Quantum Field Theory
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Module - 3
Free Filed Quantization: Spinor and Vector Filed
Lecture - 21
Fermion Quantization V

So, we are discussing the quantization of radiation, quantization of the Dirac spinor. What we have done so far is we have constructed the linearly independent solutions.

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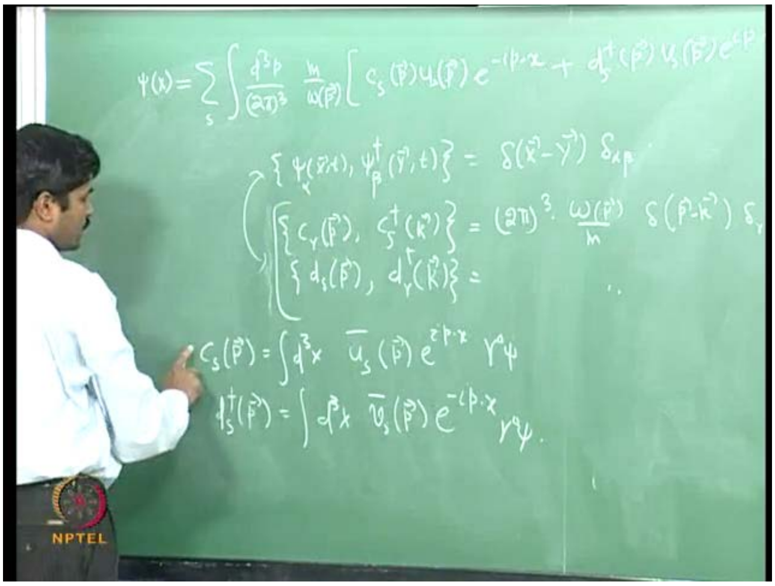

Then, we have seen that the most general solution can be expressed as a sum over s integration d cube p over 2 pi cube m over omega p, then c s p u s p e to the power minus i p dot x plus d s dagger p v s p e to the power i p dot x. Similarly, you have psi by x. What we have seen is that in order to get fermi Dirac statistics, we need to adopt anti commutation relation. Therefore, what we have seen is that psi of x t psi dagger, i times i dagger is the conjugate momentum to the t psi. Therefore, psi alpha x t psi dagger beta of y t is nothing but delta x minus y delta alpha beta.

So, this implies that the operators c s and d s daggers the satisfy the following anti commutation relation, c r p c s dagger k is 2 pi cube times omega p over m delta p minus k delta r s and a similar expression for ds and then all that anti commutation relations.

So, we shown that this relationship here this and d s p d r dagger k again is equal to the same thing. What we saw is that this is equivalent to this. You can also show that starting from this that these anti commutation relations are satisfied, all you need to do is you need to have an expression for this operator c s and d s daggers in terms of the field psi, which I will write it down here for you.

Then, you can verify that this is in fact through c s of p is equal to integration d cube x u s bar of d e to the power i gamma 0 psi and d s dagger of p is equal to d cube x u s bar of p e to the power i p dot x gamma 0 psi. So, you take this anti commutation relation or you take, you consider this, you substitute this for the c s and d s and then use these anti commutation relations. You will be able to derive these relations. Anyway, so what we will do now is that we will consider the propagator for the Dirac spinor. Then I will derive an expression for the propagator.

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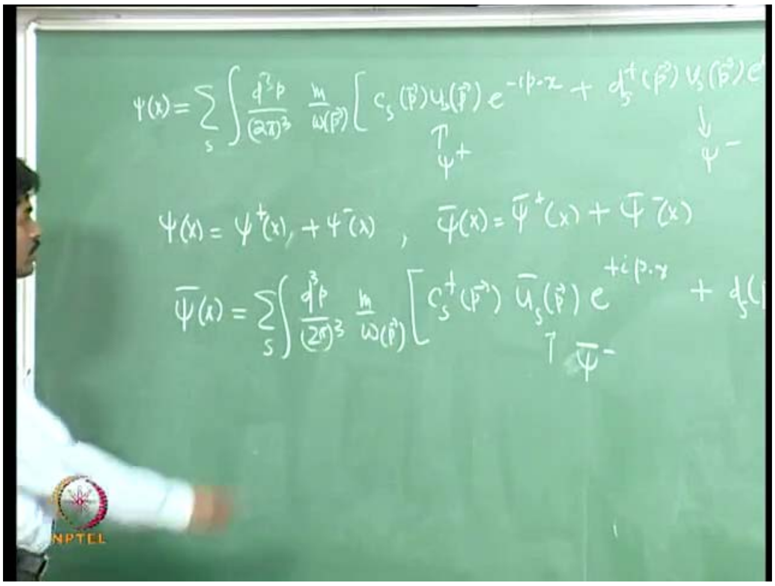

So, to do that, let us define psi plus to be this part here, and psi minus to be this, so you can write psi of x is equal to psi plus of x plus psi minus of x. Similarly, you can consider psi bar of x. It is some over s d cube p over to 2 pi cube m over omega p c s dagger of p u s bar of p e to the power plus i p dot x.

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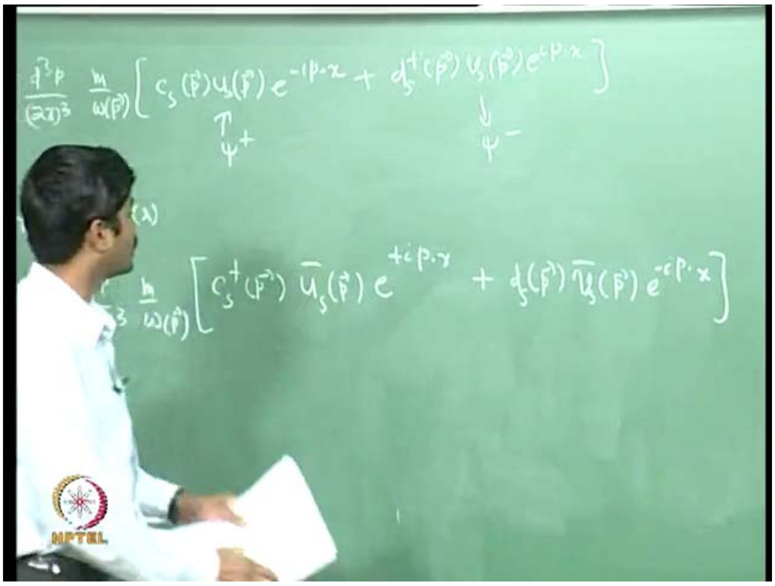

Here, you have d x of p d s bar of p e to the power minus i p dot x.

