Quantum Field Theory
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Module - 03
Free Field Quantization: Spinor and Vector Fields
Lecture - 19
Fermion Quantization III

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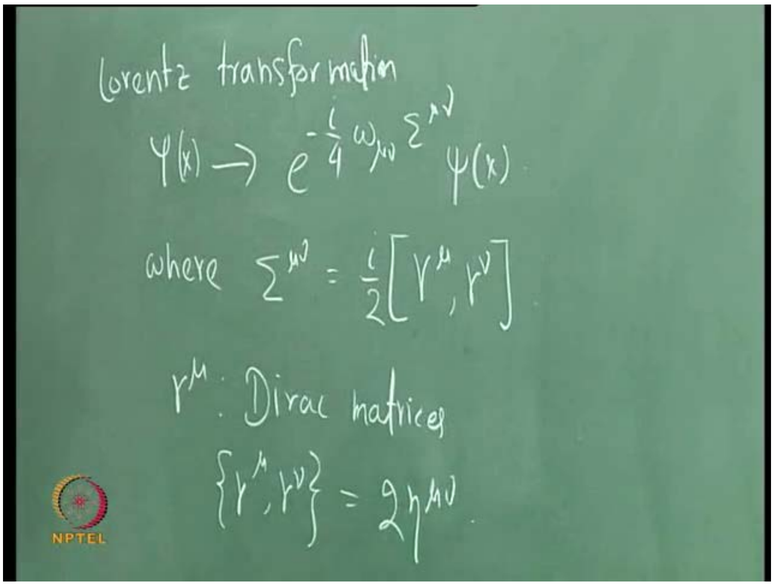

So, in the last lecture we have introduced the Dirac spinor, psi and then I have shown how it transforms under Lorentz transformation. So, under Lorentz transformation the Dirac spinor goes like psi of x goes to e to the power minus over 4 omega, mu nu sigma mu nu acting on psi of x. Where sigma mu nu is given by i over 2 commutator of gamma mu gamma nu, where the gamma mu gamma nu are the Dirac matrices. They are a set of 4 by 4 matrices, which satisfy the anti-commutation relation, gamma mu gamma nu equal to 2 eta mu nu, then we have seen that in one particular representation known as chiral representation.

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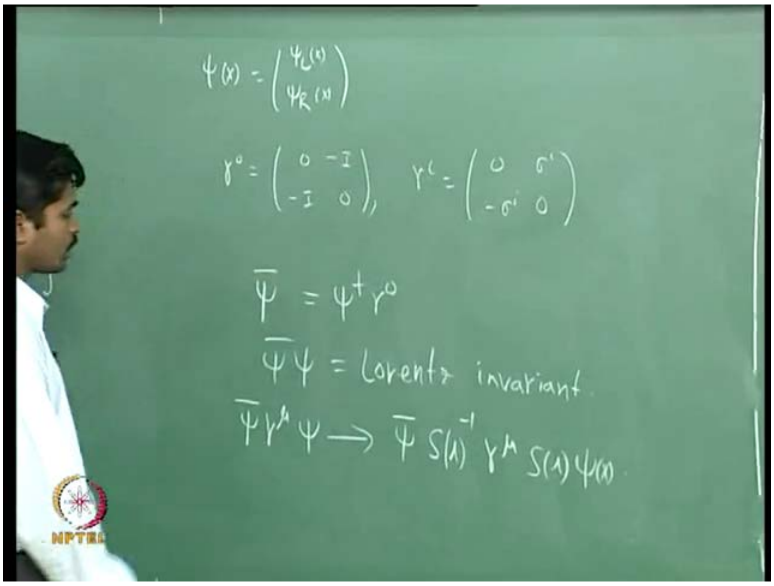

This Dirac matrix, this Dirac spinor can be written as psi l psi r into subdivided left handed and right handed components. And the Dirac matrices in the chiral representation r given by 0 minus i minus i 0 and gamma i equal to 0 sigma i equal to sigma i minus sigma i equal to 0. So, these are Dirac matrices however, we need not have this representation, any representation, any set of 4 by 4 matrices will satisfy these anti-commutation relation, will be good enough to represent the Dirac gamma matrices, the Dirac spinors.

Then what we saw is that we can try to construct various invariants from the Dirac spinor, one of them we have seen is if we define psi bar equal to psi dagger gamma 0, then psi bar psi is actually Lorentz invariant. Then we were discussing Lorentz transference property of this subject psi bar gamma mu psi. So, let us see how this subject transforms under Lorentz transformation. This transforms like psi bar goes to psi bar s lambda inverse and then, gamma mu and then psi goes to S lambda psi of x. So, this is how it transfers.

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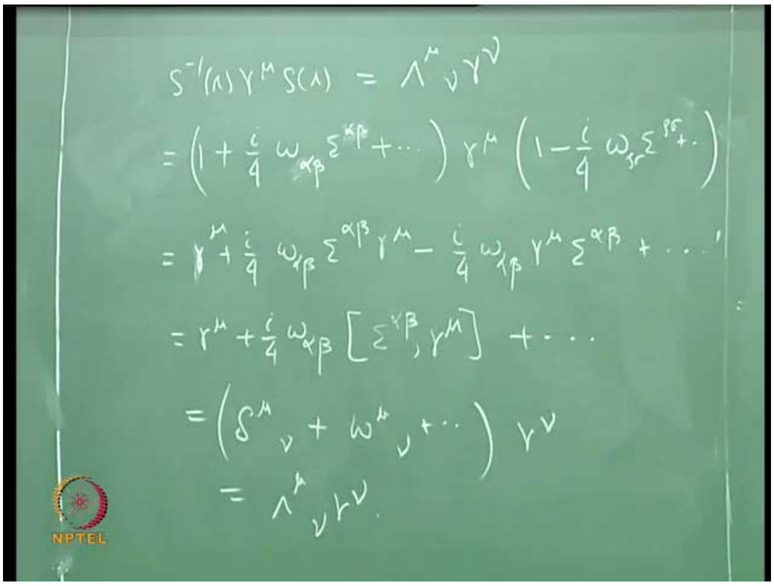

So, what we need to do is, we need to know what is this object S inverse lambda gamma mu S of lambda, we can do an infinitesimal transformation where, this is given by two order omega mu nu. This is 1 minus i over 4 omega mu nu sigma mu nu, then let me write it as alpha beta, alpha beta gamma mu and 1 minus i over 4 omega rho sigma, sigma rho sigma. So, to order omega this looks like 1 minus i over 4 omega alpha beta sigma alpha beta gamma mu.

So, this is a gamma mu to the 0th order and then, this will be plus because we are considering S inverse. So, there is a plus sign here and then you have finally, minus i over 4 omega alpha beta gamma mu sigma alpha beta and all other terms. So, this is nothing but gamma mu plus i over 4 plus omega alpha beta and the commutator of sigma alpha beta and gamma mu and all other terms.

And in the last lecture we have evaluated the commutator of sigma alpha beta and gamma mu and if you substitute it here, then what you see is you have gamma mu i over. So, finally all the factors will cancel and what you will get is, delta of mu nu plus omega mu nu plus gamma nu. So, this is simply given by lambda mu nu gamma nu. So, this object S inverse gamma mu S is nothing but when you make a finite transformation, this will go like lambda mu nu gamma nu.

