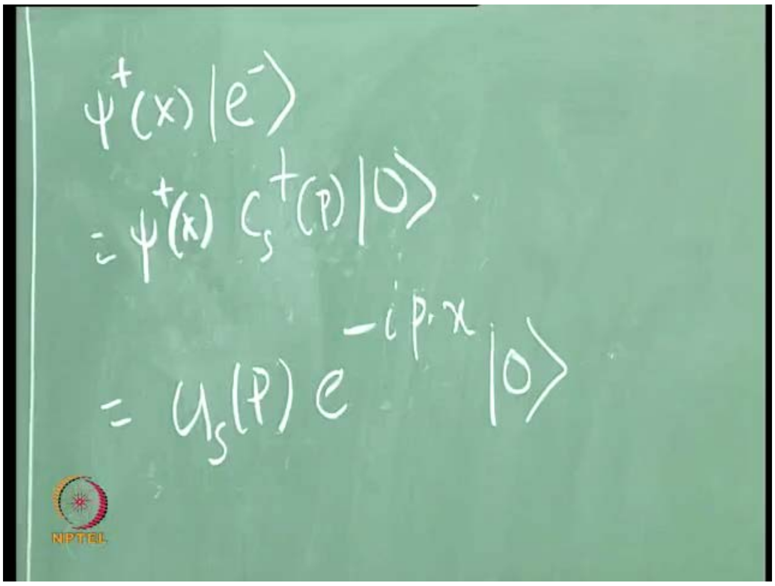


So therefore, what is saw is that, from these result, psi plus of X acting on e minus which nothing psi plus X c s dagger p on the ground state is equal to u s p e to the power minus i p dot X acting on the ground state. Similarly, you can evaluate what happens, when m u plus acts on a one photon state.

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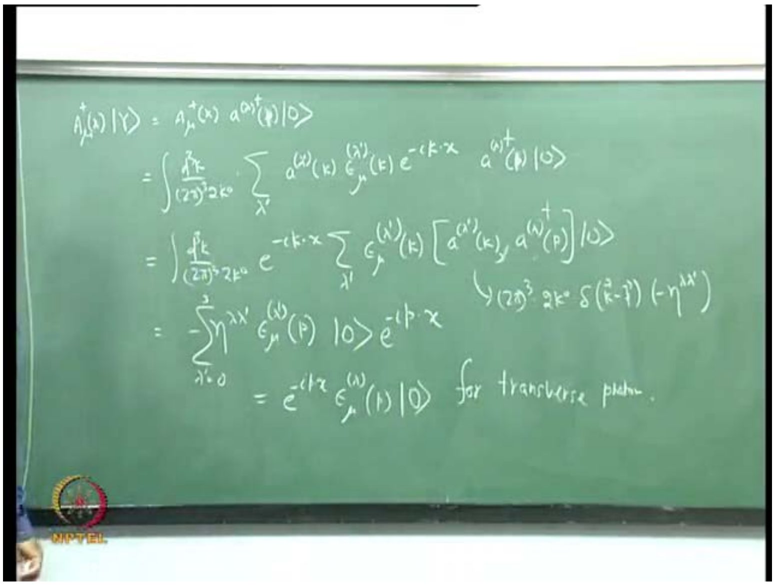


So, let us say that, you have a photon gamma of polarization lambda and momentum k and you want to know, what do you get when you consider a mu X acting on that. So, this is nothing a mu plus X acting on, so a lambda dagger of k acting on the ground state.

We can substitute the value for a mu plus, which is given by these, so this is nothing integration, let us say this is a momentum p d cube k over 2 pi cube 2 k 0 sum over lambda a.

Let us say you are considering some over lambda prime a lambda prime k epsilon mu lambda prime k e to the power minus i k dot x acting on a lambda dagger k 0. So, let us rearrange the terms, so this is integration d cube k over 2 pi cube 2 k 0 e to the power minus i k dot x and then sum over lambda prime epsilon mu lambda prime k a lambda prime k a lambda dagger p. Again you can write this term as the commutator of a lambda prime k and a lambda prime dagger minus a term with ((Refer Time: 21:10)) when acting on the ground state.