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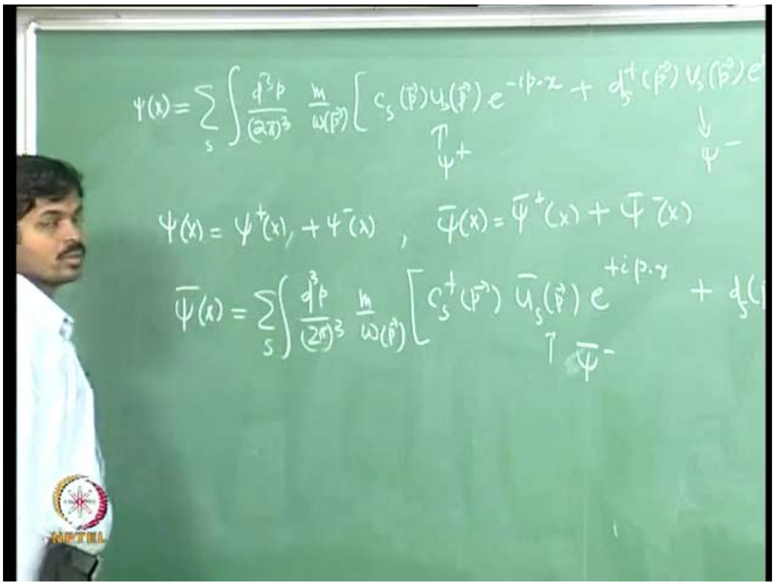

So, because of this e to the power i p dot x dependence, we denote this is to be psi bar minus and this to be psi bar plus. So, therefore psi bar of x again is psi bar minus of, psi plus of x plus psi bar minus of x.

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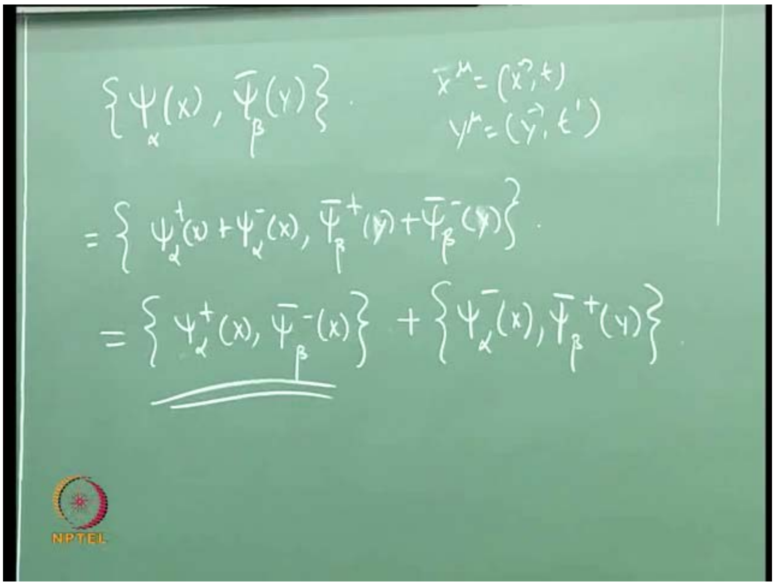

Now, what we would like to do is we would like to derive this anti commutation relation psi of x psi bar of y where x is, x mu x t, y mu just y p prime. So, these are not equal time anti commutation relations. The time here need not be the same. We will derive it in terms of the component of the spinor. So, I will write it as psi alpha psi bar beta and you can see that this here is anti commutation of psi alpha plus of x plus psi alpha minus of x with psi bar beta plus of x plus i bar beta minus of x, y. Thank you.

Now, you can see that it is of course obvious, psi plus actually here contain c s and psi bar plus contains d s. Therefore, the anti commutation of psi plus with psi bar plus is 0. Similarly, the anti commutation off psi minus with psi bar minus it is 0. So, all you will be left with this psi alpha plus of x psi bar beta minus of x and then here psi alpha minus of x and psi bar beta plus of y. That is what you will have.

So, now let us derive the expression for one of these relations, which we will compute this explicitly. Then I will just give you the answer for this expression. To do that, let us explicitly use the expressions for psi alpha plus and psi bar beta minus.

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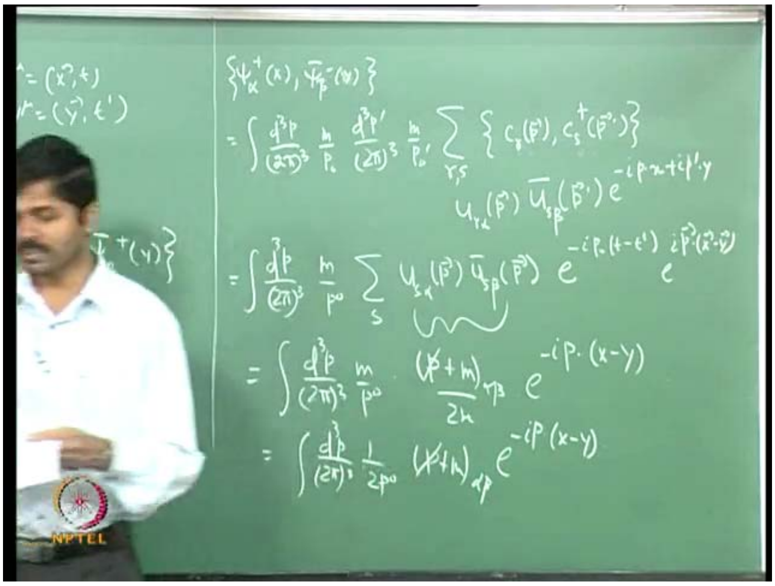

So, if we do that, then what I will get? So, evaluate psi alpha plus of x psi bar beta minus of x y is just equal to integration d cube p over 2 pi cube m over p 0 and d cube p prime of 2 pi cube m over p 0 prime. Then I have to carry out this spins some over r and s and c r of p c s of c s dagger off p prime this times u r alpha of p u s beta of p s bar beta p prime and t to the power minus i p dot x plus i p prime the y, so this psi of them by state forward off substitution.