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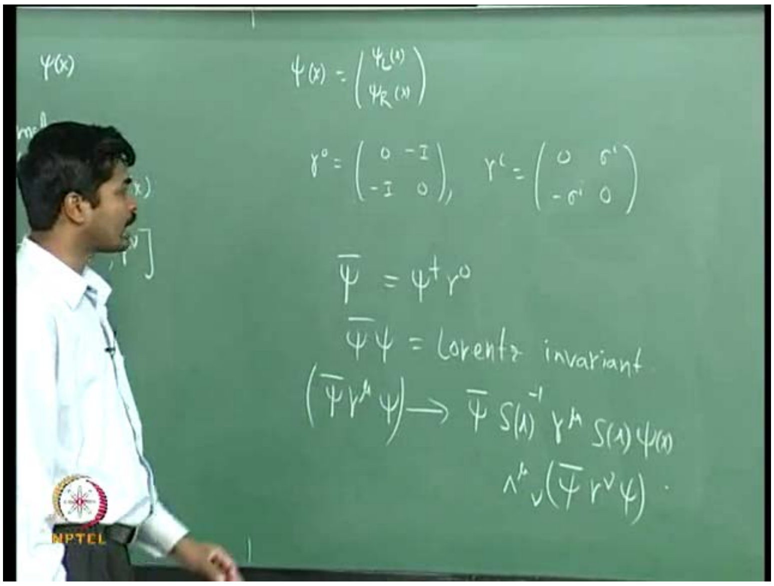

So, therefore, this object here psi bar gamma mu psi under a Lorentz transformation, goes like a vector because, this quantity, this is nothing but lambda mu nu psi bar gamma nu psi. So, therefore this is a vector under Lorentz transformation. Similarly, we can construct a tensors and pseudo-scalars and pseudo-vectors and so on, under Lorentz transformation. What we are interested is, we are interested to construct a Lorentz invariant Lagrangian density. So, this would involve psi and its derivatives.

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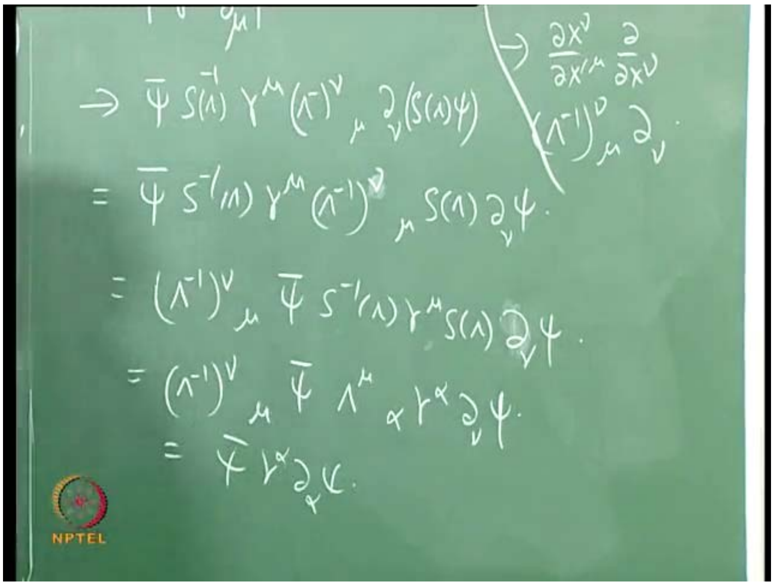

You can try to construct a Lorentz invariant Lagrangian density, which is quadratic in del mu psi but it turns out that any such Lagrangian, this is quadratic in del mu psi gives you various unphysical features. For example, the energy heads no lower bound and so on however, luckily we have a Lagrangian density which is first order in del mu psi, which is Lorentz invariant. So, let us try to construct a term which is first order in del mu psi, which is Lorentz invariant, to do that consider this term psi bar gamma mu del mu psi.

This quantity here, as you know psi bar this del mu is not there, then this would have transformed like a vector but we wanted to have one derivative of field psi. So, we have inserted this del mu psi here and then we can show that this quantity over here is invariant under Lorentz transformation. While this psi bar goes to psi bar S lambda inverse and then you have gamma mu and del mu under Lorentz transformation goes to, del mu is del over del x mu and under Lorentz transformation this goes to del x mu over del x prime mu del over del x mu.

So, this quantity is lambda inverse nu mu del mu. So, you have lambda inverse nu mu del nu of S of lambda psi, but S of lambda does not depend on space time. So, I can just pull it out, what I get is psi bar S inverse lambda gamma mu lambda inverse mu nu mu and S of lambda del nu psi, this is a number. So, again the components of lambda mu nu are numbers so, again I can pull it out.

So, I have lambda inverse nu mu psi bar S inverse lambda gamma mu S of lambda psi. This quantity is again we have computed, what it is del mu psi so, I can substitute this here, then what I will get is lambda inverse nu mu psi bar and S inverse gamma mu psi is lambda of mu alpha gamma alpha del nu psi. And finally, this is simply psi bar gamma alpha del alpha psi. So, this quantity is Lorentz invariant so, you can have Lagrangian density for a massive spin of field.

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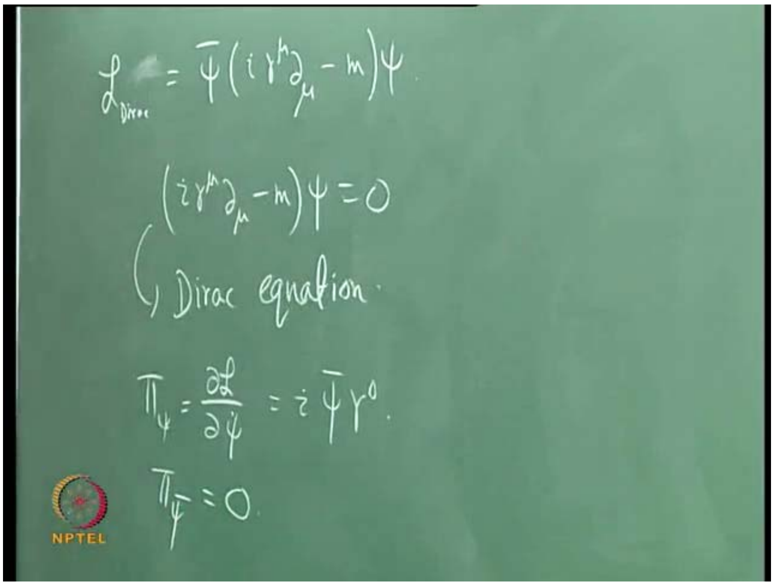

And the Lagrangian density is given by l Dirac, the l Dirac Lagrangian is psi bar i gamma mu del mu minus m psi. I have inserted a factor of i to make the Hamiltonian Hermitian, which we will soon see, how the Hamiltonian density is Hermitian, it will not be Hermitian, if there is no i here. So, we will take this Lagrangian density for the Dirac field and then, we will quantize the system. As usual, before quantizing this we have to first study the plain wave solutions for this filed here.

Let us do various things for example, let us try to construct the Hamiltonian density and so that, the Hamiltonian is Hermitian and let us look for the plain wave solutions for the equation of motion. What is the equation of motion? The equation of motion is given by i gamma mu del mu minus m acting on psi equal to 0 and this is known as the Dirac equation. And we would like to study the plain wave solution to the Dirac equation, which we will do in a moment before that, let us try to construct the Hamiltonian. So, the conjugate momentum to the field psi is given by pi psi equal to del l over del psi dot and this is nothing but i times psi bar gamma 0, what is pi psi bar? 0, right.