Therefore, I will just substitute the commutator of a lambda prime and a lambda dagger in this place. And we know what is the commutator is, this is nothing 2 pi cube 2 k 0 delta k minus p times minus eta lambda lambda prime. So, you can now carry out the k integration, when you carry out the k integration, of course this 2 pi cube 2 k will cancel and what you are left with is minus eta lambda lambda prime epsilon mu lambda prime p acting on the ground state, where there is a sum over lambda prime is transfer from 0 to 3.

So, this time says the upper e to the power i p dot x, now suppose you are considering a transfer photon here then for transfer photon, you can just consider this is to be minus delta lambda lambda prime. And what you will get is, this whole thing is equal e to the power minus i p dot x epsilon mu lambda p times the ground state. So, if you have a photon with a transfer polarization then this is what you are going to get.

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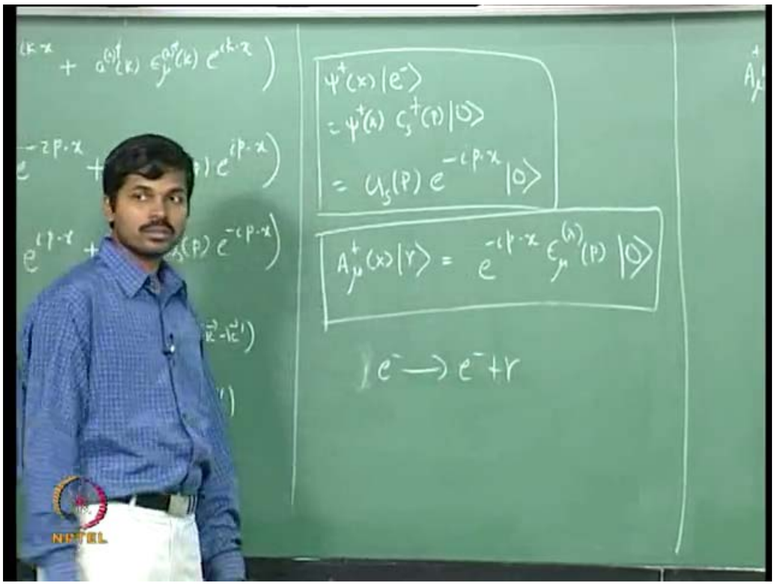


So, let me summarize, at here this is what you will get for the electron, and for photon. You will get a mu plus X gamma is equal to e to the power minus i p dot x epsilon mu lambda p and the ground state. So, we will now use these two results and their conjugates to evaluate the amplitude for this process, e minus going to e minus plus gamma.

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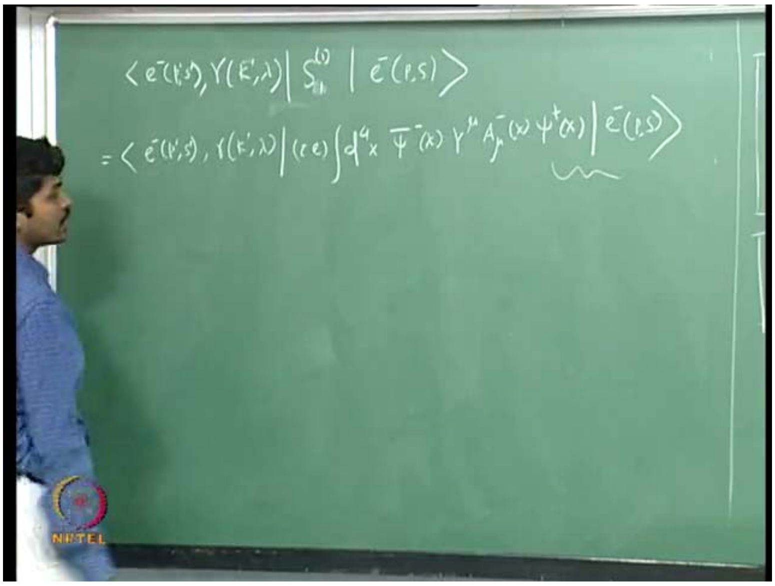

So, let us do that, so what we need is, we have an outgoing electron of momentum p prime and a polarization s and you have an outgoing photon of momentum, let us say k prime and polarization lambda, this is what we have. So, we have already seen, which term in the S matrix, that may give a non zero contribution, let us see minus p prime s prime gamma k prime lambda and i e times integration d 4 X psi bar minus of X gamma mu a mu minus of X psi plus of X acting on e minus p s.