Now, I will use the anti commutation relation for c r p c s dagger p prime. Then I will carry out the p prime integration. What I will get as a result is integration d cube p over 2 pi m over m over p 0 and anti commutation here will cancel m over p 0 prime and 2 pi cube and the delta constant that comes here will remove the d cube p integration. So, all I will have is sum over s u s alpha p u bar s beta of p e to the power minus i p 0 t minus t prime e to the power i p dot x minus y.

So, what is this quantity here? So, this of course, we have seen it many times. This we can substitute the expression for this d cube p over 2 pi cube m over p 0 and this one here is p slash plus m alpha beta component of p slash plus m divided by 2 m this times e to the power minus i p dot x minus y. So, this looks something very much similar to I mean this is just a d cube p over 2 pi cube times 1 over 2 p 0 because this m will cancel with this m. Then this p slash plus m alpha beta due to the power minus i p x minus y. If this p slash plus m alpha beta was not there, then we are very much familiar with this expression. We gave a name for this. We said that this is nothing but delta plus of x. So, if you recall the derivation of the propagator in this scalar field theory, then when you take the Klein Gordon field, this is what you get as a result.

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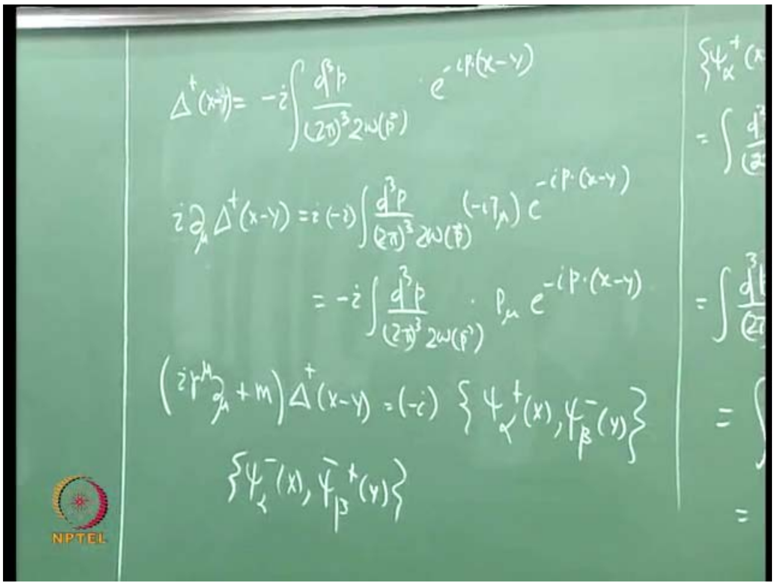

So, delta plus of x, I will use the notation that we have used earlier is equal to minus i d cube p over 2 pi cube 2 omega p e to the power minus i p dot x. So, however this is not quite. So, because I have this expression here, but let us consider this quantity. So, delta plus x minus y here is x minus y to get p slash plus m what i can do is that i can for example, if i consider i del mu of delta plus of x minus y. Then this will give me minus i d cube p over 2 pi cube 2 omega p and del mu will give me a factor of minus i p mu minus i p mu c to the power of minus i p dot x minus y. There is an additional i here. There is the minus i. Then there is i because of this i here, this minus i is because of this i p mu, there is a minus i here.

So, as a result, what I have here is a minus i d cube p over 2 pi cube 2 omega p and p mu e to the power of p dot x minus y. This is what I have got for this one. Therefore, if you consider this quantity i gamma mu del mu plus m acting on delta plus of x minus y, then what you get? So, this is precisely equal to this quantity with a factor of minus i here. There is no minus i, there is a minus i, so minus i times psi alpha plus of x psi beta minus of y. So, similarly, you can compute the other term, which is nothing but psi alpha minus anti commutator of psi alpha minus of x and psi bar beta plus of y. Most of the relations are identical. All you will get here is delta minus in the place of delta plus.

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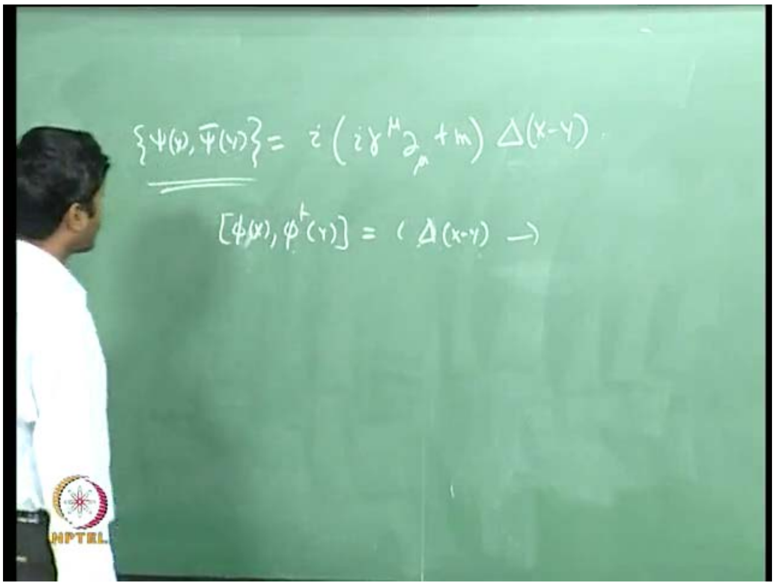

Therefore, the anti commutator psi of x psi bar of y, which is the sum of this two terms will be given by i times i gamma mu del mu plus m delta plus of x minus y plus delta minus of x minus y, which you have defined to be delta of x minus y. So, this is the notation that we have used. The reason I express it in this form is because we already know that when we considered the commutation relation, phi of x phi dagger of y in the complex scalar field theory, what we got is i delta x minus y and then this delta was 0.

