Quantum Field Theory
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Module - 3
Free Field Quantization: Spinor and Vector Fields
Lecture - 20
Fermion Quantization IV

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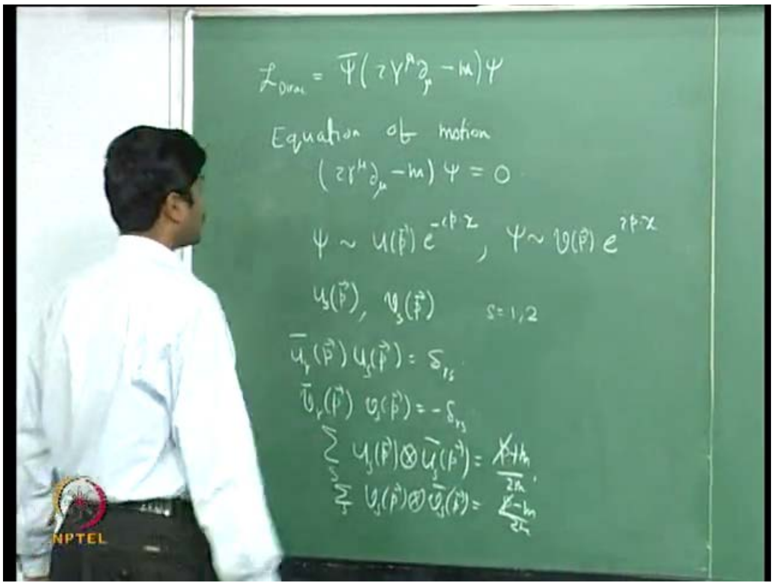

So, in the last lecture we have constructed a Lagrangian density for the Dirac spinor, in the Lagrangian density is given by psi bar i gamma mu del mu minus m psi, from which we can derive the equation of motion. And the equation of motion for psi is given by, i gamma mu del mu minus m psi equal to 0, this admitted both positive frequency as well as negative frequency modes. They are given by psi equal to u of p, e to the power minus i p dot x. So, if you try to look for a solution of this kind, then we see that there are two linearly independent solution.

So, there are two linearly independent four component spinors recycle as u s p and if I look for a solution which is of the kind psi going to v of p, e to the power i p dot x. Then again, there are two linearly independent four component spinors, which I will denote as v s of p. These, where s runs from 1 to 2 so these spinors u s p and v s p they satisfy a number of identities, which we have derived in the last lecture, which I will summarize here, u bar r p u s, p equal to delta r s. Then v bar r p v s p equal to minus delta r s and then sum over s, u s p u bar s p equal to p slash plus m over 2 m, then an analogue statement for v, which is sum over s v s of p v s star of p equal to p slash minus m over 2 m and a number of identities, in addition to all these things.

What you will do is that, we will use these solutions, these linearly independent solutions to construct the most general solution and then we will see what happens, when we quantum Dirac field. So, the most general solution will of course be a, linear superposition of solution of these kinds.

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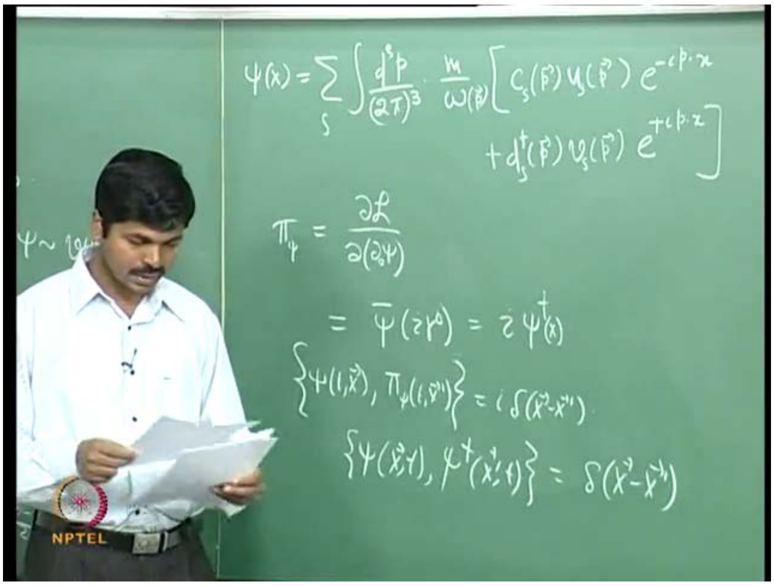

Therefore, psi of x can be written as the sum over s integration d q p over 2 pi cube and you will use a slightly different normalization m over omega p. You can see that, this is again Lorentz invariant because m is the rest mass of particles, which is a Lorentz invariant quantity. And this integration measure is you have already seen is Lorentz invariant, then the coefficient i will call them c s of p, u s p e to the power minus i p dot x and then d s dagger of p, v s e to the power plus i p dot x, this is for psi.

Now, let us try to look at what is the conjugate momentum to the field psi. This is given by pi psi equal to be momentum conjugate to psi, then this is del l over del, del 0 psi. And from the expression for the Dirac Lagrangian you can see that, this quantity is the given by, psi bar the i gamma 0, which is equal to i and psi dagger of x. So, you might think that the equal time commutation relations again will be something of this kind, psi of t and x, pi psi of t x prime equal to i delta x minus x prime. However we have already seen, if we have commutation relations of these kinds, then we get some computation relations for this operator c s and d s daggers and then this gives as a spectrum for particles width of a Bose Einstein statistics.

We do not want particles of have spin half integer to obey Bose Einstein statistics, we want them to obey the Fermi Dirac statistics. And that cannot we achieved by considering a commutation relation, but in the field and the conjugate momentum, what we will do is, we will instead of commutation relations, we will have anti communication relations instead.

And then we will see what happens, when we consider the anti-commutation relation anti commutator of the field and the corresponding conjugate momentum is i times delta x minus x prime. This is what we will assume and then we will correspondingly get some anti-communication relations, but in these operator c s p and d s dagger p, we will see that these anti commutation relations give a spectrum of particles which obey the Fermi Dirac statistics.