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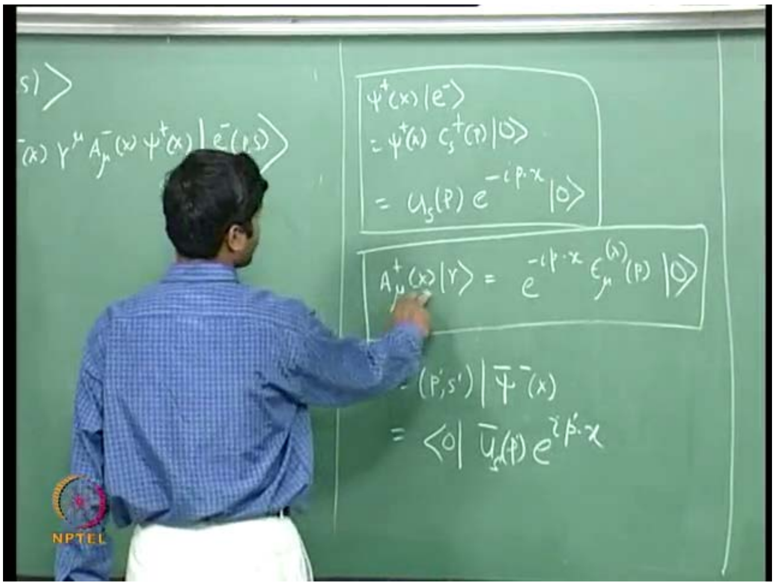

So, this part I can directly substitute it here and this one as you can see, psi bar minus acting on these will give me, from this term itself I can conclude that, if I have an electron state e minus of e prime X prime, if psi bar minus of x on it from right then this will give me 0 times u s bar of p e to the power i p dot x, p prime and s prime.

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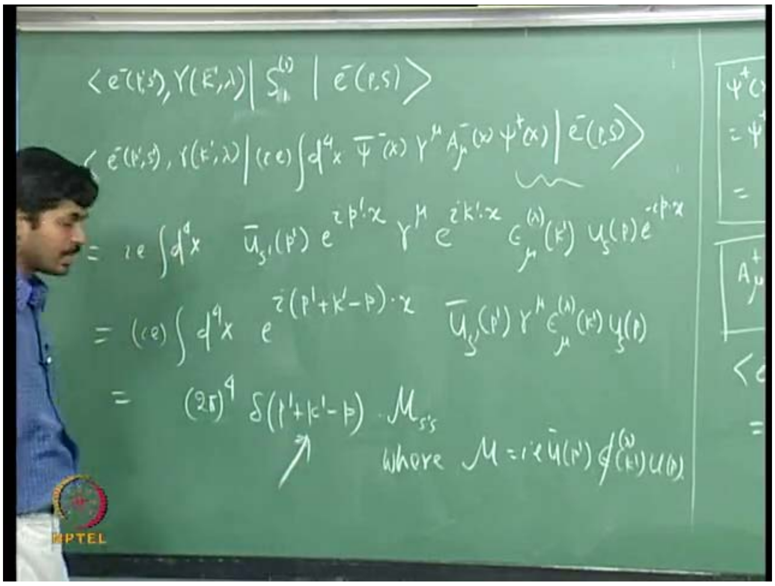

Similarly, I have, if a mu of X, a mu plus X acts on this state then what I will get here is i e integration d 4 X then these are numbers as far as the ground state is associated, because these are the operator from the ground state, but these numbers simply multiplied them. So, I can just tack these things out here, as far as this state is concerned, these are just some multiplicative coefficients. So, this is just 0 0, I will normalized the ground state to be 1 then all that is left with it is u bar s prime of p prime e to the power i p prime dot x.