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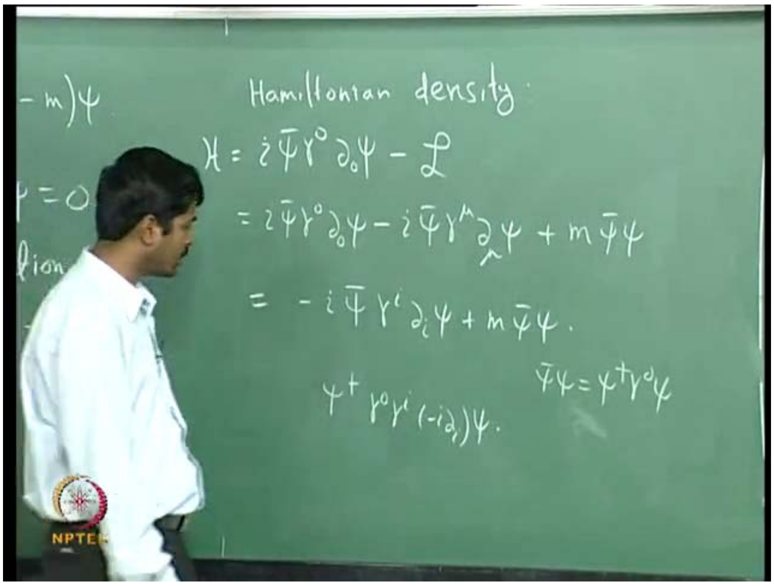

So, let us try to construct the Hamiltonian density, it is given by i psi bar gamma 0 del 0 psi minus L. So, the Hamiltonian equals to H is given by this and if you substitute for the Lagrangian density then, this is psi bar gamma 0 del 0 psi minus i psi bar gamma mu del mu psi plus m psi bar psi. So, this is simply minus i minus psi bar gamma i del i plus psi plus m psi bar psi.

Now, you know why I have inserted this symbol, this effecter of i here? Because, i del i this is Hermitian operator. So, you can write, the second term is obviously Hermitian because, psi bar psi is Hermitian, psi bar psi is psi dagger gamma 0 psi, which is the Hermitian conjugate of this quantity is equal to itself because, gamma 0 is Hermitian. And then you have psi dagger psi on the both sides, but in the first term you have psi dagger and gamma 0 gamma i and minus i del i psi and this operator inside will be Hermitian if this is Hermitian so, let us. So, let us just take Hermitian conjugate of that.

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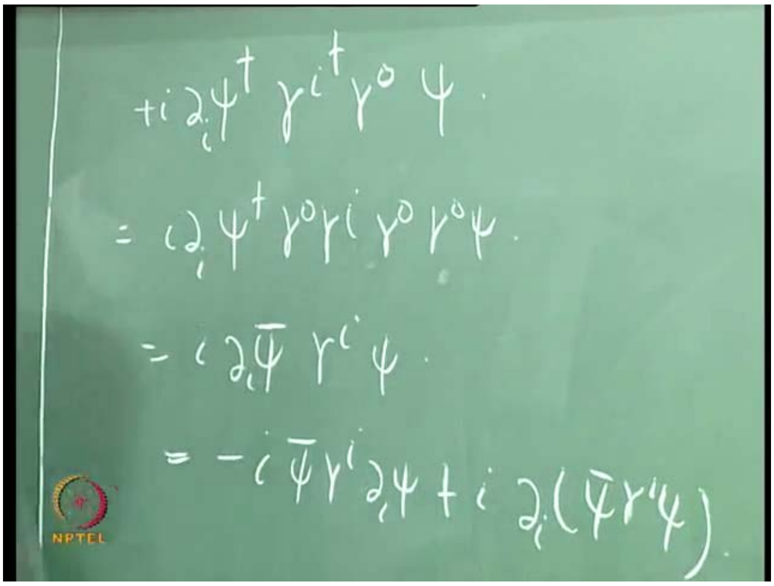

It is just minus i will give you plus i del i psi dagger and then gamma i dagger gamma 0 psi. And what is gamma i dagger? gamma i dagger is i del i psi dagger gamma 0 gamma i gamma 0 gamma 0 psi, gamma 0 square is 1. So, this is simply i del i psi bar gamma i psi and then, this will be up to a total derivative. This is just minus i psi bar gamma i del i plus i del i psi bar gamma i psi.

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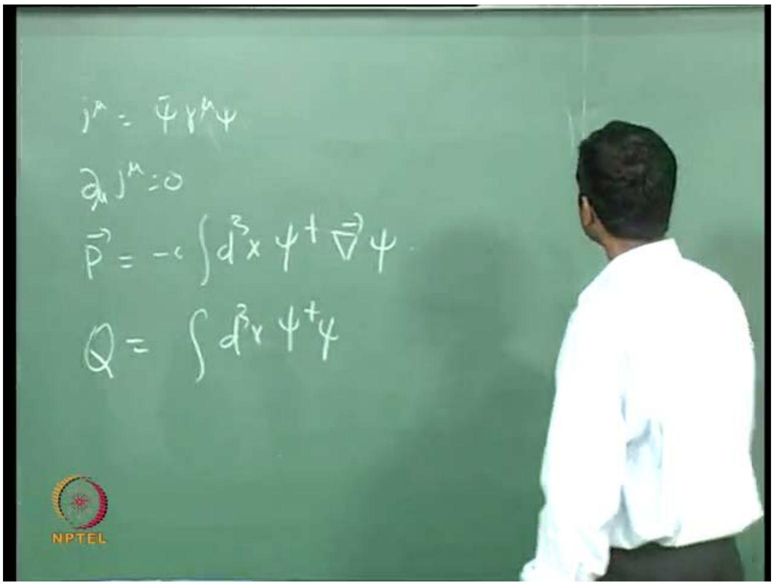

There is u and invariance and then, you can show that there exists a conjugate mu, which is given by psi bar gamma mu psi and that mu is equal to 0. I will not do this for you and you can construct various quantities like the momentum and so on. You can construct the energy momentum operator and then you will have the momentum which is given by minus i d cube x psi dagger ((Refer time: 19:11)) psi. The total charge Q will be d cube x psi dagger psi and so on. What we will be studying is, we will consider plain wave solutions to the Dirac equations so, let us do that. As usual, we will consider both positive frequency as well as negative frequency modes.

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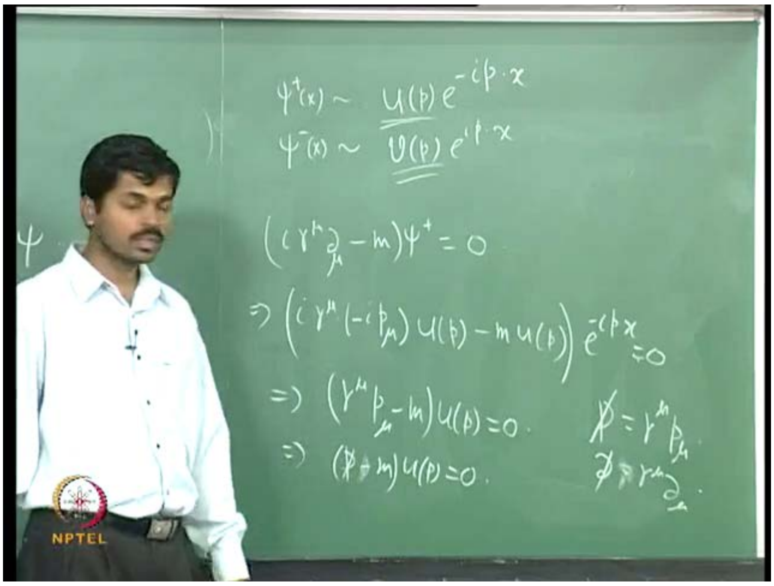

The positive frequency modes, I will denote them by psi plus of x, which will go like u of p e to the power of minus i p dot x. And the negative frequency modes, I will denote them as psi minus of x, which will go like v of p e to the power of i p dot x. I will substitute these two forms and then I will see what, then we will discuss, what are the expressions for these spinors u of p and v o p?