What we have shown earlier is this delta of x minus y is actually 0 when x minus y is actually or x and y are actually separated by space like separation. So, here we can use the same argument, exactly the same argument. If x and y are space like with respect to each other, then this delta actually vanishes. Therefore, the whole thing in the right hand side will vanish. So, causality is not violated here because of this. On the other hand, if you had in this theory, if you had considered commutation relations in addition in place of anti commutation relations, you would have got the relative minus sign in between. Then you would not have got this delta of x here.

If you explicitly evaluate the commutation by using the fact that they obey equal time commutation relations instead of the anti commutation relations, then you will get a different answer here, which will not be 0 when x and y are space like separated.

Therefore, the causality will be violated if you consider for a spin of particle, if you consider equal time commutation relations instead of the equal time anti commutation relations. That is one of the reasons. So, there are various reasons. So, one of them is of course the spectrum is not bounded from below.

Then, the second thing is the spin of particles will not obey the form Dirac statistics. Finally, the causality will be violated if you consider commutation relations instead of the anti commutation relations. So, we will denote this quantity to be whatever we have here the anti commutation, anti commutator of psi alpha plus and psi beta minus, I will denote this quantity as s plus of x minus y, so x plus alpha beta of x minus y.

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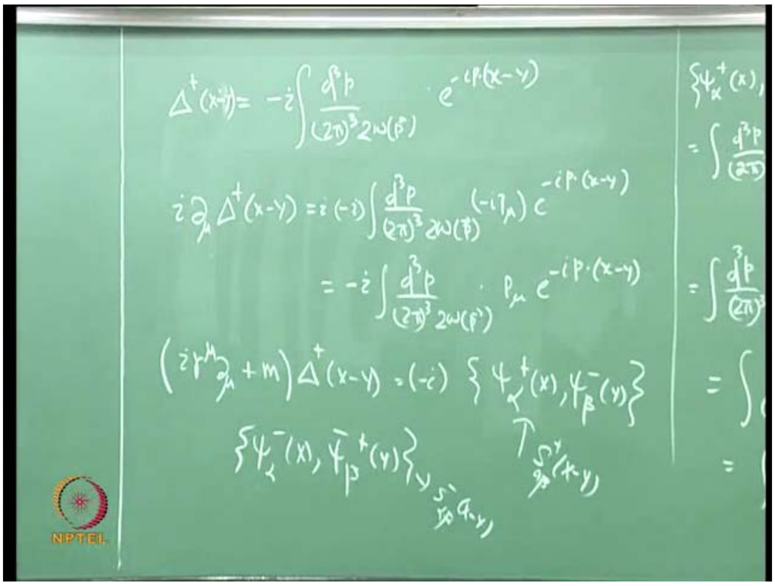

Similarly, here you will have s minus s minus alpha beta of x minus y for this.

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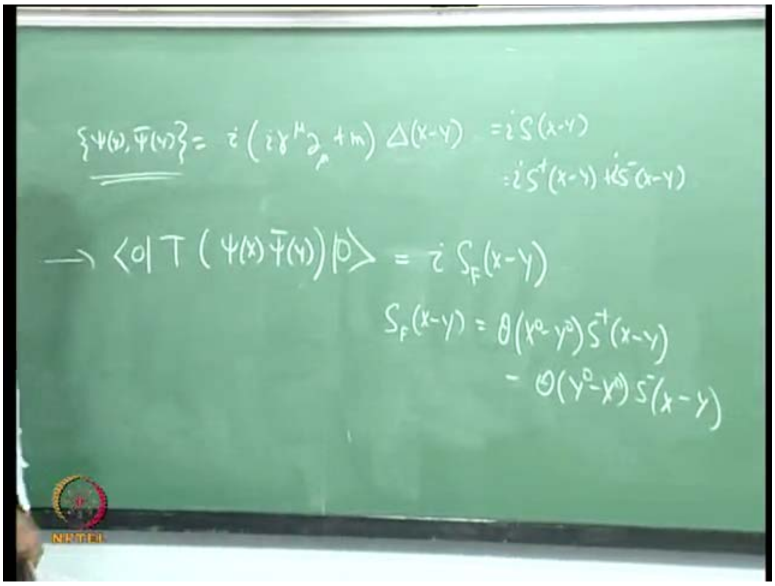

So, this quantity here I will denote has s of x minus y where s of x minus y is simply s plus x minus y plus s minus of x minus y. So, you take delta plus and then you act this operator here on it. What you will get is denoted by s plus and similar expression for s minus. You can do analogous calculation for the vacuum expectation value of time ordered product. So, time ordered of psi of x psi bar of y, this is what you can consider.

This will give you the Feynman propagator for the Dirac spinners, which I will denote as i. So, there is i here in all this definitions, i times SF of x minus y where SF of x minus y is the Feynman propagator for Feynman. This is nothing but theta of x 0 minus y 0 s plus of x minus y plus or minus theta of y 0 minus x 0 is minus of x minus y. This is the final propagator. So, I leave it as a home work for you to compute the vacuum expectation value of this time order product and on. So, this is in fact equal to this quantity. Any question? Now, let us look at this vacuum in more detail. Then let us see what the quantum numbers that we need to label these states are?

