Quantum Field Theory

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Module - 3
Free Field Quantization Spinor and Vector Fields
Lecture - 18
Fermion Quantization II

So, what we did yesterday is the following, we started with a, with a vector transformation and then I showed that we can represent it as a bi spinor.

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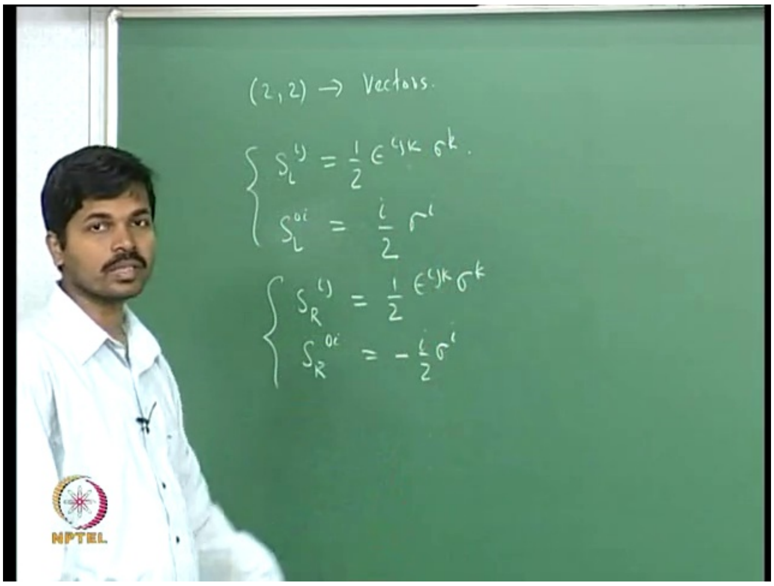

So, 2 comma 2 representation, where Lorentz group is a vector and then we went on defining spinor transformations. So, you have introduced what are the spinors? Especially what are the left-handed weyl spinors and right-handed weyl spinors? What we will do today is, we will first of all we will use the slightly different notation.

So, I will tell you what we been by left and right spinor, and then we will introduce the four component Dirac spinor and construct an invariant action, involving the Dirac spinor. So, what we did is we have introduced this s l i j, this is half epsilon i j k sigma k and s l i 0 or s l 0 i which equal to i over 2 sigma i. So, i just flipped it with a minus sign that is one thing and similar expressions for the right-handed spinor.

We can rewrite this left and right-handed spinor in a slightly different notation, which will help us to construct the Dirac spinor. So, let us rewrite this one or what we can do is, we can also introduce the right-handed generators s r i j equal to half epsilon i j k sigma k and s r 0 i is minus i over 2 sigma i. So, the left-handed and right-handed spinor's transfer under the Lorentz transformation, such that the generators are given by s l and s r s l and s r are given in terms of the power matrices by these formulae.

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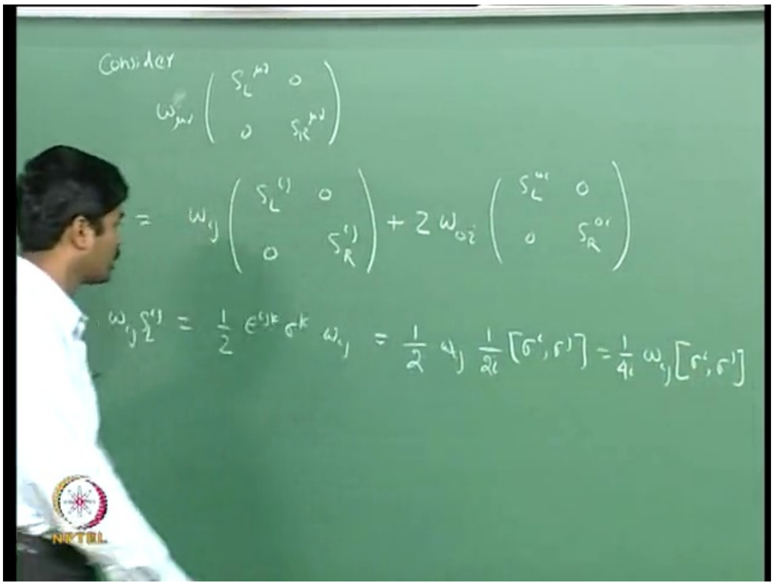

Now look at this subject, which is omega mu nu so consider s l mu nu 0 0 s r mu nu. So, this is nothing but omega i j s l i j 0 0 s r i j plus 2 omega 0 i s l 0 i 0 0 s r 0 i. What is omega i j s l i j? It is given by half epsilon i j k sigma k omega i j and epsilon i j k sigma k, I can write it in terms of commutators of sigma i and sigma j. This is nothing but 1 over 2 times omega i j 1 over 2 i commutator of sigma i and sigma j. So, this is 1 over 4 i omega i j sigma i sigma j commutator. I can substitute it here.