So, let us consider the Dirac equation, the Dirac equation is given by i gamma mu del mu minus m psi. For positive frequency modes, this will be psi plus equal to 0 and this simply implies i gamma mu times minus i p mu u p minus m u p times e to the power of i p dot x equal to 0. This simply implies that gamma mu p mu minus m acting on u p equal to 0. Contraction of gamma mu with various operators will appear quite frequently, I will use the symbol slash to denote contraction with gamma mu.

For example, I will denote p slash for gamma mu p mu and so on. And del slash for gamma mu del mu, whenever I use a slash it is just gamma mu del mu and so on. So, this, this I can write it as p slash plus p slash minus m acting on u of p equal to 0. By solving this equation, we will find the most general expression for u of p. Similarly, if you look for the negative frequency modes, then, you will get an equation for v of p.

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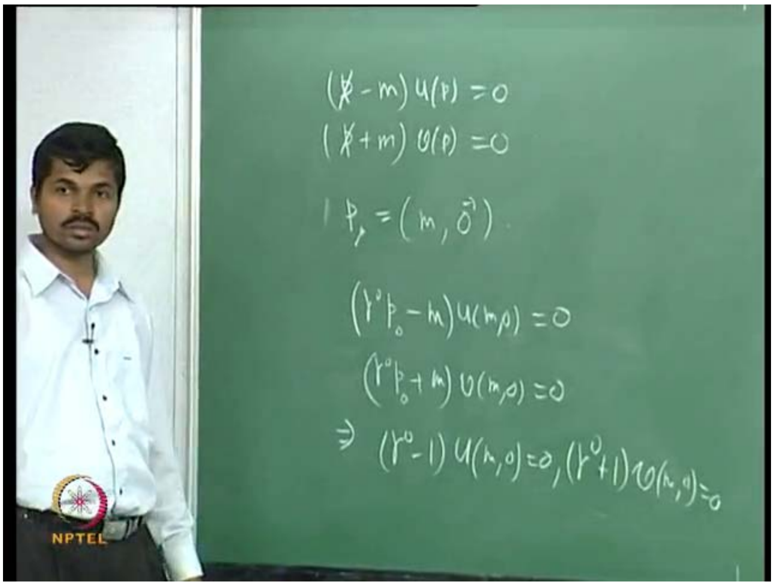

I will summarize these two equations p slash minus m u p equal to 0, p slash plus m v p equal to 0. These are the two equations separated by u of p and v of p, as you can see directly from the Dirac equation or in other words, these are the Dirac equations for the spinors u of p and v of p. We want to solve these two equations, to solve that, note that if we consider massive fields, then we can go to the rest frame. Where, the momentum, the for momentum p is simply m and 0 in the rest frame because, there is free momentum is 0 in the rest frame. We already know that, these equations are co-variants under Lorentz transformation.

So, you can solve it in the rest frame and then we will find the solution for the u of p and v of p, in the rest frame and then we do a boost to find the general solution, in any frame. So, that is what we will do, in the rest frame these two equations will look like the following. This equation looks like gamma 0 p 0 minus m acting on u of p equal to 0 and the second equation looks like gamma 0 p 0 plus m acting on v of p equal to 0, but since p 0 equal to m. Therefore, the same plus gamma 0 minus 1 u, p again here is m 0 is equal to 0, here again m 0 equal to 0 m 0 equal to 0 gamma 0 plus 1 v of m 0 equal to 0. To solve this, we will choose one particular representation for the gamma matrices.

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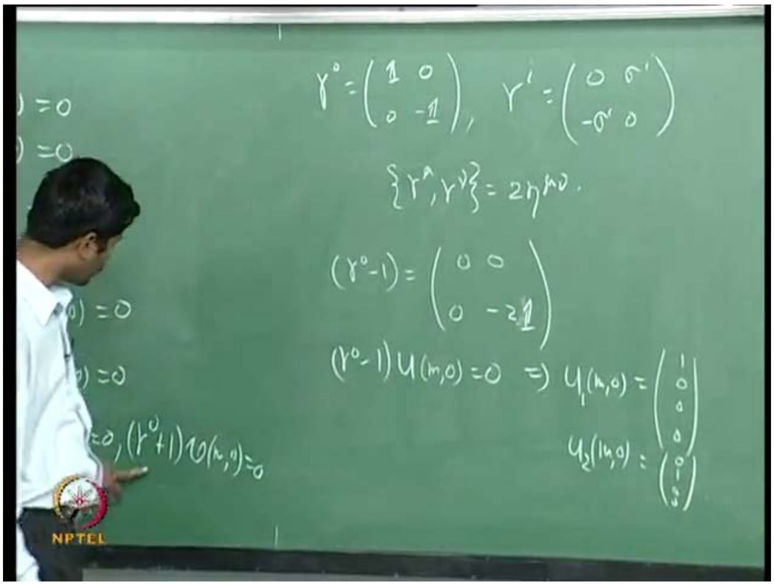

And then the solutions look very simple in this representation where, gamma 0 is 1 0 0 minus 1 and gamma i are 0 sigma i minus sigma i 0. You can see that this set of gamma matrices also satisfy the Clifford algebra gamma mu gamma nu equal to 2 eta mu nu. So, they can be represented for gamma matrices, the Dirac gamma matrices, in this representation these two equations have very simple solution.

We can see that, there are two linearly independent solutions for this equation and two linearly independent solutions for this equation. Gamma 0 minus 1 will have equal to 0 0 0 minus 2, 2 times identity. So, this suggests that the linearly independent solution therefore, gamma 0 minus 1 u m 0 equal to 0 m plus. There are two linearly independent solutions for u m 0, which I will denote as u 1 m 0 which is 1 0 0 0 and u 2 m 0 and u 2 m 0 is 0 1 0 0. Similarly, you can solve the other equation here gamma 0 plus 1 v equal to 0 again there will be two linearly independent solutions for v.

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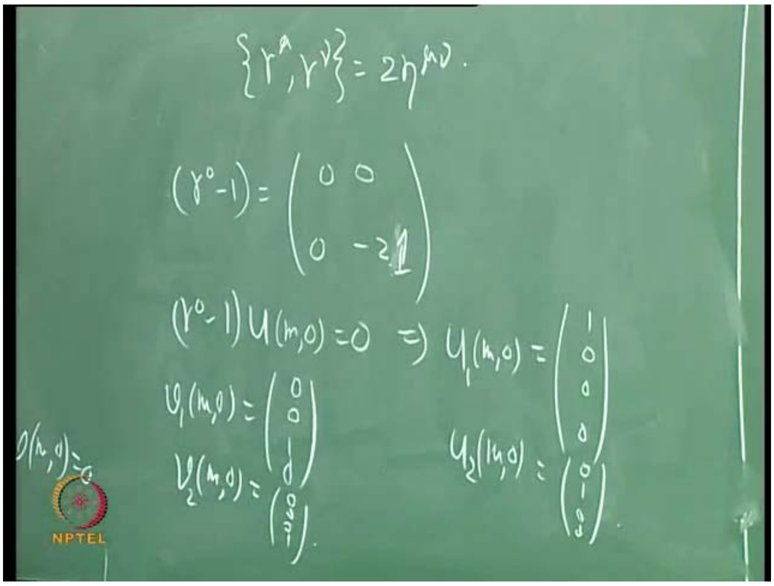

This is given by v 1 m 0 equal to 0 0 1 0 and v 2 m 0 equal to 0 0 0 1. You can find the most general solution, but even without finding the solution in the general frame, by performing a Lorentz boost, we can deduce a number of properties for this Dirac spinors of u of p and v of p. Now since there are two linearly independent solutions for u and two linearly independent solutions for v.