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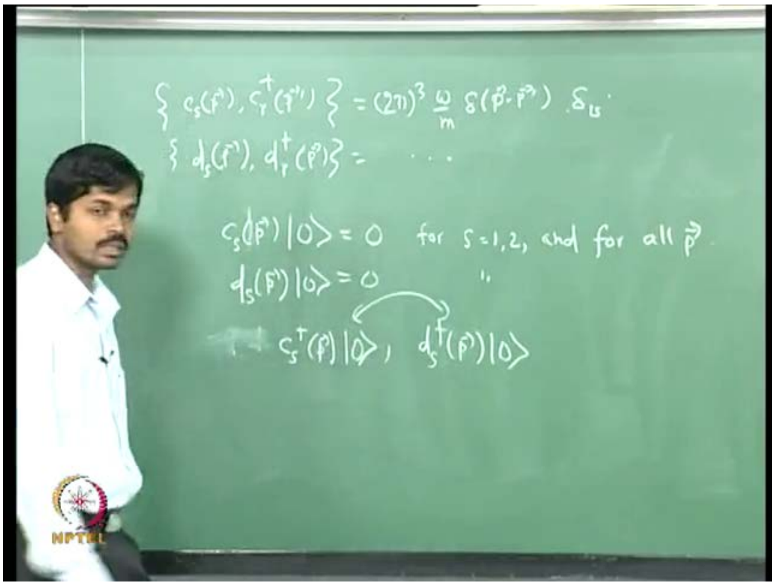

So, we have already the anti commutation relations c s p c r dagger of p prime this 2 pi cube whatever, 2 pi cube and omega over m delta p minus p prime at similar relation for d s of p d r dagger of p prime. This suggests that this c s is the new lesion operator and the c dagger at the creation operator. Then vacuum is annihilation by c s p. This is equal to 0 for s equal 1, 2 and for all p.

Similarly, the vacuum is also annihilated by d s p for s equal to 1, 2 and for all values of p. There is delta r s. Thank you. The particle states are for example, created by acting this creation operators c s dagger of p acting on the this and d s dagger of p acting on the vacuum. I have already briefly stated in the last lecture that this appearance of two different types of creation and annihilation of operator implies that there exist two kinds of particles. These two types of particles, there is an additional quantum number to distinguish between these two particles and this quantum number comes from the charge operator.

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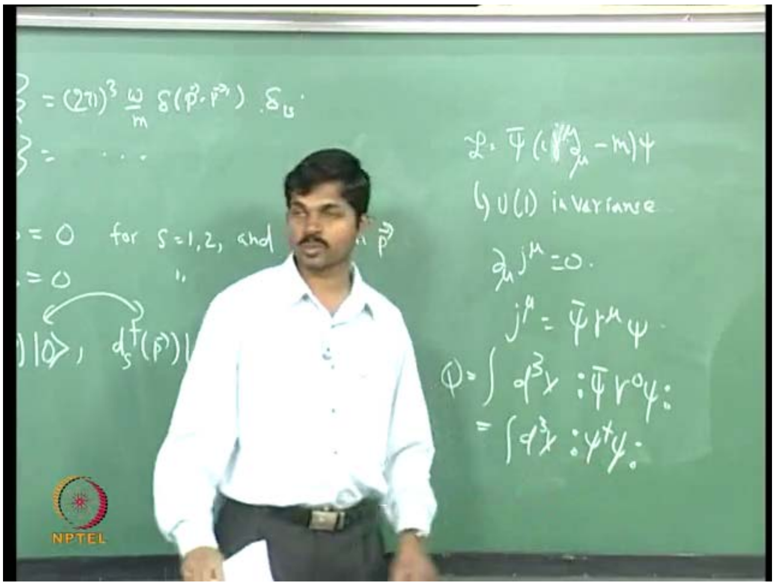

We have already seen that the Dirac Lagrangian is in fact psi bar i gamma mu del mu minus m psi. This has u1 invariance, global u1 invariance. If you consider this transference psi going to e i theta time psi, then this Lagrangian density is in variant under transformation. As a consequence, we get a conserve del mu j mu equal to 0 where j mu is equal to psi bar gamma mu psi. This quantity is the charge operator, Q. The total charge is given by d cube x.

All physical quantities are considered within the normal ordering. So, this is psi bar gamma 0 psi or this is nothing but d cube x normal order product of psi dagger psi. We know the mode expansion for the psi and psi dagger for these two fields. So, you can substitute the mode expansion. We can use the anti commutation relations here. Finally, what we will get is I will write down the expression for the charge operator.

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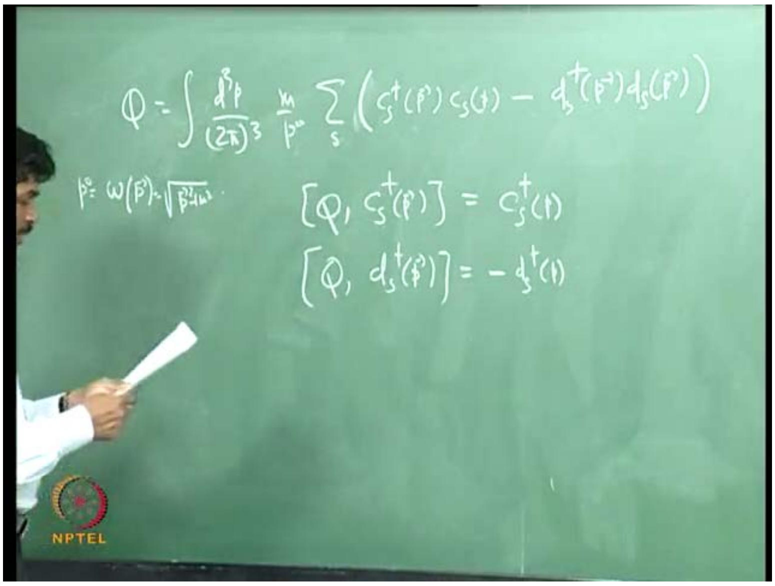

So, after you substitute the mode expansion, what you will get is q is d cube p over 2 pi cube m over p 0. Although we write p 0 in all these expressions, I assume implicitly that this p 0 is not independent, but it is omega. It is square root of p square plus m square. This is the condition that we are imposing and then sum over is c s dagger of p c s p minus d s dagger of p d s of p. Because of the minus sign here, you can show that this implies the commutation relation between q and c s dagger of p is nothing but c s dagger of p. The commutator of Q d s dagger of p comes with a minus, d s dagger of p.