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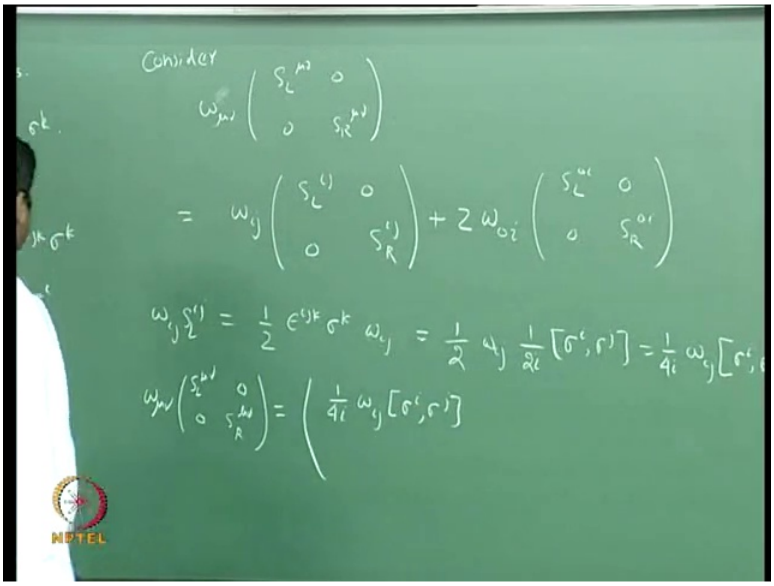

Then, what I get for omega mu nu s l mu nu 0 0 s r mu nu is equal to 1 over 4 i. So, here 1 over 4 i omega i j sigma i sigma j commutator.

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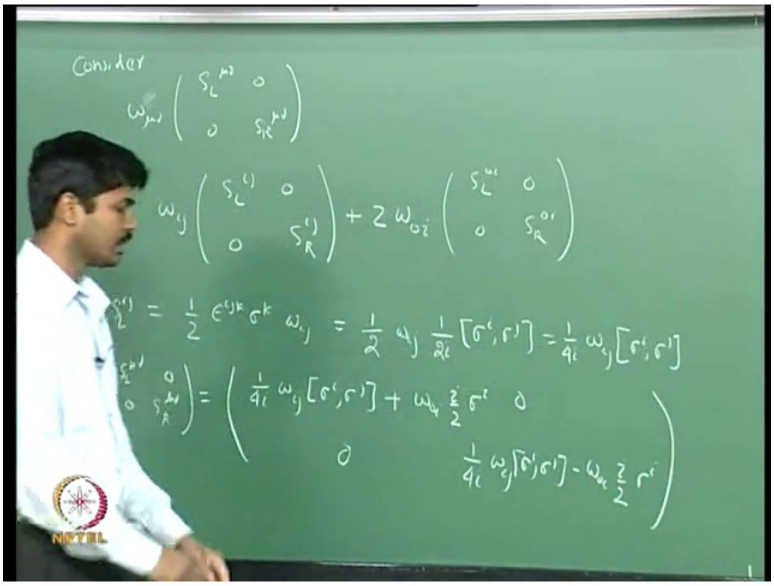

And then plus i over 2 and 0 0 minus omega 0 i, i over 2 sigma i. So, this is the transformation rule.

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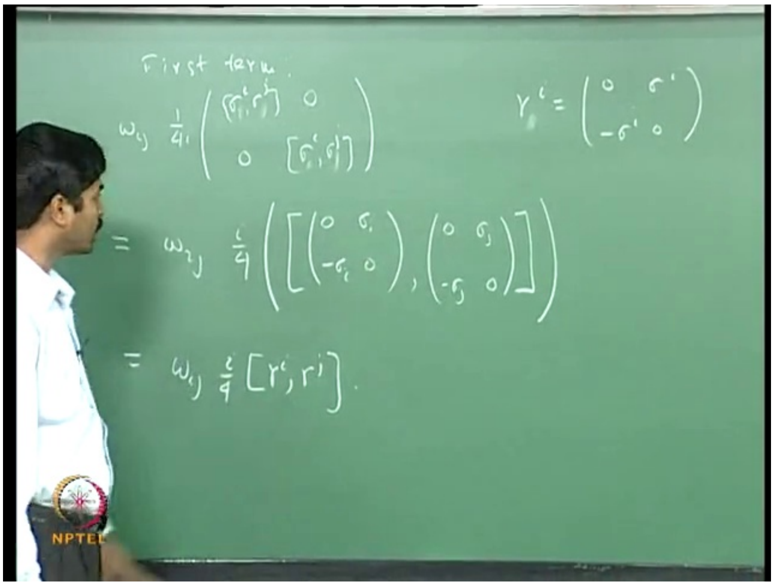