So, this, a must saying that psi of x t psi dagger of x prime, t the equal time anti commutation relations is given by delta x minus x prime. We can use the anti-commutation relation and derive the corresponding anti commutation relations per c s, p and d s p. Instead what I will do is, I will give the anti-commutation relations for c s and d s. And then we will check that this anti commutation relation is satisfied, provided this operator c s and d s dagger satisfy the following anti-commutation relations.

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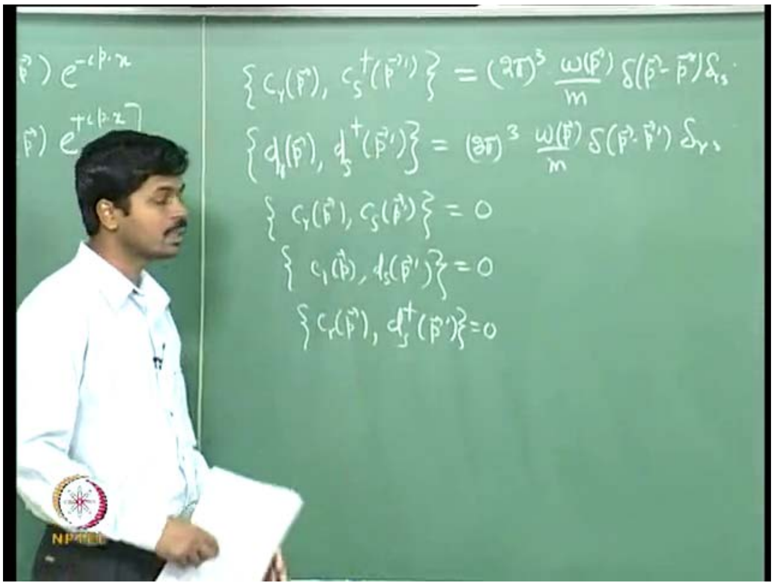

So, the anti-commutation relations satisfied by the c s and d s is given by c r p c s dagger of p prime, anti commutator is given by 2 pi cube omega of p divided by m delta p minus p prime delta r x. And similar relation for the d's d r of p, d s dagger of p prime is equal to 2 pi cube omega of p divided by m delta p minus p prime delta r s. Whereas the anti-commutations relation between c r of p c s of p is equal to 0, c r of p d s of p prime equal to 0 and c r of p d s dagger of p prime is equal to 0. And then the conjugate of these two equations, they have might conjugate of these equations.

So, these are all the anti-communication relations satisfied by this operator c's and d's. What we will do in the following is, we will assume this anti commutation relations and then from there we will derive the anti-commutation relations for psi and psi dagger. So, let us do that.

Student: What will happen to the anti-commutation relation on this pi end side?

Then you will get here, you can I mean just like a, the case of a complex scalar field, you will have commutation relations here instead of anti-commutation relations. Then you try to contract multi particles states from the vacuum.

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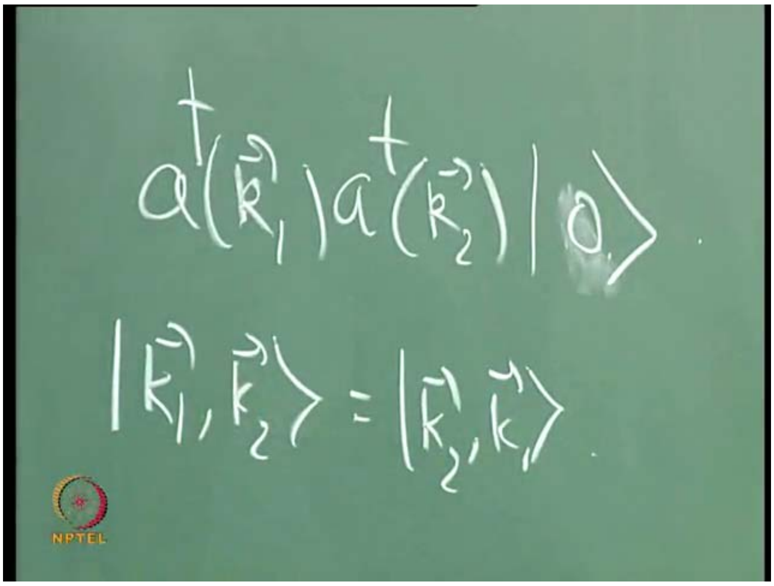

So, how to multi particle states are constructed a dagger of k 1, a dagger of k 2, let us consider a two particle state, constructed art of vacuum. Because, a dagger and a commute therefore this k 1, k 2 is same as k 2, k 1. This state does not change under the exchange of the quantum number of shear. So, that why this particle like states actually obey the Bose Einstein statistics. What we will like to is, we will like to if this particle like states obey Fermi Dirac statistics and hence you would think that, this operators should actually obey some anti-commutation relations.

That is the reason we start with the anti-commutation relations for the field and the corresponding conjugate momentum. So, let us, let us derive this equation from these relations, I will consider the components of the spinor shear it will be easy for us to do the competition intense of the components. So, the components here the both psi and psi dagger are four component spinors.

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And what I would like to consider is psi of psi alpha x t commutating with psi dagger beta x prime t, the equal time anti commutation relations. I already know what is the expression for psi.

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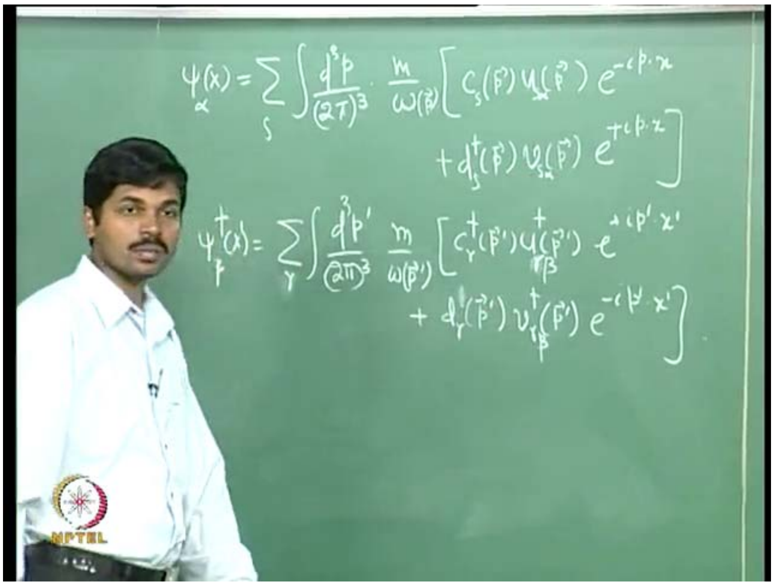