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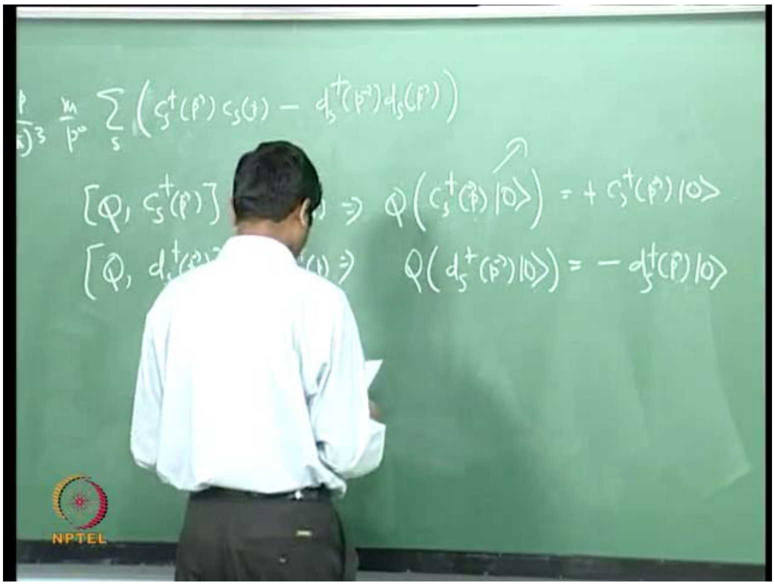

As a result, if you consider a state like this, so this implies that Q acting on c s dagger p vacuum is a plus c s dagger p acting on vacuum, whereas this equation here implies that the charge operator acting on the d s dagger of p 0 is nothing but minus d s dagger p of 0. Therefore, these are in our convention, positively charge particles and these are negatively charged particles because of the plus and minus sign here. Similarly, we can in addition to charge, we also have spin. So, what you can show in fact is that you can choose an appropriate basis, and then I will not derive it again here.

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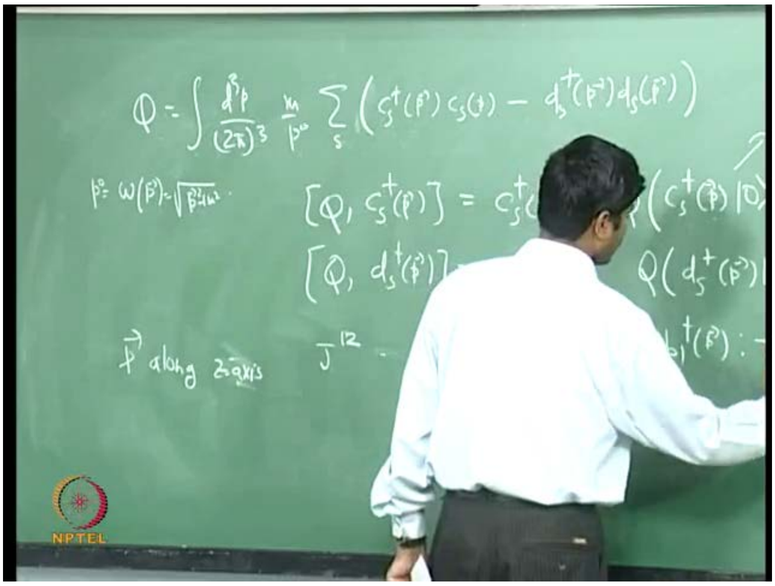

You can consider p to be along z axis. Then in some suitable basis, you can consider the Eigen states of J 12 operator.

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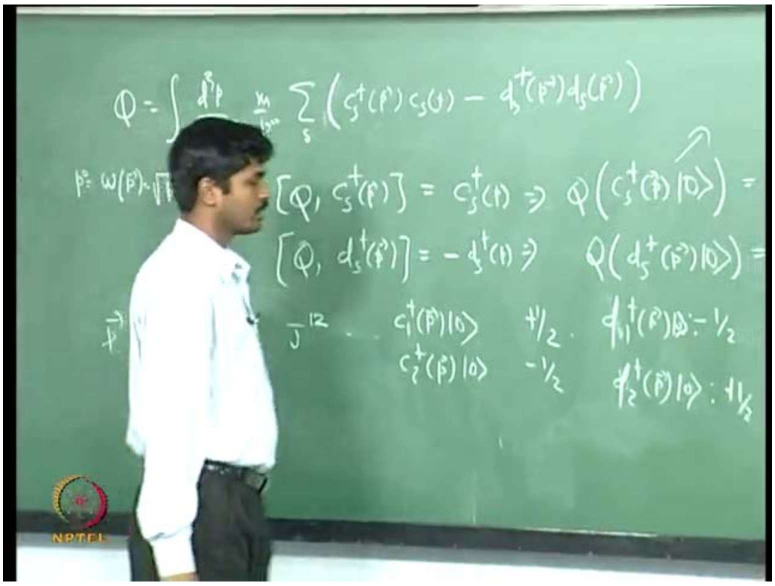

Then, I will say, I will just say that c 1 dagger of p, this have Eigen values plus half, c 2 dagger of p is a state with spin minus half, b 1 dagger of p is a state with spin minus half and b 2 dagger of p acting on ground state is a state with spin plus half, d 1. Thank you, d 1 and d 2. So, this appearance of the particles of two different charges, the charges positively and negatively charged particles suggest that there must be some symmetric with actually relates this to particles.

If you look at their energy Eigen values both, the energy Eigen values are both particle will be the same, but they have opposite charges. So, there should be some symmetry, which relates particles of positive charge, but particles of negative charge. It is known as the charge conjugation. Charge conjugation is a discrete symmetry. It is not a continuous symmetry. It is a discrete symmetry. There are over there discrete symmetries. For example, parity is also discrete symmetry. Time reversal is a discrete symmetry and charge conjugation is also discrete symmetry. So, what we will do in next few minutes is I again I will briefly discuss about this discrete symmetries that is parity, charge conjugation and time reversal. So, let us discuss parity first. What is the parity transformation?