And then gamma mu then when m u minus acts on this, what I have is e to the power i k prime dot x and epsilon mu lambda k prime and then this one is just u s p e to the power minus i p dot x. Epsilon star, so let us consider the photon to be all transverse and rear. So, here write this in principles sort of a star, but I will not, I will assume from now on that, all the incoming and outgoing photons are represented by real polarization as a transverse.

So, what I have is i e d 4 X e to the power i p prime plus k prime minus p dot x and then you have u s prime bar p prime gamma mu epsilon mu lambda k prime u s of p, this quantity I will call this to be. So, i e times this quantity, I will represented it by m, which I will cal as the Feynman amplitude. So, what I get is 2 pi 4 delta, the four dimensional delta function p prime plus k prime minus p times m, where m s prime s where m means, i e times u bar p prime epsilon slash lambda of k prime u p.

So, this is what I get for this amplitude, but now we will see, because this delta function is there, whether it can actually be satisfied for this process. So, this is what is imposed by the delta function.

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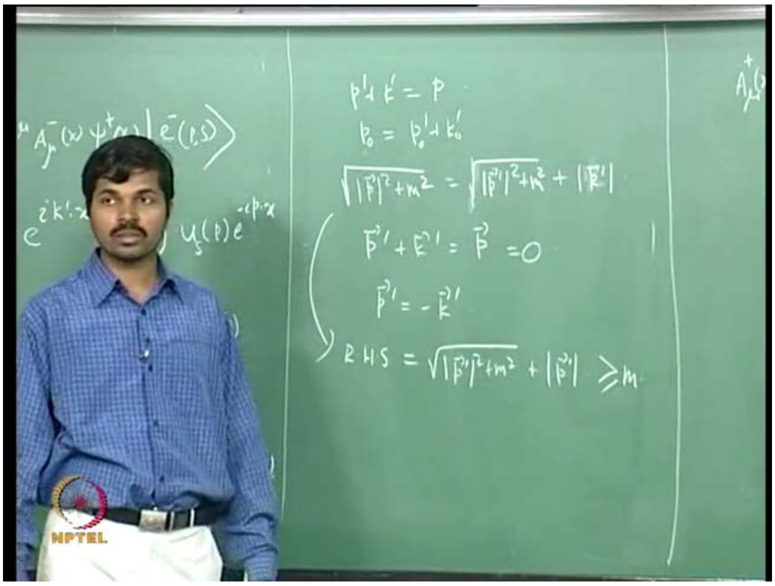

It p prime plus k prime, so it actually be equal to p, therefore p 0 will have to be equal to p 0 prime plus k 0 prime or since p 0 is the energy of the electron, it is nothing mod p square plus m square, should be equal to square root of mod p prime square plus m square plus mod k 0 is more simply mod k prime, because photon is massless. What about the i th component of the momentum, p prime plus k prime is equals to p, now since the incoming electron is a massive particle, you can go the rest frame of the incoming electron and then you can do this calculations here.

So, in the rest frame of the incoming electron, p equal to 0, therefore in the rest frame, p prime is equal to minus k prime. Now, what is this equation gives me in the rest frame, this gives me, so the RHS is simply equal to mod p prime square plus m square plus mod p prime. This quantity is as you can see, this must be greater than or equal to m and the equality will be saturated, this inequality is saturated only when p prime is equal to 0. So therefore, this will be satisfied only when p prime and k prime is equal to 0 or in other words, when there is no outgoing photon at all.