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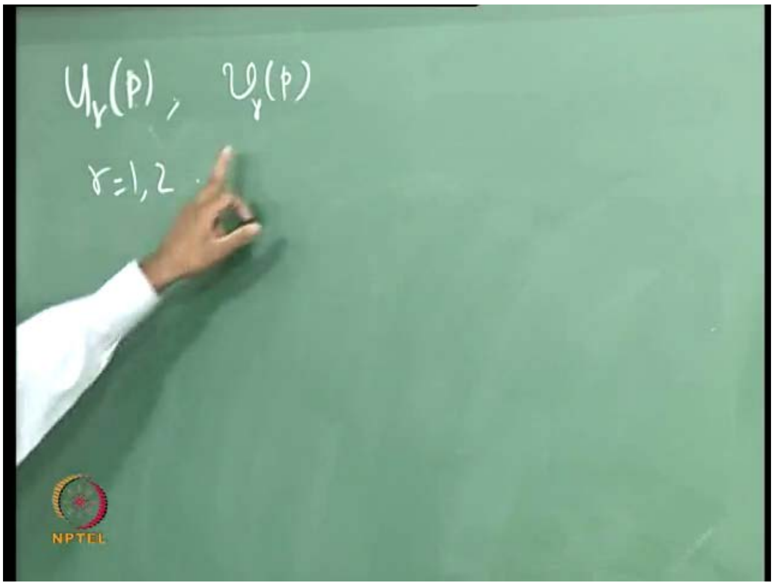

I will denote them as u r of p and v r of p where, r runs from 1 and 2. Therefore, I should clarify this symbol once more, this is a four component spinor for r equal to 1, as well as 2. And similarly, v r p again for equal to 1 and 2, these are four component spinor.

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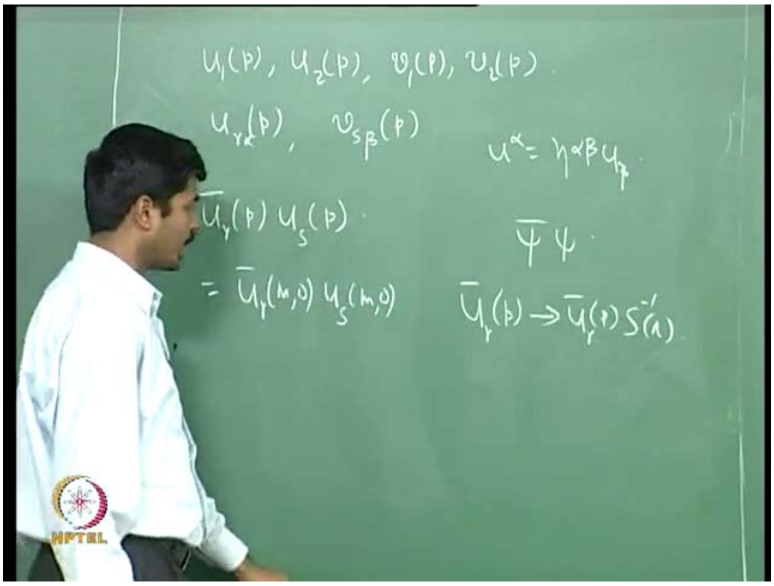

So, we have four, four component spinors which are u 1 p u 2 p v 1 p and v 2 p, they are the components. So, these are just columns you have four entries for u 1, four entries for u 2 and so on. So, I will denote their components by alpha beta and so on. So, I have u one alpha of p where alpha runs from 1 to 2 and so on. So, these indices I will denote as 1, 2 and then the components of u 1, u 2, u 3, u 4 etcetera those I will denote by using this symbol alpha. So, you have u r alpha of p for the components of these spinors and similarly, you have v s beta p. We would like to know, what is this notation here, u bar r of p, u s of p.

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They are as usual, just like here, you can just contract them by the usual metric eta mu nu and so on. So, u alpha is simply eta alpha beta u beta and so on, but these are not raised lower, they are raised and lowered r and s indices are raised and lowered by the unit matrix. So, let us look at this quantity here.

This is Lorentz invariant quantity because, we have already seen earlier that psi bar psi is invariant under Lorentz transformation. When you perform a Lorentz transformation, u in fact because this they just differ by a number, u transforms the same way psi transforms. So, you get a u bar goes to u bar u r bar of p under Lorentz transformation goes to u r bar p S inverse lambda. So, this quantity is Lorentz invariant so you can compute this in any invariant especially, we can go to the rest frame and evaluate this quantity here. So, this is same as u bar r of m 0, u s of m 0 and you know what are these u 1, u 2 you already know, from this expression here.

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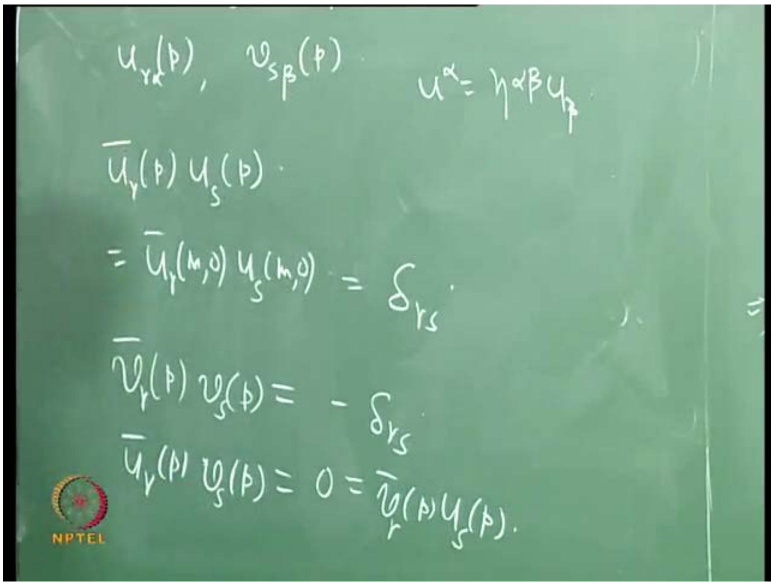

So, you see by putting explicitly the values for u bar and u s that, this is nothing but delta r s. Similarly, you can show that v bar r of p, v s of p is equal to minus delta r s. Whereas, u bar r of p, v s of p is equal to 0, which is also equal to v bar r of p u s of p. So, by explicit computations and using Lorentz invariants so, we have seen that the momentum space, the spinors in the momentum space obey these following relations.

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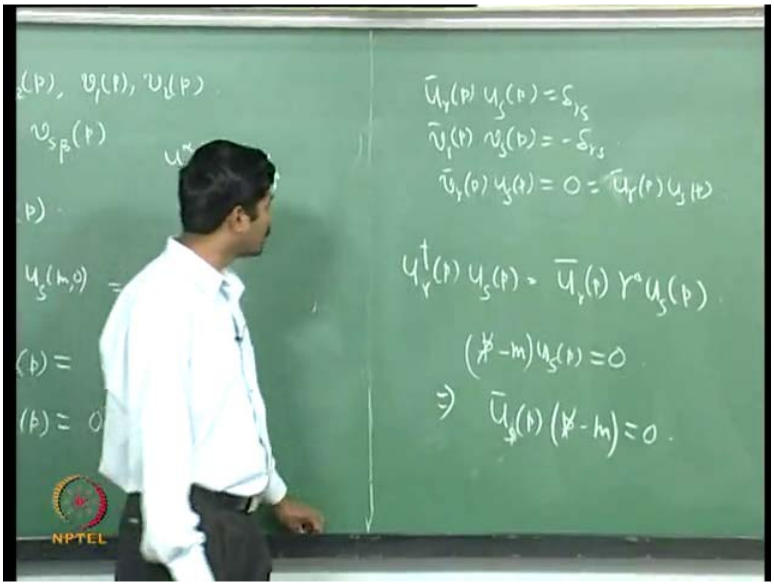

u bar r p u s p equal to delta r s v bar r p v s of p is minus delta r s v bar r p u s p equal to 0, equal to u bar r p u s p. We will extensively use all this relations later on. Unlike u bar p u s p, this quantity u dagger r p u s p is no longer Lorentz invariant. So, however we can just use the Dirac equation and then we can evaluate, let us say, this quantity by using the Dirac equation.