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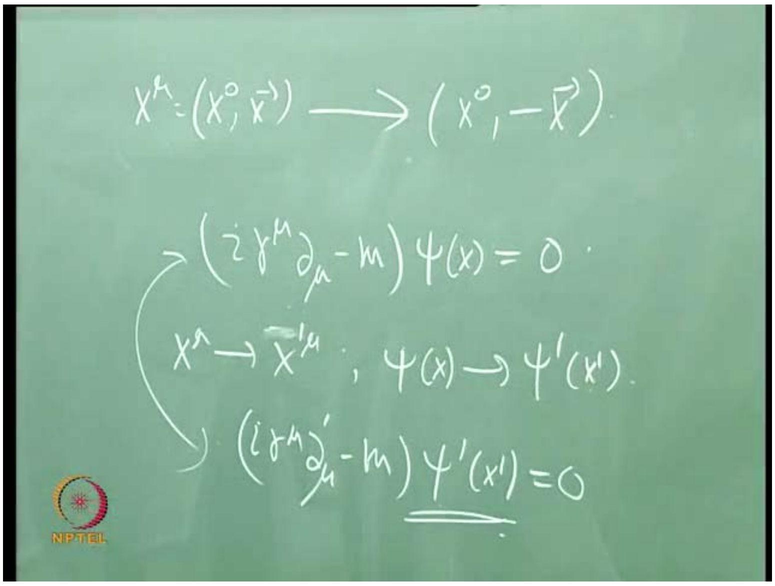

You consider x mu, which is x0, x under parity. Parity transformation is transformation which just splits. It keeps the x0 co ordinate the same, but it changes the sign of the spatial coordinates. So, what happens to the Dirac equation under parity? We will like the Dirac equation to be covariant under the transformation. What is the Dirac equation? It is i gamma mu del mu minus m psi of x equal to 0.

So, suppose x mu goes to x mu tilda x mu prime, which is just this. Then this equation has to be psi of x goes to psi prime of x prime, then psi prime of x prime. So, it satisfies this equation, i gamma mu del prime mu minus m acting on psi prime of x prime equal to 0. So, we need to know. So, the question is given this transformation, what is psi prime of x prime so that these equations are consistent with each other? So, let us consider the second equation.

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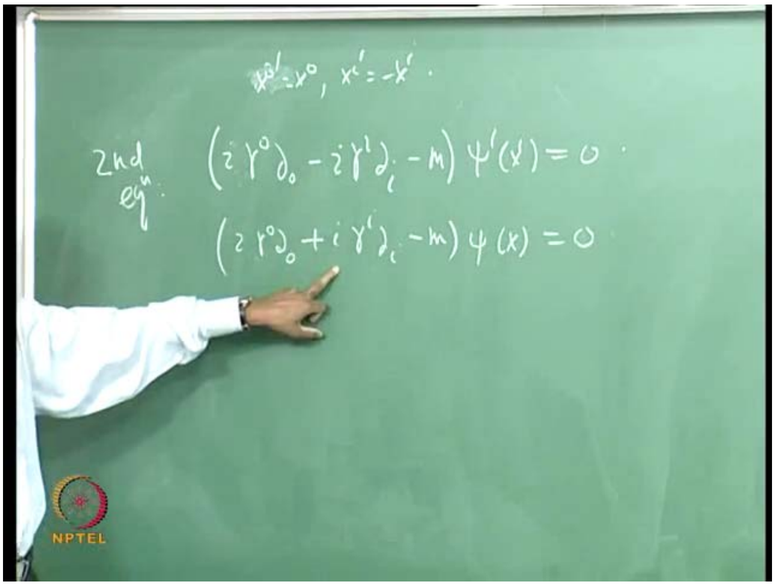

It is just i gamma 0 del 0 and minus i gamma i del 1, minus because I am flipping the sign of x and under the transformation, x0 goes to x0. So, x0 prime is x0 and xi prime is minus xi under parity. Therefore, the second equation is this minus minus m psi prime of x prime equal to 0. Now, what I need is I, let us compare this with this equation, i gamma 0 del 0 plus i gamma i del i minus m psi of x equal to 0. So, if you compare these two equations, then you would see that I mean there is this of course this crucial sign difference here.

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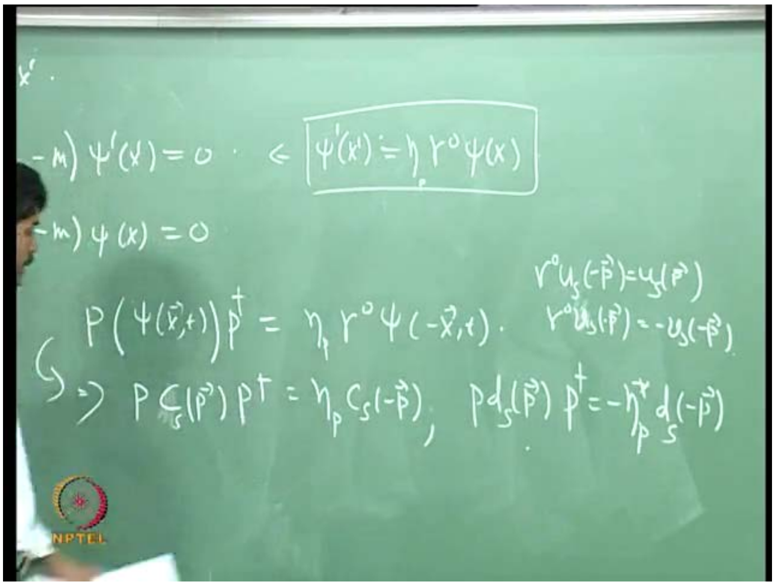

These two equations will be said consistent with each other provided psi prime of x prime is gamma 0 psi of x because if you substitute here psi prime of x prime gamma 0 psi of x, then here this minus sign will be as it is. This is because this is m times reversal identity operator, but here if you take the gamma 0 to the left, it will flip a sign. So, it will become plus, but here the sign will not be seen because the gamma 0 appears here again.

Therefore, psi prime of x prime must be some phase, which I call eta p. It must be equal to gamma 0 psi of x. If psi prime of x prime is equal to this, then the Dirac equation is covariant under parity. Now, what you do is you just use this relation and then you can derive. So, the point is that under parity p, if is exists, so you say that there exists a unitary operator p says that it takes psi of x t p dagger to psi prime of x prime p prime, which is nothing but eta p gamma 0 psi of minus x t. So, this will tell you how the parity operator acts on the creation and annihilation operators.