This trivial tells me what is the expression for psi dagger, what is psi dagger it is sum over x, integration d q p prime i will use for the integration variable. And for this one, I will use r 2 pi cube m over omega p prime c r dagger of p prime u s dagger of p prime, u r dagger of p prime, e to the power of minus i plus i p prime and x prime here. I am considering x prime and this one plus d r dagger of p prime, d r of p prime, v r dagger of p prime e to the power minus i p prime dot x prime. So, when I take the anti-commutation of this with this, I will consider psi alpha of x. So, there will be a u s alpha here, there will be a v s alpha here and similarly, I will consider psi dagger beta. So, there will be a beta, beta th component of u r p prime and beta th component of v dagger r of p prime.

Now, what will I get for this anti commutator here, it will. So, the first step and this it will give some anti commutator here, which is none 0, then anti commutator of this term and this term trivially vanishes, because of this relation here. Similarly, anti commutator of this with this will vanish, because of the conjugate of this equation, conjugate of this equation. And then the anti commutator of this, with this which is not going to vanish so these are the terms which are going to survive, the anti commutator step this with this, the anti commutator term this, with this term.

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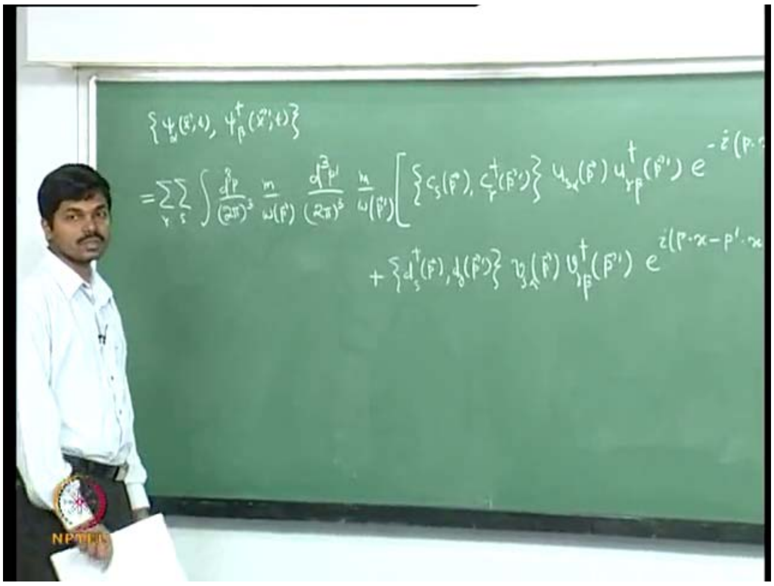

So, let us look at these two terms, in fact, i can call this to be psi plus and this to be psi minus, then psi plus psi dagger minus and psi. So, these, the commutation relations with the non-vanishing, these are the anti-commutation relations, which are not vanishing. So, this is sum over r, sum over s and then you have integration d cube p divided by 2 pi cube m over omega p. And again, you will have integration number p prime, which is p d cube p prime over 2 pi cube m over omega p prime and then there will be two terms.

And I will erase this and then so this times the anti-committer of c s of p, c r dagger of p prime and then u s alpha p, u r dagger beta p prime and then e to the power minus i p dot x minus p prime dot x prime this is the first step. And similarly, I will have one more term, which is given by d s dagger of p, d r of p prime, v s alpha of p, v r dagger beta of p prime and then e to the power plus i p dot x minus p prime dot x prime.

So, this is what we will get these are the two non-vanishing terms. Now let simplify this you know, what is anti-commutation relation here, it just up to some normalization its delta r s times, delta p minus p prime. So, let us substitute that we will substitute that and then you will carry out the p prime integration.

So, if we substitute that because of the chronicle delta, delta r s the sum over r we can just carry out and then it will be a sum over u s alpha p, u s beta dagger p prime, but there is a chronicle, there is a Dirac delta, delta p minus p prime, which will make both the p and p prime equal, if you carry out the p prime integration.

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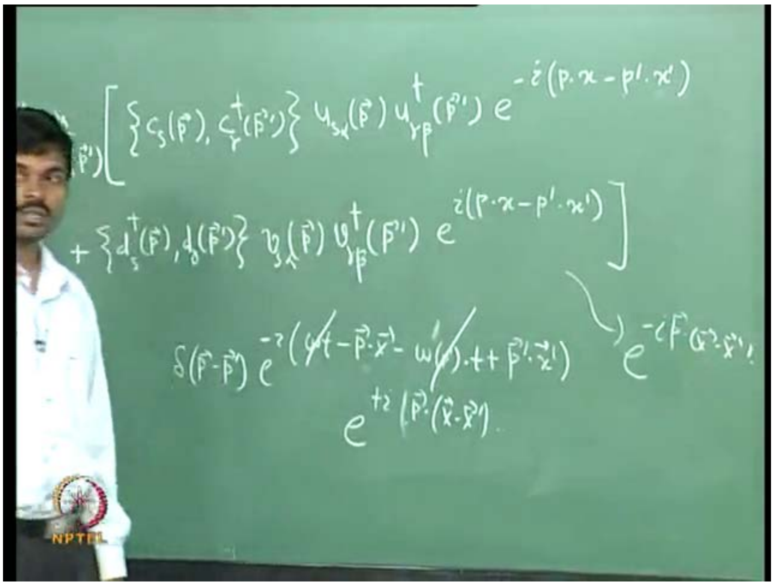

Now look at this term here, this term is e to the power i, the e to the power minus i, p dot x is omega t minus, this three vector product p dot x. And then here, p prime dot x prime is minus omega prime of omega p prime is what this 0th component of p prime dot, 0th component of x prime means t because here we are considering equal time and the commutation relations.