This object here, the first term which is given by 1 over first, which is omega i j 1 over 4 i sigma i sigma j commutator noise 0 0 sigma i sigma j, I can rewrite this term as omega i j times i over 4 0 sigma i minus sigma i 0. So, what I did is I instead of this commutator here I took two matrices 2 4 by 4 matrices, there I will denote them as gamma i.

So, gamma i is 0 sigma i minus sigma i 0 then the commutator of gamma i gamma j is given by this, you can already see this 0 sigma I, will give you, commutator here will give you, commutator of sigma i sigma j with a minus sign. This minus sign will make, this minus 1 over i will make it i over 4. So, and when the commutator works so if I introduce gamma i to be this, then what I have here is omega i j times i over 4, commutator of gamma i gamma j. Here I have sigma i sigma j sigma i sigma j and that gives me the commutator of gamma i gamma j gamma with a factor of i over 4. Now I can similarly, look at the second term here.

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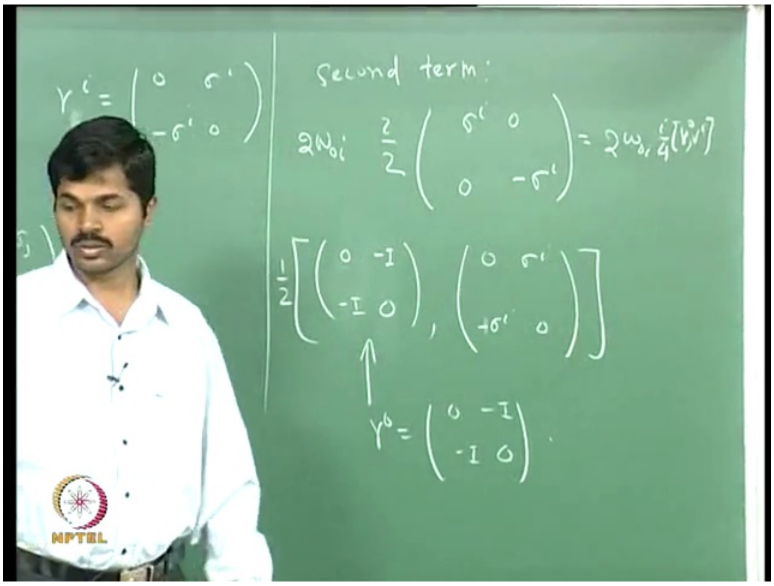

The second term is nothing but 2 omega 0 i s l 0 i s r 0 i. So, the second term can similarly, be i over 2 sigma i 0 0 minus sigma i this quantity here, is nothing but the commutator of minus identity 0 0 minus identity. If you take this with 0 sigma i minus sigma i 0 this commutator then this is nothing but with a factor of half. Then this will give you sigma i 0 0 sigma i, this I have done slightly a mistake, this is a 0 minus i minus i 0 commutator of identity with anything is 0.

So, that is not actually true, when you consider this you can see that if you multiply this with this you get a sigma I, which is here. And this one will, with this will give you minus sigma i. The other term and the commutator, will give you minus of that so both of them will get this half vector will cancel. So, that will give you this so this matrix here is half times commutator of these two matrices. I can define something which I will call as gamma 0 gamma 0 is 0 minus i minus i 0, then this quantity here is nothing but 2 omega 0 i times i over 4 commutator of gamma 0 gamma i.

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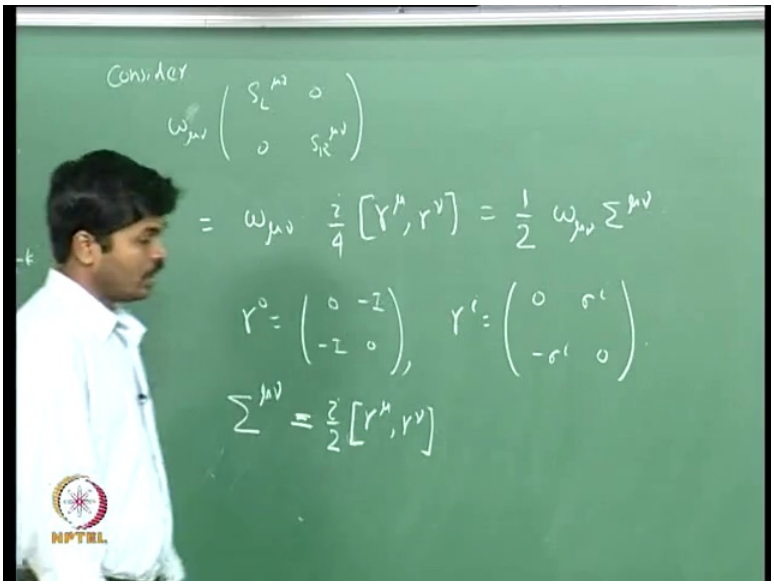