So, let us see what are the, what is the expression for this quantity, this is nothing but u bar r p gamma 0 u s p. By definition of u bar this is simply given by this. Now, we use the Dirac equation for u s, u s of p obviously this Dirac equation p slash minus m u s p equal to 0. If you take the Hermitian conjugate this equation, then you will get the Dirac equation obeyed by u bar s of p and the Dirac equation for u bar is given by u bar s of p, p slash minus m equal to 0.

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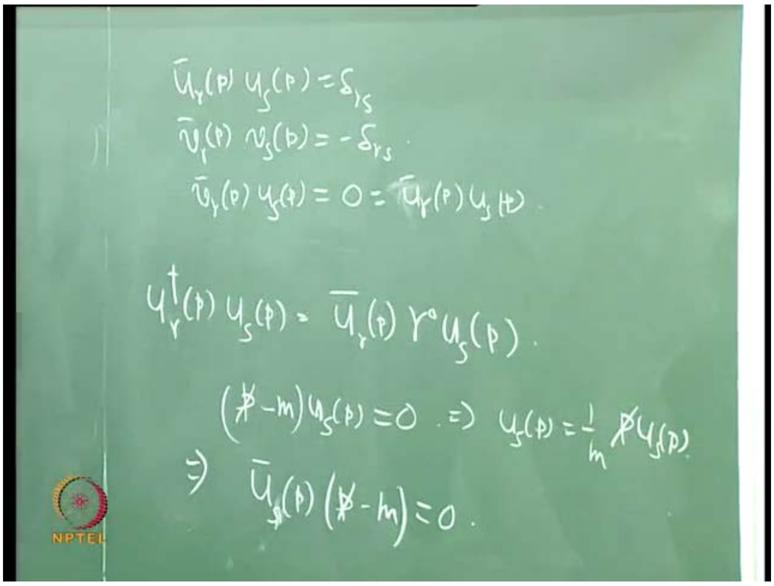

So, to evaluate this quantity u dagger r p u s of p what I can do is, that I can use these two equations and especially, for u s of p I will substitute 1 over m p slash u s p. And for u bar is, similarly we can do, we can write it as u bar.

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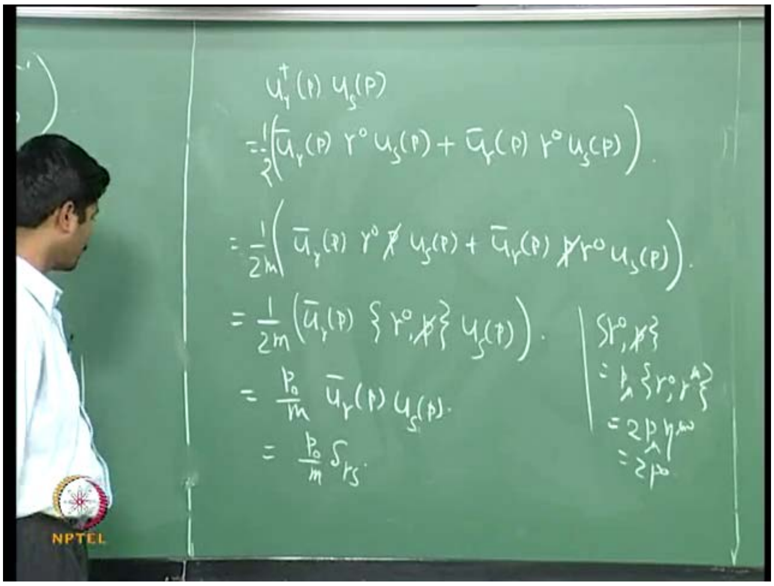

So, this is u r bar p gamma 0 u s of p, which is half of the same quantity this plus u bar r p gamma 0, u s of p. And in the first equation, I will write for u s p 1 over m p slash u s and in the second equation for u bar r p, I will write 1 over m u bar r p slash. So, this is nothing but 1 over 2 m u bar r p gamma 0 p slash u s p plus u bar r p slash gamma 0 u s

p. It is straight forward and this implies you have 1 over 2 m u bar r p anti-commutator of gamma 0 p slash u s p. And what is anti-commutator of gamma 0 p slash? Gamma 0 p slash anti-commutator is p mu gamma 0 gamma mu anti-commutator. So, this is equal to 2 p mu eta mu nu which is 2 p 0. So, what we have got here is, this is p 0 over m which is just a number. So, I can just pull it out u bar r p u s p so, this is nothing but u bar r p u s p is simply delta r s. So, you get p 0 over m delta r s.

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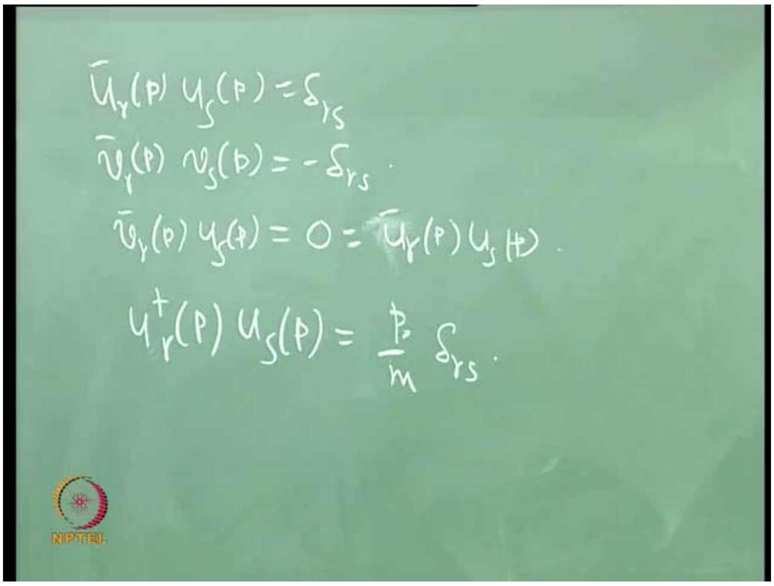

So, to this I can add this relation u dagger r p u s p equal to p 0 over m delta r s. So, what we did is that, we have, we found the solution in the rest frame and then from using these, the four linearly independent solutions in the rest frame, we have derived some important identity.

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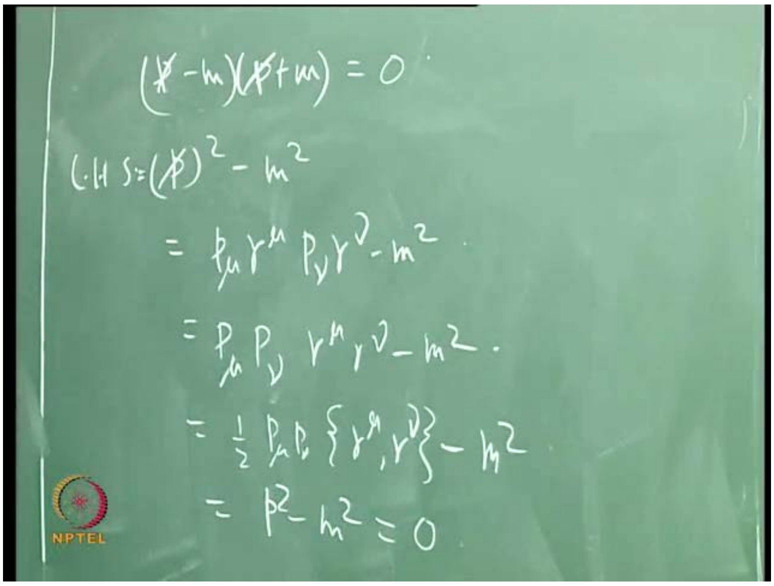

What we will do now is that, we will find the general solution for u s p in any general frame, this you can do by using two ways. First is of course, can use a boost and then you can construct the solution, the general solution for any arbitrary momentum. We will not do that instead what we will do is, we will use some identities like p slash minus m times p slash minus m times p slash plus m equal to 0. Using this identity we will construct the most general solution for any arbitrary p, before that let me prove to you that this is in fact this is 0. This is nothing but p slash square minus m square, but p slash square is p mu gamma mu p nu gamma nu minus m square.