So omega prime of p t and then plus p prime dot x prime, this three vector product, so because the whole thing is multiplied by, multiplied by delta, Dirac delta p minus p prime therefore, I can substitute here instead of, in the place of p prime I can just write p. Then the t dependents will cancel from here and here so what you have is, e to the power plus i p dot x minus x prime. That is what you are going to get here when you, when you carry out the p prime integration shear and state when you carry out the p prime integration, because of the opposite sign here. What you will get here is, everything will be the same except that, you will get e to the power minus i p dot x minus x prime. This is what you are going to get so let us let us carry out the commutation.

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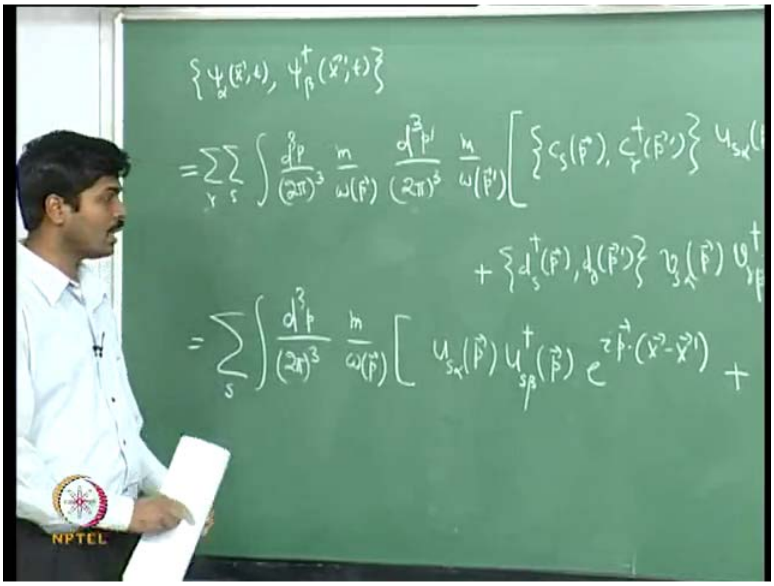

So, this sum over r will go away because of the chronicle delta. So, you will have only sum over s and then integration of d cube p over 2 pi cube, m over omega p, this m over omega p prime will cancel here. This normalization omega over m and this 2 pi cube again will cancel, this 2 pi cube and I am carrying out this d cube p prime integration. So, the Dirac delta will go away. Finally, what we will be left with is u s alpha of p u s beta dagger of p again, because of the integration over p prime and then e to the power of i p dot x minus x prime.

Whereas, in the second term I will have a plus v s alpha p, v s dagger beta of p, e to the power minus i p dot x minus x prime. This is what you will get, when we substitute for the anti-commutation relations for c s and d s and so on. Now you look at this sign here, here it comes with the plus sign, but here it comes with the minus sign.

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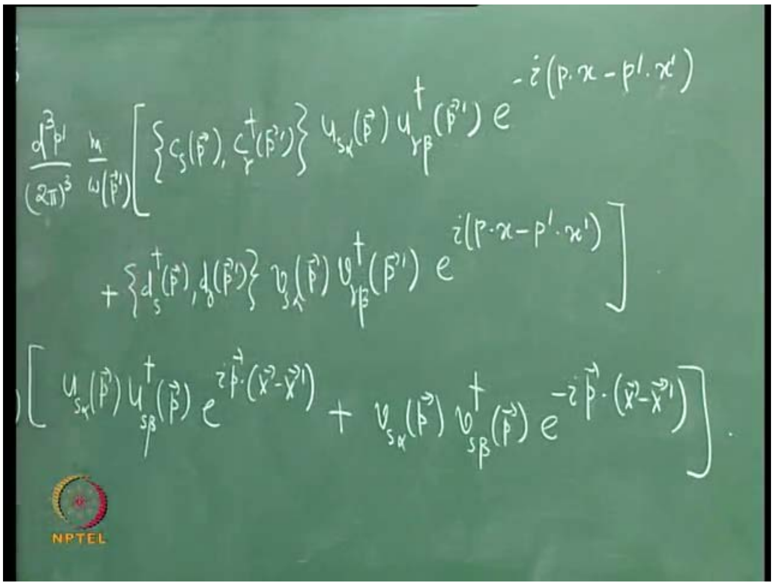

If I want to take this is a common factor, what I can do is in this second term. For example, I can seem this integration variable p to minus p and when I do that, here I will have a plus sign, but in the arguments v s will change this sign. Yes or no?

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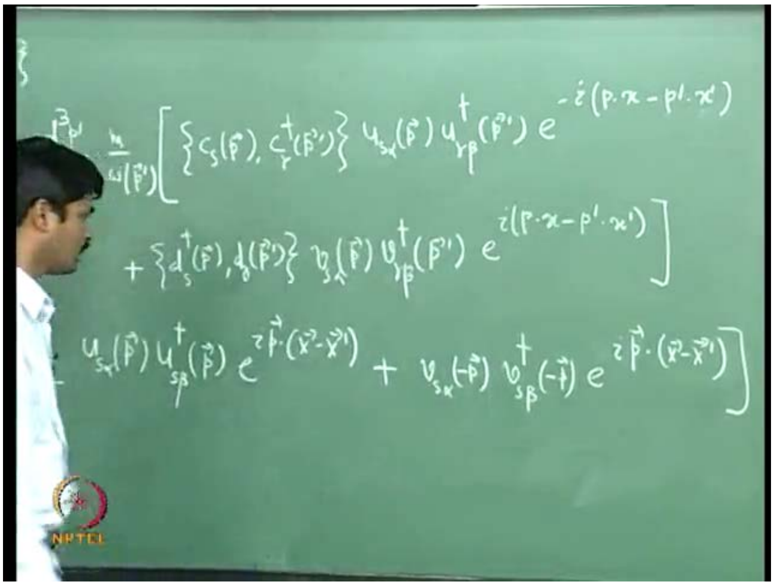

So, let us let us do that, therefore, what I will have here is v s of minus p and v s dagger beta minus p and here, I will have a plus.