So, what you have seen is, if you consider this term here, then this is nothing but omega mu nu times i over 4 gamma mu gamma nu commutator. Where gamma mu is such that gamma 0 is 0 minus i minus i 0 and gamma i is 0 sigma i minus sigma i 0. With this definition of the gamma matrices, this quantity here is written like this. I will introduce this generator sigma mu nu sigma mu nu is nothing but i over 2 commutator of gamma mu gamma nu. So, if I define sigma mu nu to be this, then this quantity here is half omega mu nu sigma mu nu.

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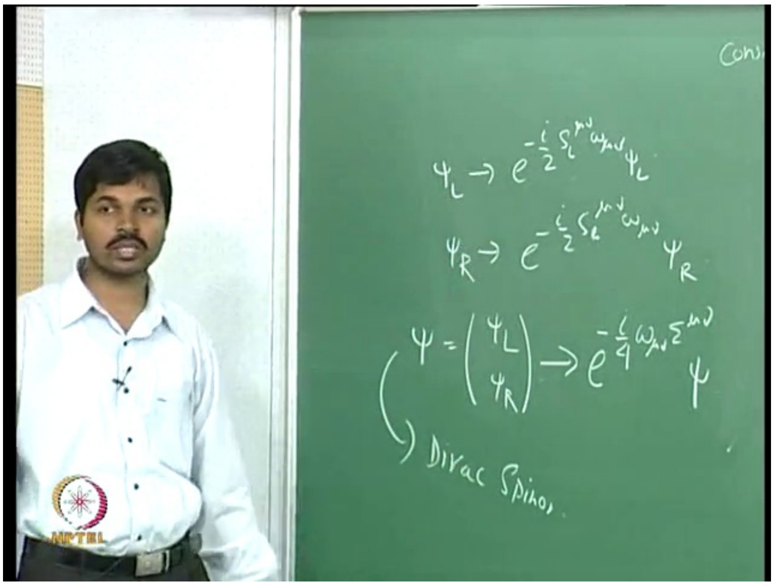

Now, if you look at how left and right spinor 's transfer psi L goes to e to the power minus i over 2 s l mu nu omega mu nu psi L and psi R goes to the e to the power minus i over 2 s r mu nu omega mu nu psi r. So, if I, if I define the four component spinor psi to be this psi L and psi R, then this four component spinors psi under Lorentz transformation, transfers in the following manner e to the power minus i over 4 omega mu nu sigma mu nu psi. So, what we are shown doing all this exercise is that, if you consider two component spinors, two component weyl spinors psi L and psi R, then first of all psi L with this transformation becomes a left handed spinor.

Psi R is a right handed spinor in the sense that, here we will show later that left handed spinors are spinors of positive velocity in the sense that, their spin is along the direction of propagation. Whereas, these are the ones with negative velocity, in this aspect the spin is minus of the direction of propagation say, it is, it is the opposite of the direction of propagation.

So, if psi L and psi R are defined to be this, then we can introduced a four component Dirac spin, which is just psi L psi R which under Lorentz transformation transfers like this. So, this sigma mu nu's at the generators of the Lorentz transformation. The psi is known as a Dirac’s spin obviously, it does not form an irreducible representation under Lorentz transformation. It reduces to these two left and right handed representation, which is obvious in the calculation that we have done here.