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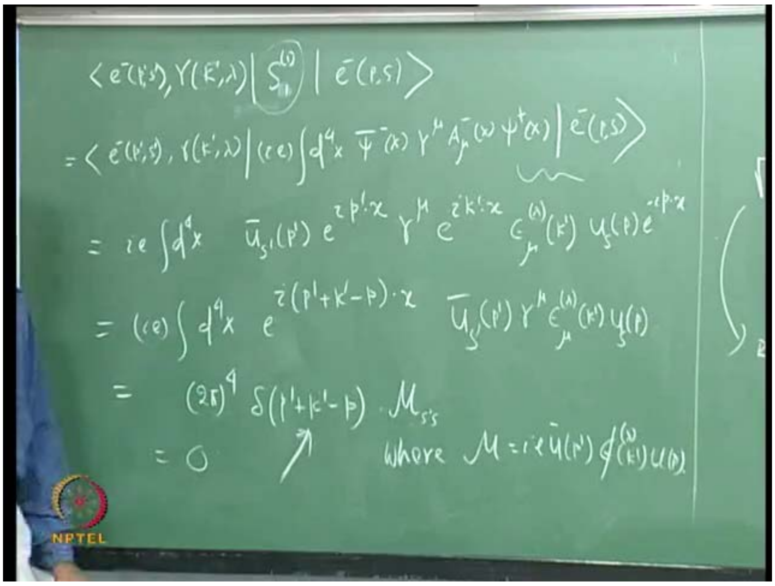

For k prime non zero, this will not be satisfied, therefore this delta function, because of delta function, this amplitude will always 0 for any outgoing real photon. So, you can see, you can try to consider what are possibilities in the S matrix, and you can see that, this first attempt does not give any contribution at all, because you will always get delta function like that and then you will see that, it will never be satisfied at all. So, there is no physical process, to which the first order term in the S matrix gives any contribution. Now, what we will do is, we will consider as simple example at the second order. Let us say, let us consider the example of Compton's scattering.

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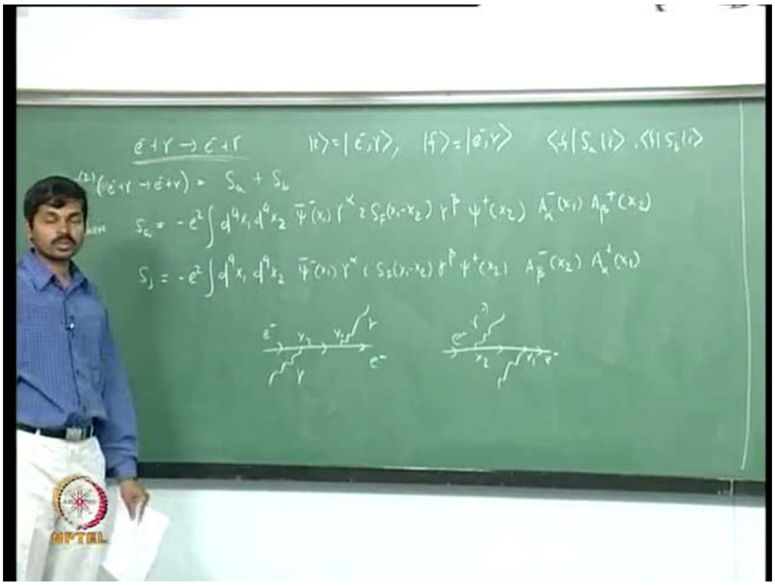

So, in the case of Compton's scattering, we have a e minus plus gamma going to e minus plus gamma and we saw that, at the second order, there are two terms which can contribute. So, S 2 e minus plus gamma going to e minus plus gamma as two terms, which I will call as S a plus S b, where S a is equal to minus e square d 4 X 1 d 4 X 2 psi bar minus X 1 gamma alpha i S F X 1 minus X 2 gamma beta psi plus of x 2.

And S b is minus e square d 4 X 1 d 4 X 2 then psi bar minus X 1 gamma alpha i S F of X 1 minus X 2 gamma beta psi plus X 2 times A beta minus of X 2 A alpha plus X 1, this is what we will get. Strictly significance, because there is a normal ordering, I should have written it here, but it does not matter, because this A minus commutes with psi plus, same thing here. And there are two diagrams, which actually contributes to these two processes, one of them is given by there is an incoming photon and there is an outgoing photon at X 2 X 1, this is gamma, this is e minus, e minus.