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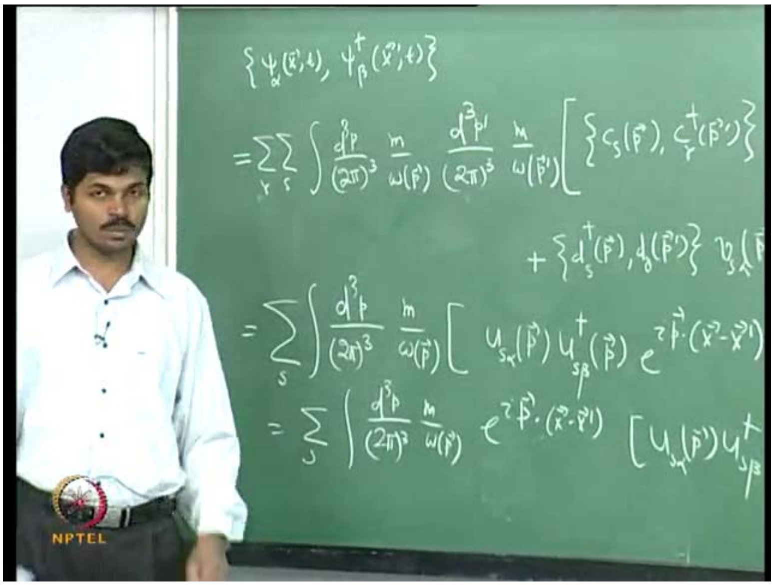

Now, we can just take this out so sum over s, d cube p over 2 pi cube m over omega p and then e to the power i p dot x minus x prime and then u s alpha of p.

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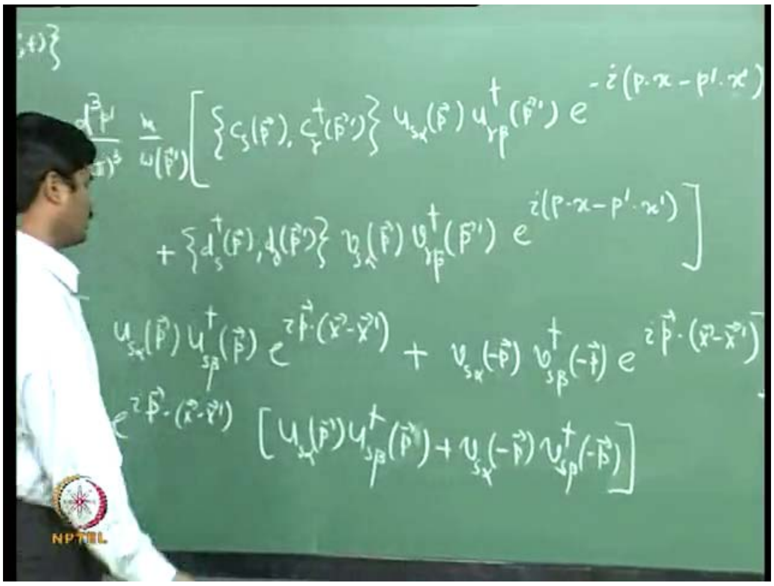

u s beta dagger of the p prime, p again plus v s alpha minus p v s dagger beta of minus p. So, it is almost look like we are having a Dirac delta function here, but it is not a, not yet because of this term mainly inside this square bracket. What is this quantity here? When you carry this sum over x what you get? So, in the beginning of this lecture I have, I have written down some of the identities that you have proof in the last lecture.

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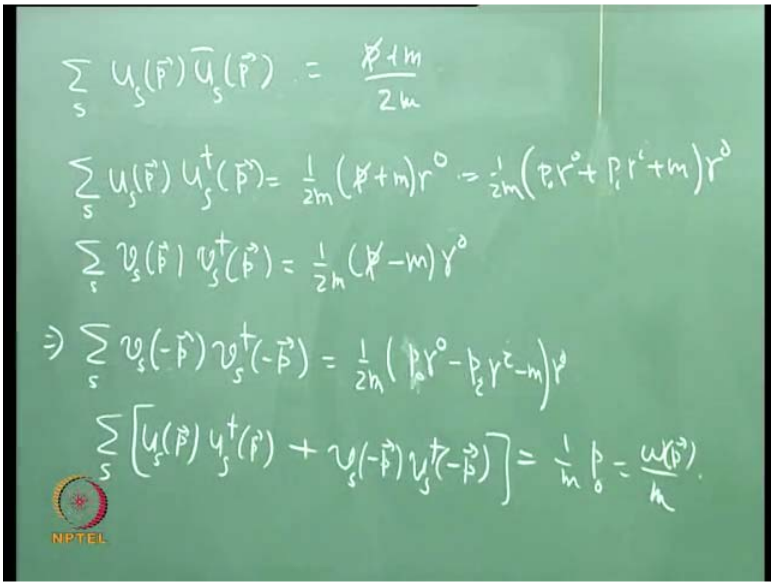

One of them is if you carry out the sum over s, u s p u s bar of p then this quantity is p slash plus m over 2 m here instead of u s u s dagger. So, if you want to get an expression for u s u s dagger, you get some, you just multiply gamma 0 from the right. So, this is simply u s p u s dagger of p is 1 over 2 m p slash plus m gamma 0. Similarly, sum over s v s of p v s dagger of p its 1 over 2 m p slash minus m right? gamma 0. What do you want instead is, instead is v s of minus p, v s dagger of minus p. So, the second line implies sum over s v s minus p, v s dagger of minus p is equal to 1 over 2 m p slash gamma 0, p 0, gamma 0 minus p i gamma I, because of the minus here.

So, you cannot simply have a minus sign here because this 0th component does not change the sign. And minus m, whole thing multiplied by gamma 0, where as this one here, you can see is given by 1 over 2 m p 0, gamma 0 plus p i gamma i plus m, whole thing multiplied by gamma 0 from the right. So, if you add this term from this term then what you get is, the last two terms cancel is so there and the first term adds.

So, you have sum over s u s of p, u s dagger of p plus v s of minus p, v s dagger of minus p is given by 1 over m p 0, p 0 and there is this gamma 0 square which is identity. So, it is just simply remember p 0 is nothing but omega p divided by m. If you want put the alpha, beta etcetera then this is just delta, alpha, beta.