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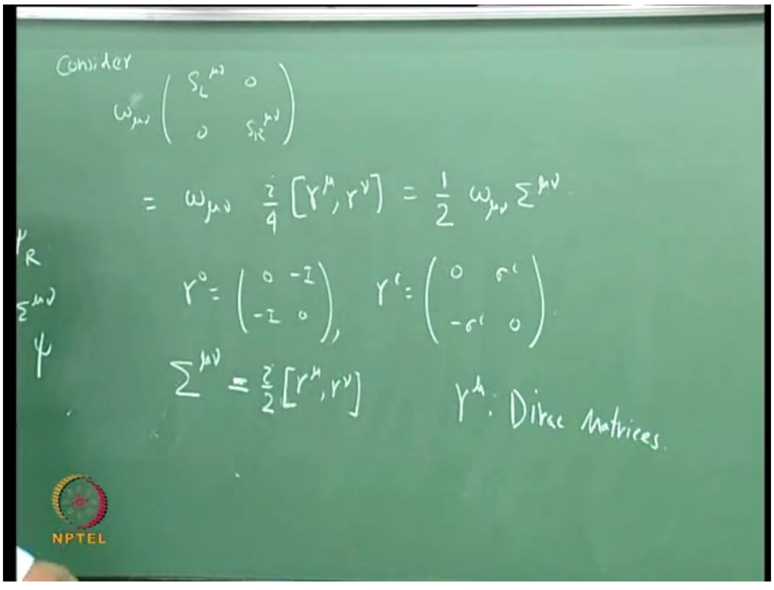

These gamma matrices, which are gamma 0 and gamma i are known as the Dirac matrices. So, gamma mu are the Dirac, Dirac matrices.

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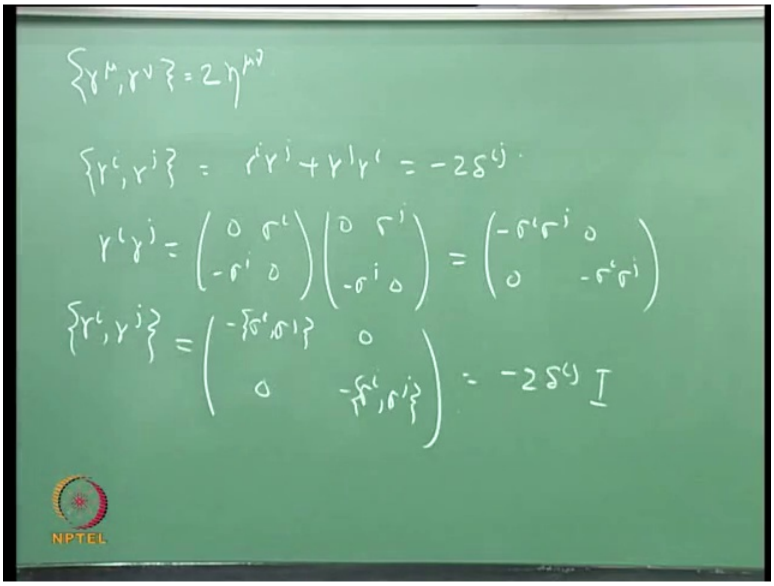

These Dirac matrices satisfy the following relation. This you can easily verify let us consider this relation gamma i gamma j anti commutator. This is nothing but gamma i gamma j plus gamma j gamma i gamma i gamma j is 0 sigma i minus sigma i 0 0 sigma j minus sigma j 0. So, this is minus sigma i sigma j 0 0 minus sigma i sigma j.

So, the anti commutator here is gamma i gamma j anti commutator is nothing but minus sigma i sigma j anti commutator. And here 0 0 minus sigma i sigma j anti commutator what is sigma i sigma j anti commutator? So it is delta i j, twice delta i j. So, this is minus 2 delta i j times the identity matrix 4 by 4 identity matrix. So, this anti commutator here is minus two delta i j, you can show that gamma 0 square is identity. From this definition it is obvious that, gamma 0 square is identity and gamma 0 gamma i anti commute.

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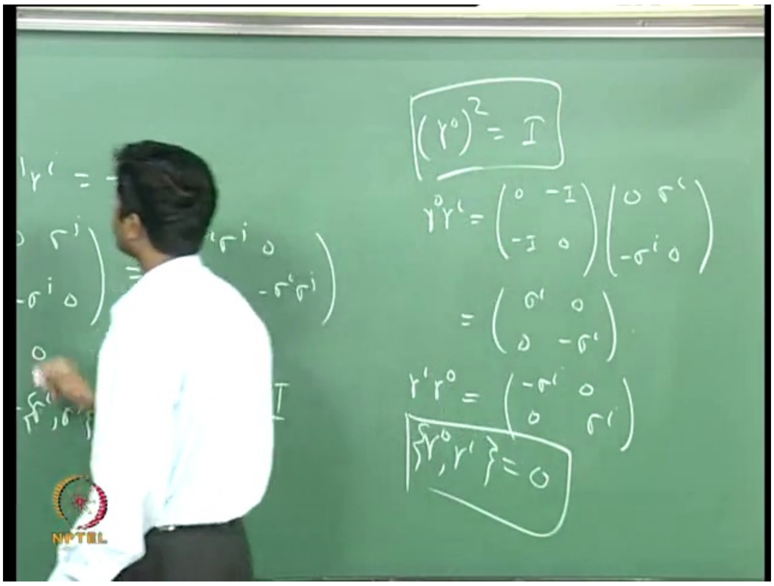