So, another one is, there is an outgoing photon at X 2 e minus and there is the incoming photon is observed at X 1, these are the two processes, a physical processes. So, what we need to do is, we need to consider the instate, which is it is one electron and one photon and the out state or final state f again, there is one electron and one photon. We need to consider the spin and the polarization momentum of the photon as well as the electron in initial and final state.

And we need to consider, we need to find what is the amplitude for this process, what is f S a i and what is f S b i. And these two matrix elements we need to consider and we need to take this sum of these two matrix elements to compute the amplitude for this process at lowest order. So, let us evaluate first this quantity here, we already know, what we get when psi plus X 1, the initial state and so on. So, let us do that and see, what we get for the amplitude.

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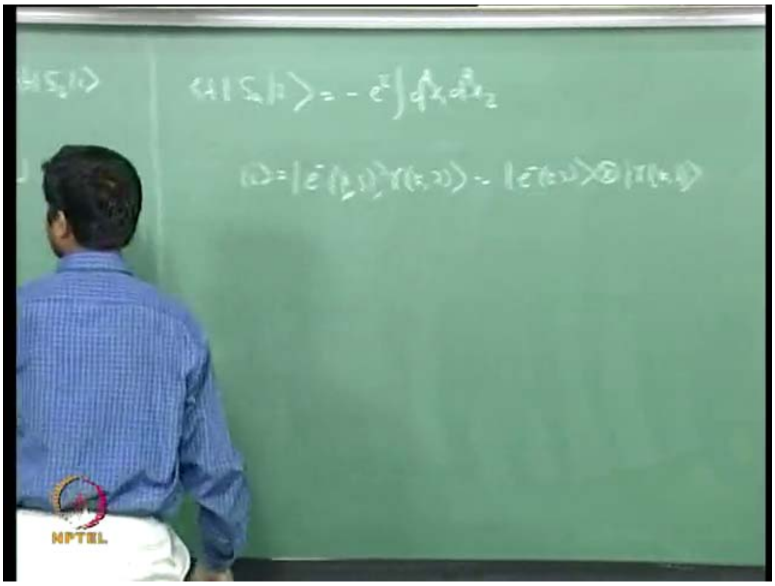

So, I will write f S a i, this is minus e square d 4 X 1 d 4 X 2 and then so this initial state here i and again has some electron of momentum p and polarization I will call this to be s and it is a photon of momentum k and polarization lambda, this I can take is e minus p s and gamma k lambda and this state, this psi plus will act on this one. Whereas, the A plus will act on this state here.

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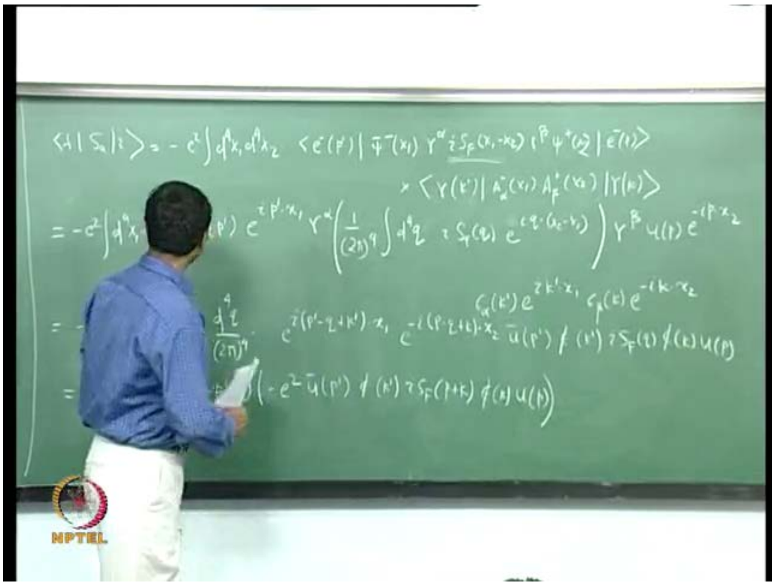

So therefore, what I can do is, I can just consider the inner product e minus p prime, let us suppress the spin index for the moment, we will keep track of this spin and this is later on, psi bar minus X 1 gamma alpha i S F X 1 minus X 2 for the Fermi on propagator then gamma beta psi plus of X 2. This acts on the incoming electron and it is momentum p and spin s, this times the photon of momentum k prime A alpha minus X 1 A beta plus X 2 acting on the photon state of momentum k.