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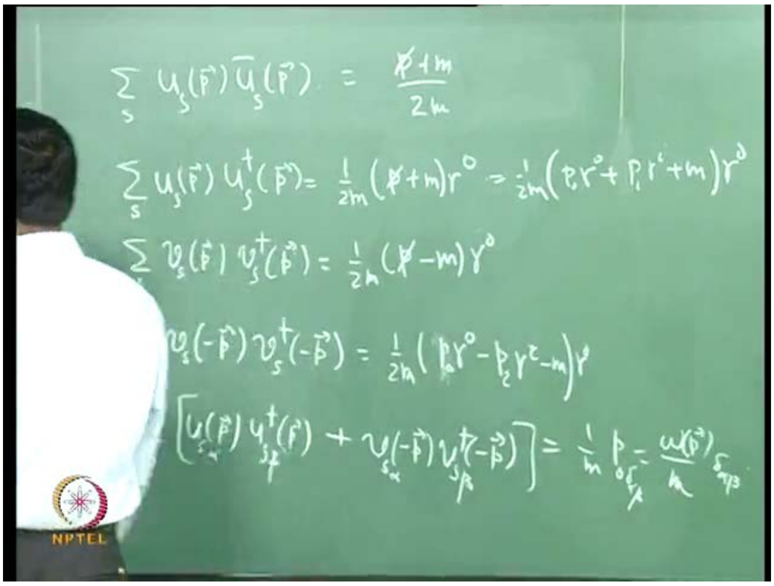

So, u s alpha, u s dagger beta and v s alpha minus p v s dagger beta of minus p is simply given by this times delta alpha beta so there will be delta alpha beta here. So, what we have seen here is, after doing all these calculation, this thing here inside the square brocket is nothing but omega over m delta alpha beta.

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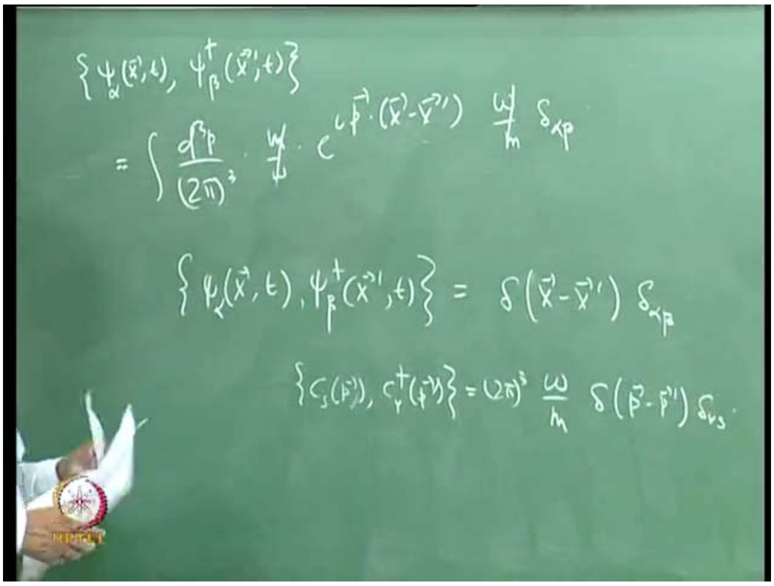

So, let us substitute the here, sum over s is gone so d cube p over 2 pi cube, m over omega and then e to the power i p dot x minus x prime. And the whole thing is given by, omega over m delta alpha beta so m over omega cancel omega over m. And hence, I can carry out the integration over p, when I do that, what we get is psi alpha of x t, the equal time and anti-commutation relations, psi beta dagger of x probability is nothing but delta x minus x prime delta alpha, beta.

So, in order to have the four components spinor, the Dirac spinor and its conjugate momentum. To obey the equal time anti commutation relation, what we saw is that we need the coefficients c s of p c r dagger of p prime to obey some equation like 2 pi cube and omega over m delta p minus p prime delta r s and so forth. So, now as you expected already that, the c s it like any lesson operators. So whereas, the c r dagger at like creation operators again.

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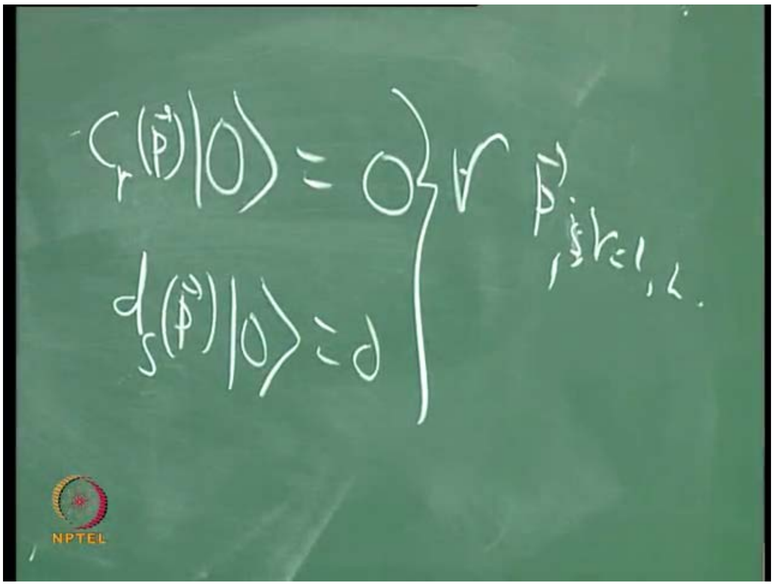

We will have a vacuum, will define 0 to be the vacuum state which is any letter by c r p for all p and for all equal to 1 and 2 and also its any letter by d s of p equal to 0 for all p and for r and s to the 1 and 2.