Gamma 0 square is identity, gamma 0 gamma i is 0 minus i minus i 0 whereas, gamma i gamma 0 is minus sigma i 0 0 sigma i. So, this simply implies that gamma 0 gamma i anti commutator is 0. So, this and this amongst the saying that, the anti commutator of gamma i, gamma mu and gamma nu is 2 delta mu nu. So, this is the anti-commutation relation that is satisfied by the Dirac matrices. Now what you can do is, we can, given a four component Dirac spinor, we can recover this gamma L and gamma R by making a projection. So, the projection is defined in terms of something, which is known as gamma 5.

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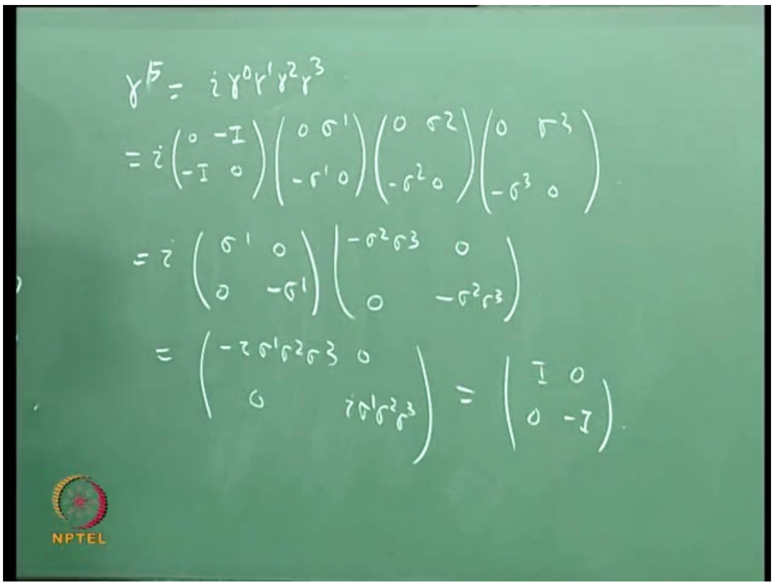

Gamma 5 in our notation is i gamma 0 gamma 1 gamma 2 gamma 3, you can, very straight forward, it is very straightforward to multiply all this. And then find an expression for gamma 5, this is 0 minus i minus i 0 0 sigma 1 minus sigma 1 0, multiplication of these two gives me this. Whereas, here minus sigma 2 sigma 3 0 0 minus sigma 2 sigma 3 and this is nothing but minus i sigma 1 sigma 2 sigma 3 0 0 i sigma 1 sigma 2 sigma 3. What is sigma 1 sigma 2 sigma 3? Sigma 1 sigma 2 is i sigma 3, sigma 3 square is 1. So, this is rest sigma 1 sigma 2 sigma 3 is i times identity matrix therefore, this one will give me i 0 0 minus i.

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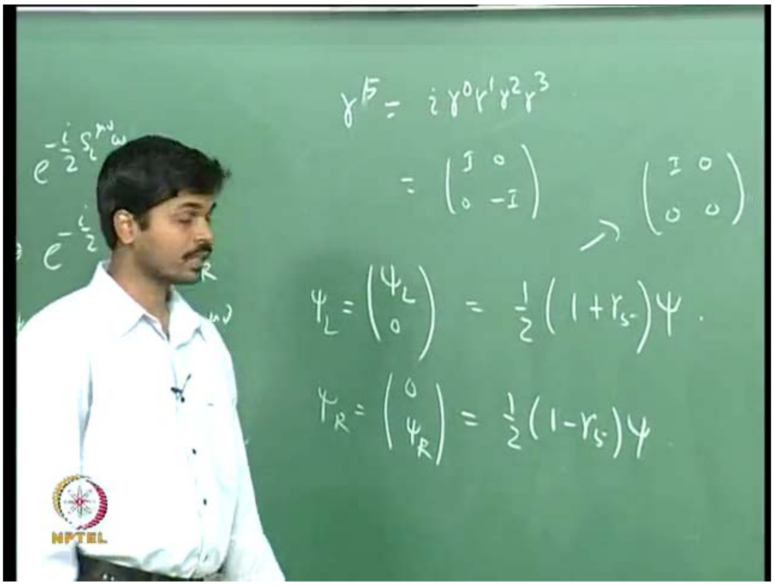

So, our gamma 5 is simply given by so you can see that psi L 0 is nothing but half times one plus gamma 5, acting on psi. Because, 1 plus gamma 5 divided by 2 will be simply identity 0 0 0 therefore, it projects out the right handed spinor and only the left handed part is left out. Whereas, 0 psi r is half 1 minus gamma 5 psi, this gamma 5 is known as the chirality operator and you can see that this quantity is an Eigen state of gamma 5. I will call this to be by abuse of notice, I will call this to be four component psi L and then I will call this one to be psi R.