So, this is p mu p nu gamma mu gamma nu minus m square, but because of the symmetry here you can write it as half p mu p nu anti-commutator of gamma mu gamma nu, for the first term. So, this is minus m square, but this is nothing but twice eta mu nu so, this is p square minus m square and p square is m square therefore, p square minus m square is equal to 0 and hence, p slash minus m times p slash plus m equal to 0.

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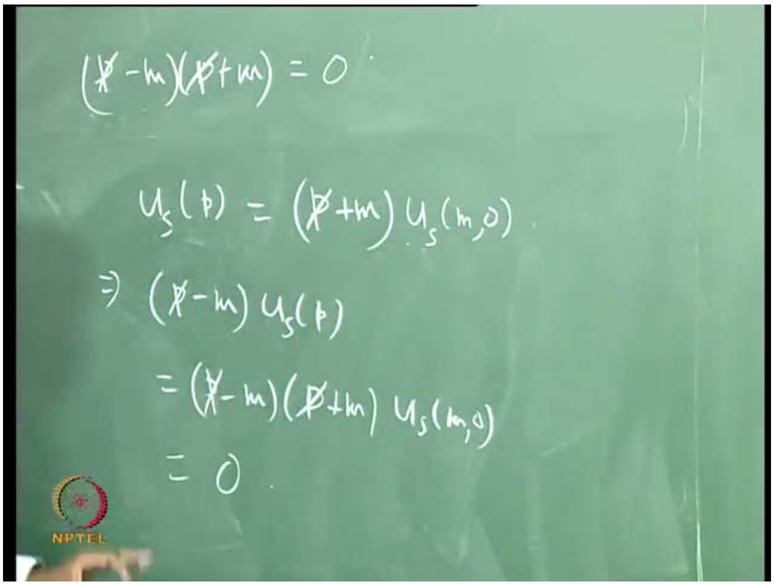

Therefore, if you consider this u s m 0 p slash plus m, this quantity is a solution to the Dirac equation. Using this identity, if I denote this to be, if I define u s of p slash, if I denote u s of, if I define u s of p to be p slash plus m u s of m 0. Then this quantity, then this is a solution to the Dirac equation and for s equal to 1 and 2, there are two linearly independent solution here, for arbitrary momentum. Because, obviously this simply implies that p slash minus m acting on the u s of p is nothing but p slash minus m, p slash plus m u s of m 0 and because, this is 0, this quantity is 0. So, for any u s of this type it is actually a solution to the Dirac equation. So, therefore there are, I have two linearly independent solutions for s equal to 1 and 2, constructed from u s m 0.

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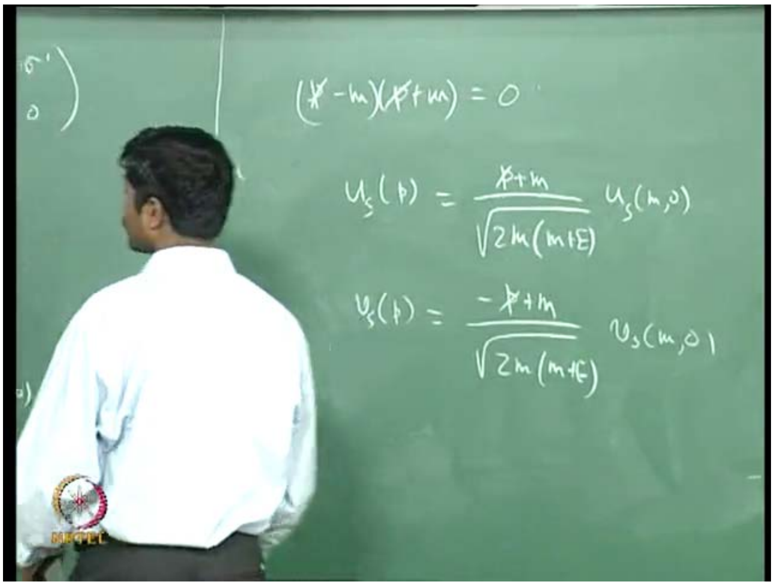

Only thing that we need to, I need to make sure is that, this normalization condition is satisfied. And this will be normalized, if you define u s to be p slash m divided by square root of 2 m times m plus E, I guess u s m 0. Then, this u s is normalized to delta r s, it is normalized such that u s p by u r p is equal to delta r s. So, I have the most general solution, which I constructed by using this identity. Similarly you have, you can construct solution for v s of p, which is minus p slash plus m divided by 2 m, m plus E v s of m 0.

Student: Can you just repeat, how it is the most general solution?

It is just, it is the solution is satisfied for any value of momentum and then there are two linearly independent solutions, that is all. And you can in fact show that this solution in fact here is, they can be obtained by Lorentz boost from the solution in the rest frame.

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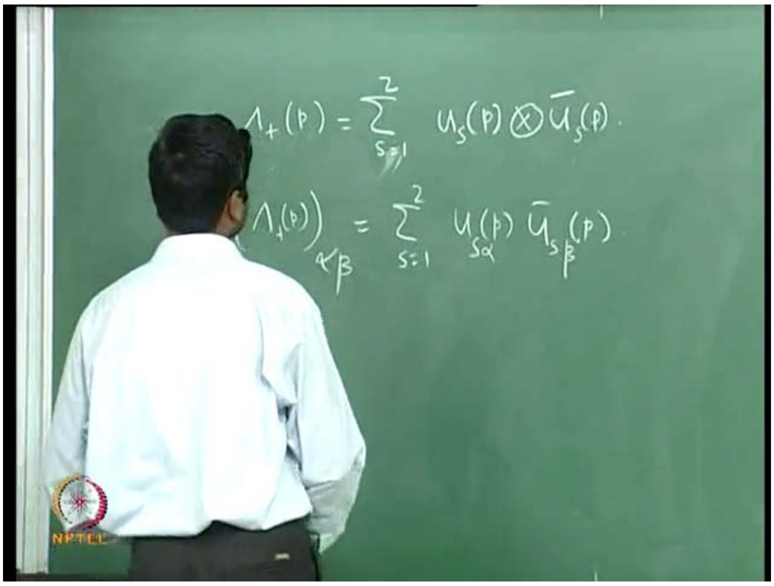

Now what I will do is that, I will introduce a projection operator, which I will define to be this, lambda plus p plus sum over s equal to 1 2 u s p u bar s p. I will show that this is in fact a projection operator, but before that I will explain what do you mean by this symbol. You already know what is u bar u, this is just a number because, this is a row and this is a column and then you can multiply it by using ordinary matrix multiplication, this is number. This product simply can be used to construct a 4 by 4 matrix out of u and u bar. And I define it in terms of its components, in the sense that lambda plus p alpha beta, this symbol simply means that s sum over s equal to 1 to 2 u s alpha of p u bar s beta p, that is all, that is what I mean by this.