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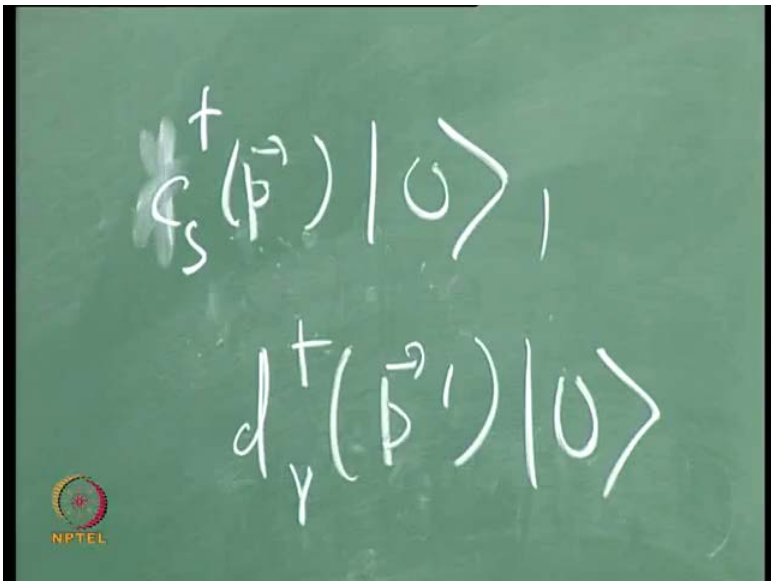

Then, the multi particle states are constructed by c s dagger of p on the vacuum and so on, d r dagger of p prime and vacuum and then a bunch of c's and d's. So, this way you can construct the entire Hilbert state and then you can see that, because of anti-commutation relation like this, this particle states it can be obey the Fermi Dirac statistics, instead of the Bose Einstein statistics. You can construct the energy momentum operator and these are and shown because the Dirac Lagrangian as we have seen has a u r in variance, it is a global u r invariance.

So, there will be ((Refer Time: 36:26)) r's, you can construct this r's operator, you can see that these are, I mean these r's, for these particle are actually opposite to these r's, this particles of the second kind. So, if we identify these with the positive these r's particles then these will be the negative charge particle or vice versa. So, for example if these are the electron states, these states represent the electron then these states will represent depositors. So, the important thing is that you must note is the following, for example when you consider the normal ordering, in the case of bosonic fields.

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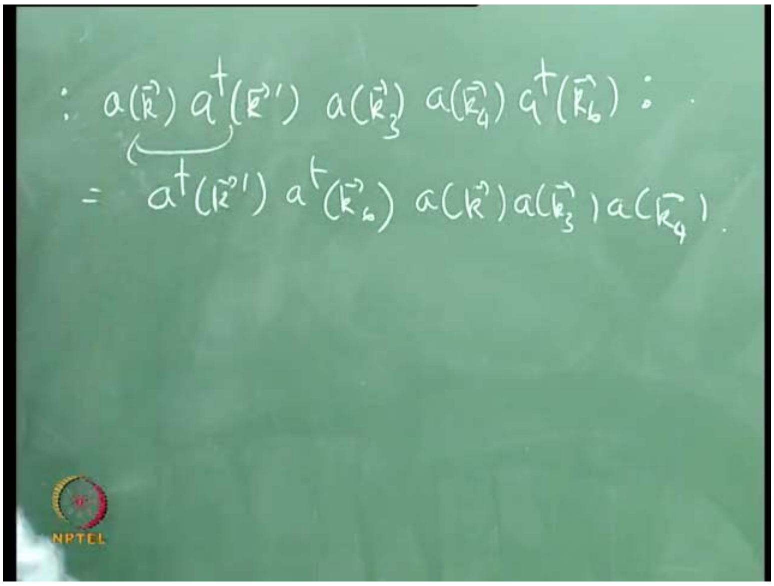

We were seeing that if you have a bunch of operators and then we just consider the normal ordering of these operators, then what we have is this, a of k, a dagger of k prime, a of k 3, a of k 4, a dagger of k 6 something like this. Then what do you have? What do you get? You just for Bosonic fields this is just given by a dagger of k prime, a dagger of k 6 and then a of k, a of k 3, a of k 4 this is all you have. Now in the case of Fermionic operators, you have to be a little bit careful about moving this to the left, if it crosses one of these operators, then it is changes the sign.

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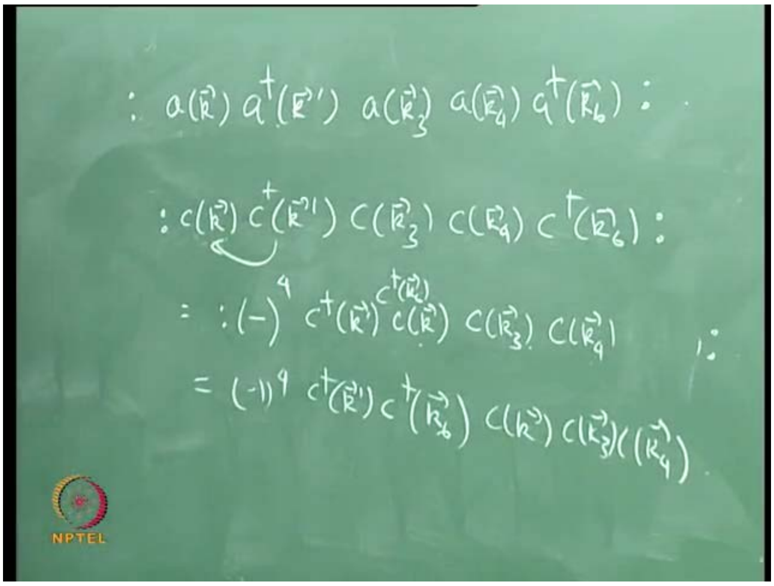

So, an identical, a similar relation for, for the Fermionic operator will have this, c I surprising the spin indices c of k, c dagger of k prime, c of k 3, c of k 4, c dagger of k 6. Let us say you have normal ordering of these operators, that you see here this operator will have to from once therefore, it will change the sign, this is just normal order product of minus c dagger of k prime c of k. And then c of k 3, c of k 4, c dagger of k 6, but again this if you want to bring it, to the extreme left, inside the normal order product it will change the sign three times, this is minus 1 to the 4 and this is c s dagger, c dagger of k 6.

So, this is simply minus k 4 c dagger k prime c dagger of k 6 c of k c of k 3 c of k 4 without any normal ordering. If there is one more operator here, then for the bosonic fields it will not make any c's, but for the Fermi harmonic field, it will gain a minus sign. So, you have to keep track of this minus signs also, these operators, if you cross these operators once they have a minus sign, so that you need to be careful about, when you do the normal ordering.