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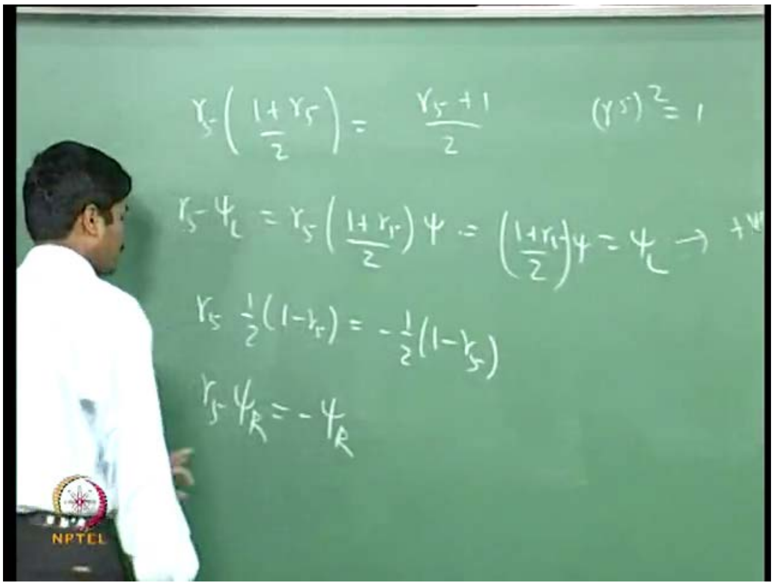

Then, you can see gamma 5, 1 plus gamma 5 over 2 this is nothing but gamma 5 plus 1 divided by 2. So, this is same because gamma 5 square is equal to 1 because of this reason gamma 5 psi L, which is nothing but gamma 5 times 1 plus gamma 5 divided by 2 psi. But gamma 5 acting on this quantity is same as itself so this is 1 plus gamma 5 over 2 psi, which is psi L. So, psi L is gamma 5 Eigen value plus 1 whereas, gamma 5 times half 1 minus gamma 5 is minus half 1 minus gamma 5. Because of this reason, 5 acting on psi R is minus psi R, that is the reason. So, it is it is called as negative chirality.

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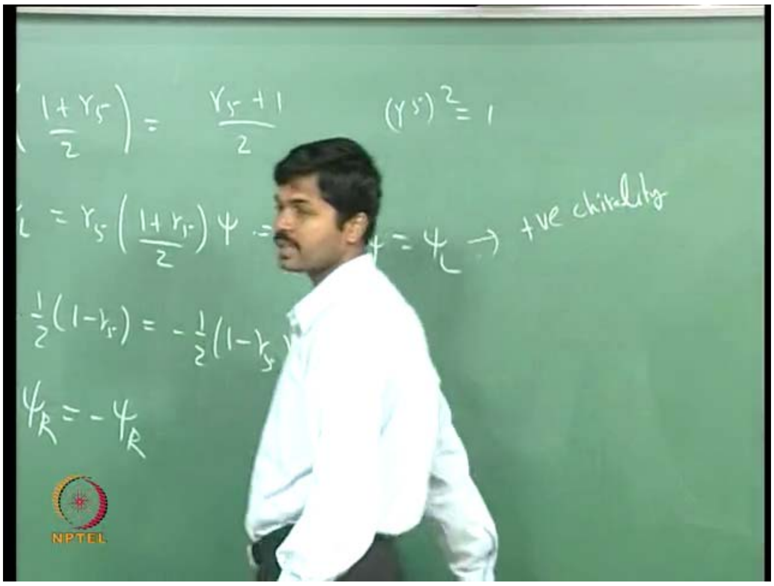

So, the left handed spinors are alpha numerals positive spinors with positive chirality whereas, the right handed spinors are spinors with negative chirality. These are known as the chiral spinor and left handed one is for, the one with positive chirality whereas, the right handed spinor is one with negative chirality.