So, this will not derive the relation again. This implies p acting on b s c s p, p dagger is eta p c s of minus p and p d s of p p dagger is minus eta p star d s of minus p where I have used the basis v gamma 0 u s of minus p is u s p and gamma 0 u s p is gamma 0 v s of minus p is minus v s of minus p. So, you can choose spin of basis where u s and v s satisfy this relation. Then this implies the creation and annihilation operator satisfy this. So, the question is what is this operator p so that this relation is satisfied? You can check if you consider.

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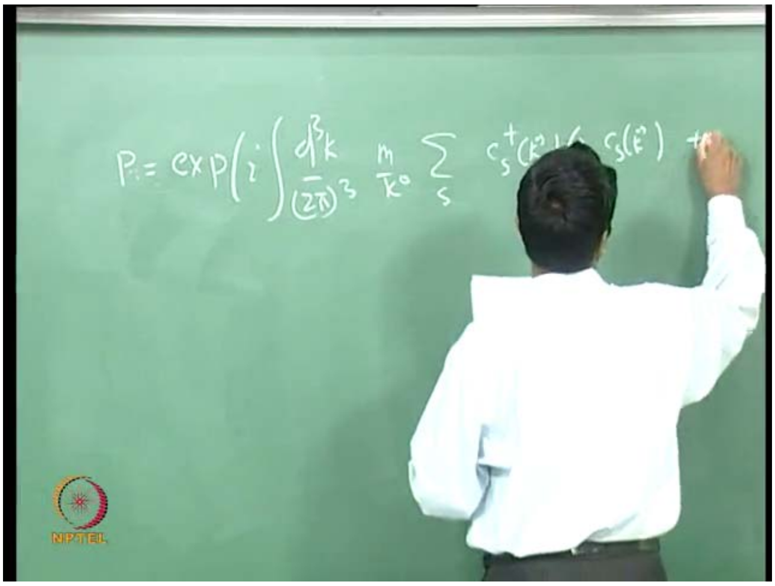

So, this is again a homework for you p equal to exp i d cube k over 2 pi cube m over k 0 sum over s c s dagger of k lambda c s of k plus pi by 2 c s of minus k…

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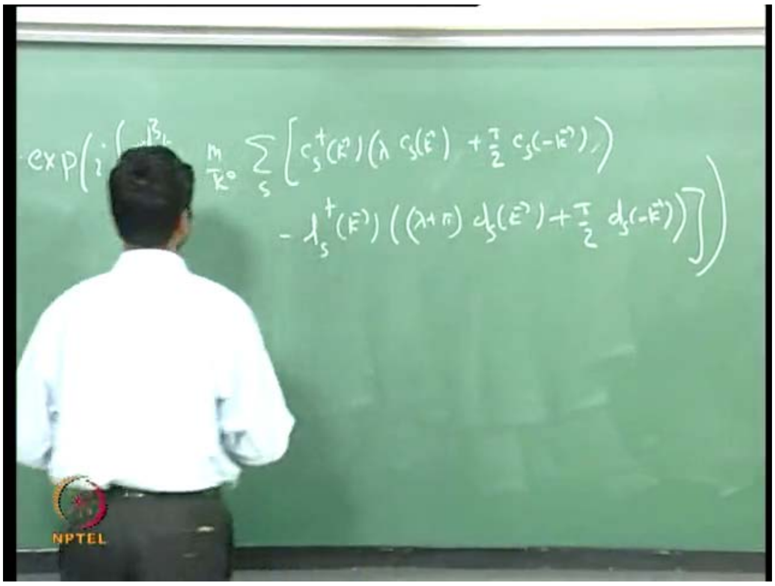

Minus d s dagger of k lambda plus pi d s of k plus pi over 2 d s of minus k. So, you can explicitly check that this is in fact unitary operator and this operator is this property. So, you check this. You find what the expression for eta is.

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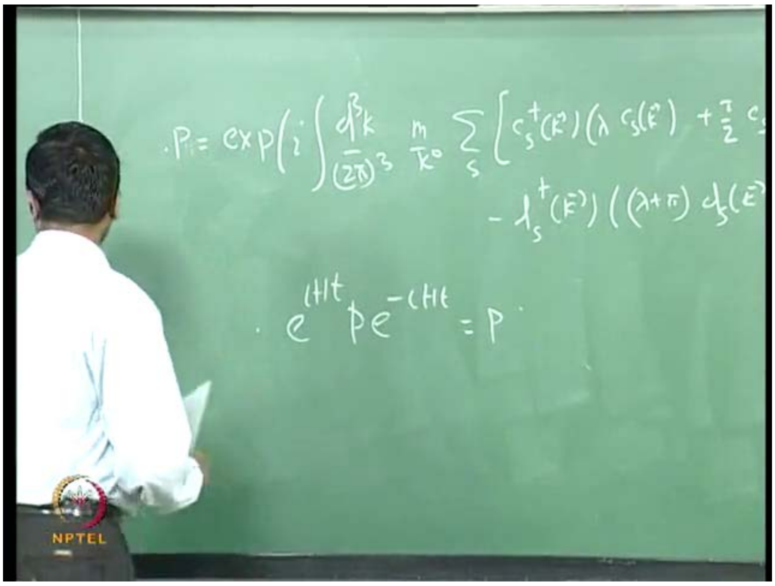

Also, you can show that this in fact is the symmetry of the theory in this sense that if you consider e to the power i H t p e to the power minus i H t, it keeps the parity operator invariant. So, this is equal to p. So, just check this. I mean that if this is your parity operator, then this in fact satisfies this relation and also this relation here. So, this is just show that I mean you have I mean there exists a parity operator which is the in fact the symmetry of theory that we considering. So, in the next lecture, we will discuss charge conjugation and time reversal. Then we will discuss interaction of the fermions with the electromagnetic field.

Student: Sir, what is lambda here?

Lambda is some the arbitrary parameters. So, what you will do is that you will get an expression for these pairs in terms of lambda.