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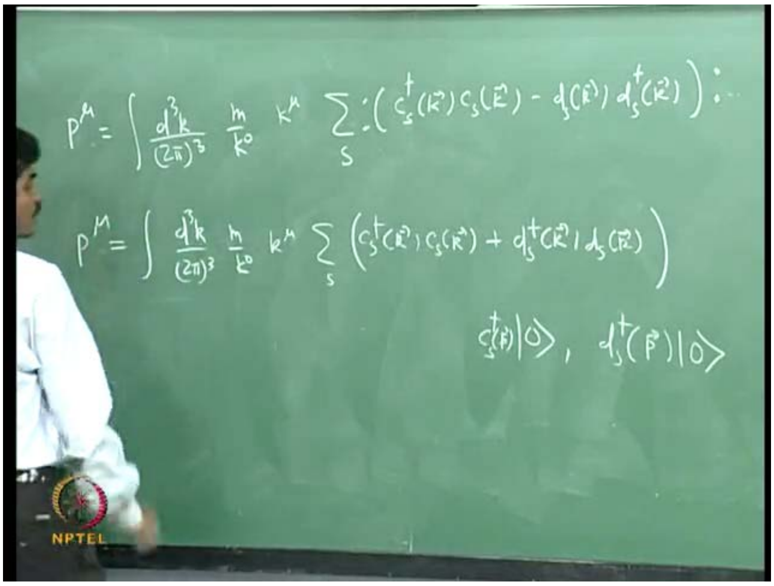

You go and work out the expression for the energy momentum denser, what i claim is the total energy momentum has the expression it is given by, d cube k over 2 pi cube m over k 0 k mu sum over s d s dagger of, c s dagger of k c s of k minus d s of k, d s dagger of k.

Now you want to consider it, insert the normal ordering so you have normal ordering and then the normal ordering makes this to move to the left, but it crosses one's therefore, it will change the sign. And hence, after you take the normal ordering you will get d cube k over 2 pi cube m over k 0 k mu sum over s c s dagger of k c of k prime plus c s dagger of k d s.

So, you can take the energy operator for example and then you add term, particle states like this c s dagger of p, one particle states or d s dagger of p, at a vacuum are so on. And then you will see that, this will have one quantum of energy and so will this state have, there is nothing negative about this thing. It is not a, not a negative energy state.

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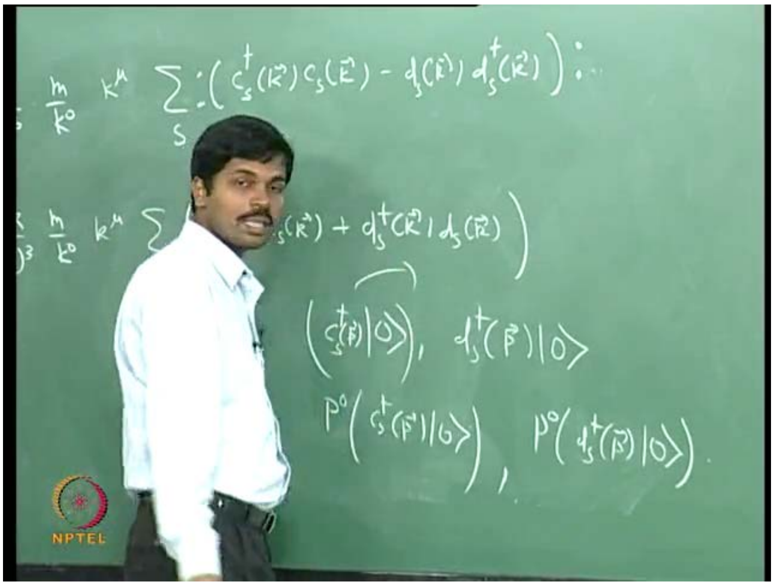

You can just, at you can, this is the p mu, you consider a state like this or you can consider a state like this p 0 on c s dagger of p 0 or p 0 acting on d s dagger of p on vacuum. You can see that both type of these particles have positive energies. So, the energy they are Eigen states of the energy operator and the Eigen value is always positive.

So, these are not negative energy states, only thing is that if you construct the charge operator, you construct the charge operator and then you at, on this, as a as well as on these then the Eigen value of states of this kind, will have opposite Eigen value for the state of the second kind. So, it is only these charge which is negative of this, charge of this, this state is minus of this charging states, but the energy Eigen states are not negative, the energy is always positive.

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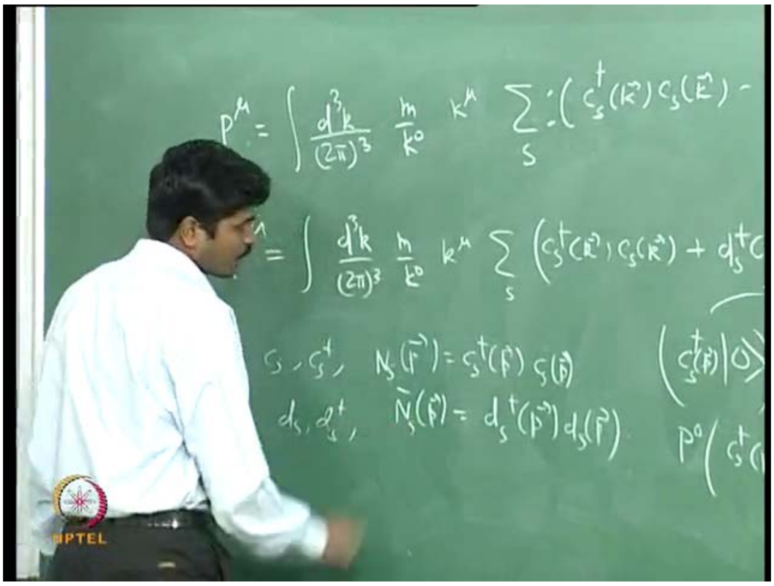

You can write it into terms of the number of operator. For example, you can introduce an s of p is equal to c s dagger p c s p and n s bar of p to be the number operator for the anti particles, d s dagger of p d s p. Then this will have simply n s plus n s bar so these are basically the c s c s dagger n s are basically the annihilation, creation and number operators or particle.

Whereas, d s d s dagger n s bar are the annihilation, creation and number operator for the anti-particles. So, what you will do in the next class is, we will consider some of the discrete symmetries and also will derive an expression for the Dirac propagator. And then subsequently, we will learn about the interaction of the fermions with the electromagnetic field.