So, what is this, this we will already know, we have evaluated this piece, we have a evaluated this piece, we have evaluated this one and as well as this one. So, all these thing we already know, so we can put all these things together and then we will see what we get. So, let us do that and finally, what I will do is that, this Fermion propagator I can write it in terms of it is four year component. So, this quantity is minus e square d 4 X 1 d 4 X 2 and then this will give me, as we have already seen u bar of p prime e to the power i p prime dot x one.

And then this one gamma alpha, this one in terms of it is four year component is given by 1 over 2 pi, fourth the integration d 4 q i S F q times e to the power minus i q dot x 2 minus x 1. You will see very quickly, why I wanted to write it this way and then I have gamma beta and then psi plus acting on e minus will give me u p e to the power minus i p dot x 2. Then finally, I have this which is nothing epsilon alpha k prime and I am suppressing this spin as well as the polarization indexes, e to the power i k prime dot x 1 and epsilon beta of k e to the power minus i k dot x 2, that is all I have.

So, let us rearrange all these terms and then see what do we get, what I will do is, I will collect the coefficients of e to the power, I will put all these exponential terms together, I will write it in terms of two terms. One collecting in the coefficients of x 1 and then I will carry out the X 1 integration here. Then the second exponential, I will also again I will carry out these this second integration, so this is nothing minus e square d 4 X 1 d 4 X 2 and then d 4 q over 2 pi fourth.

And finally, e to the power i p prime minus q plus k prime dot x 1 e to the power minus i t minus q plus k dot x 2 and u bar of p prime epsilon slash of k prime i S F q epsilon slash k u of p. So, as you can see, you have e to the power i p prime plus dot x 1 e to the power i k prime dot x 1 and then e to the power, so let us write it as plus and then e to the power minus i q dot x 1 here. Whereas, in the second term, I have if I just look at the x 2’s, I have e to the power i q dot x 2, here i minus i p dot x 2 and minus i k dot x 2, that when I put together I get here.

And then I took out the q integration outside and then remember this S F is an operator which is inserted between gamma alpha and the gamma beta. So, I do not have the freedom of moving it anywhere else, I have to do that, but this epsilon alphas are numbers, so I can write, I can bring this epsilon alpha here, that will give me epsilon slash k prime. So, you have this u bar p, this is a number, I just took it outside, but this row is already there.

So, u bar p and then gamma alpha, epsilon alpha will give me epsilon slash k prime then I have S F q which is here and finally, I have the gain this gamma beta, I can take this number epsilon beta k, that will give me epsilon slash k and finally, I have u p. Remember, i do not have the freedom to move these things anywhere other than where they are, of course I can write them terms of their components and finally, I can move the components around, because they are just numbers, but these are not numbers, so they are there.

Now, I can carry out the X 1 and X 2 integration and also I can carry out the q integration. First let us carry out the X 1 and X 2 integration, that will give me the two delta functions. Here I get delta p prime minus q plus k prime, here I will get delta p minus q plus k and when I integrate out over q, I will get delta p prime plus k prime minus p plus k. So, this is what I get and then there are this factor of 2 pi to the power fourth is here.

But, when you carry out two X integrations, you will get two factors of 2 pi to the power fourth in the numerator. One of them will cancel with this one and the remaining is given by 2 pi to the power fourth delta p plus k minus p prime minus k prime, times this quantity, minus e square u bar p prime epsilon slash k prime i S F of, q is nothing p plus k epsilon slash of k u p, so this is what is the Feynman amplitudes. So, what I did is, I evaluated this and then I wrote in the form of a delta function, times sum matrix here.

I will do this similar thing for S b and then finally, to get the full amplitude for this process, I will add them up. But, again I will give you a set of rules so that, just by looking at the Feynman diagrams, you can evaluate this matrix element here, instead of going through all these algebras, which we will do in the next lecture.