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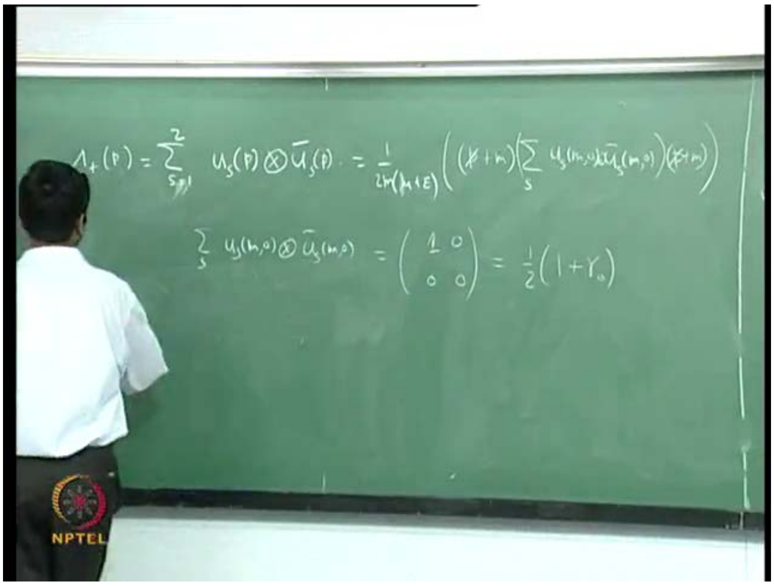

So, now let us consider this u s of p, I already know is given by this and you can show that this is 1 over 2 m times m plus E times, this t slash plus m sum over s u s m 0 u bars m 0 p slash plus m, this is straight forward. Now what is this quantity here? Again I can use the definition of this, that I have introduced and it turns out that s u s m 0 u bar s m 0 is nothing but this. So, this in our representation, in the representation that, in which we have solved for u s m 0 this is simply equal to half times 1 plus gamma 0. So, it is very straight forward because, u s the lower two components are 0's. So, only thing that is non-zero is this so, you have 1 0 1 0 with 1 0 plus 0 1 which is nothing but, identity matrix in the first sector and then all the other things are 0.

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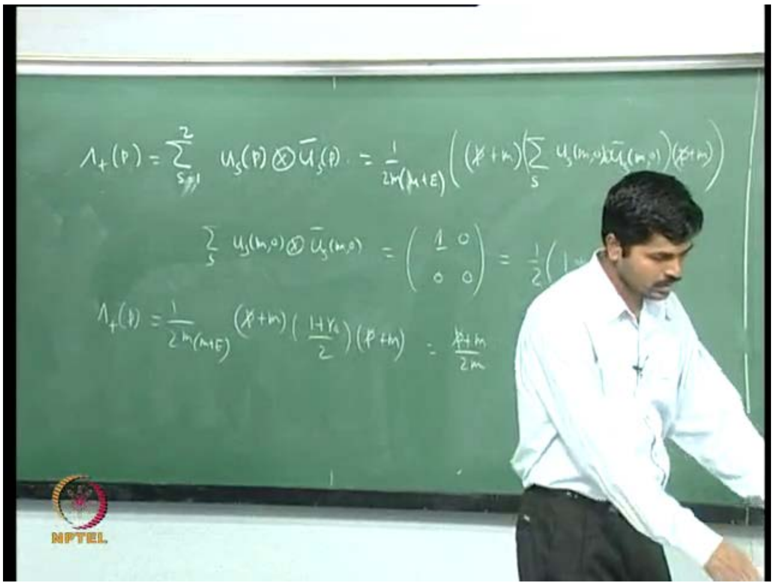

So, I will use this identity here, then I have lambda plus p equal to 1 over 2 m, m plus E P slash plus m 1 plus gamma 0 over 2 p slash plus m. And you can use, you can simplify this algebra here to show that, this is nothing but p slash plus m divided by 2 m. Similarly, I can introduce lambda minus.

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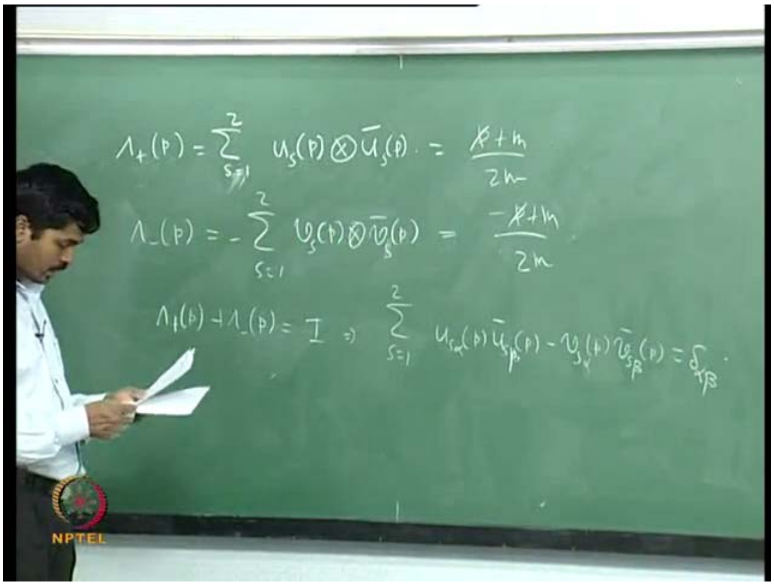

So, I leave it as an exercise for you to show that this is p slash m over 2 m and if I define lambda minus of p equal to minus sum over s equal to 1 to 2 v s of p tensor with v s bar of p. Then, this quantity will be equal to minus p slash plus m divided by 2 m. So, from here I get a complete next relation which is lambda plus of p plus lambda minus of p is identity or if I want to write it in terms of components. Then it simply means that sum over s is equal to 1 to 2 u s alpha p u bar s beta p minus v s alpha p v bar s beta p is equal to delta alpha beta.

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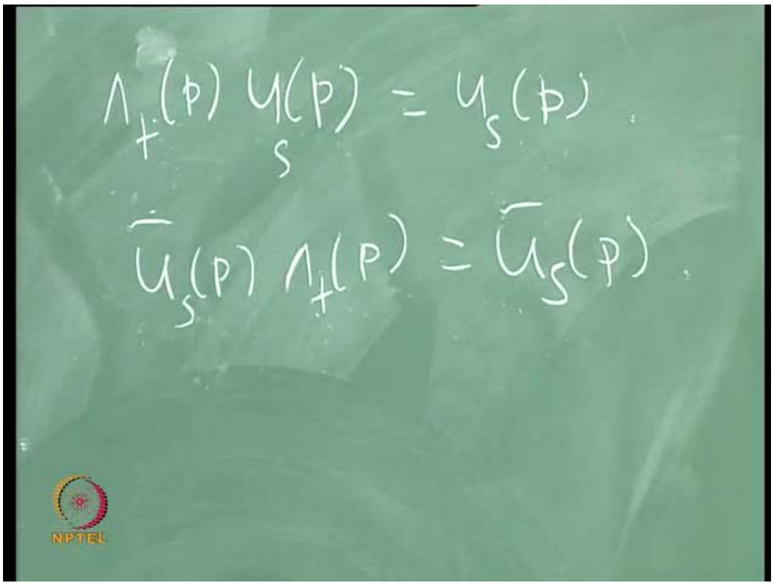

I will also leave it as a home work for you to show that, lambda plus acting on lambda plus p acting on the u p u s p equal to u s p, that is why it is called as a projection operator. Similarly, u bar s p lambda plus p is u bar s p and similar relations for lambda minus acing on v. So, what we did is, we constructed the plain wave solution for Dirac equation. So, the most general solution is of course, is a superposition of these plain wave solutions. What we will do is, in the next lecture we will take the most general solution and quantize the Dirac filed and then we will see what are the implications of this, alright?