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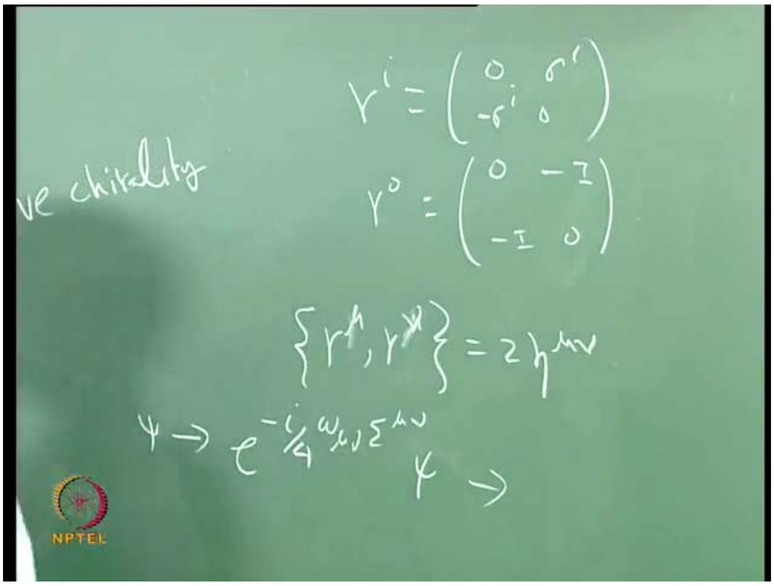

In fact although, we have used these particular form for gamma i's and gamma 0, gamma i is 0 sigma i minus sigma i 0 and gamma 0 is 0 minus i minus i 0. Any set of matrices which 4 by 4 matrices, will satisfy this relation gamma mu gamma nu equal to twice eta mu nu will be as good as these one's. This is only a particular representation for the Dirac gamma matrices, we have used this representation for convenience so that, when we take two copies of this left handed and right handed spinor, we can get a Dirac spinor. You can work in the reverse way in the sense that, you can start with this transformation for the Dirac spinor which is e to the power minus i over 4 omega mu nu sigma mu nu psi.

In any representation then you can show that, this in fact is reducible and it reduces to left handed and right handed representation, because of these it basically reduces to left handed and right handed representation. Both left handed and right handed representations for two reducible representations of Lorentz group. So, this reduction is quite straightforward, in the chiral representation in the sense, if we use the chiral representation, then you can straightaway see that the psi here is nothing but psi L psi R.

In any of the representation it will be, it will not be so straightforward in the sense that, the expression for gamma 5 will be usually different from this one in, in, in, in any of the representation. You use a different representation, you will get a different expression for gamma 5 whereas, this relation is true in the sense that, this actually this operator projects out the right handed part of the Dirac spinor. So, this one is left handed Dirac left handed spinor, in any representation and this one is a right handed spinor in any representation.

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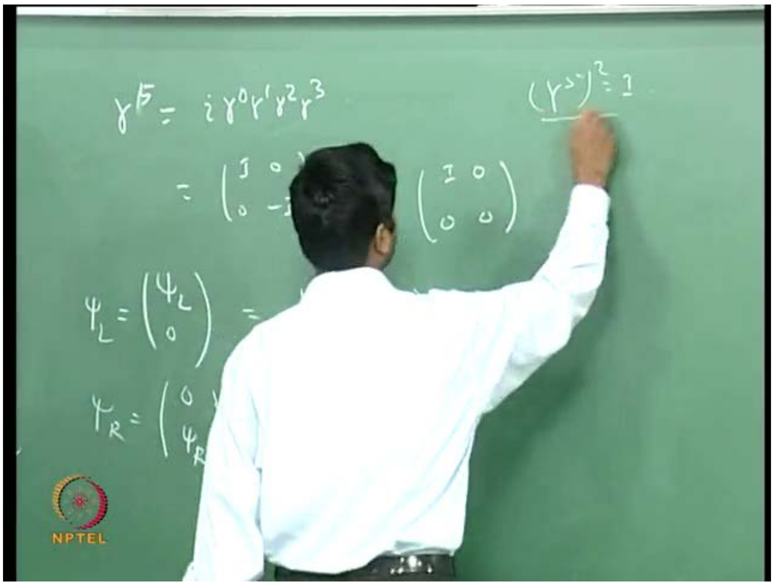

Because of the remarkable property of gamma 5, which is gamma 5 square equal to identity, in any representation. So, because of this property this is always left handed whereas, this is always right handed. And you use a different representation, you will not get this to be psi L 0, this simple form will be lost.

Alright, any question? No?

So now, now that we have understood how the Dirac spinor transfers under Lorentz transformation.

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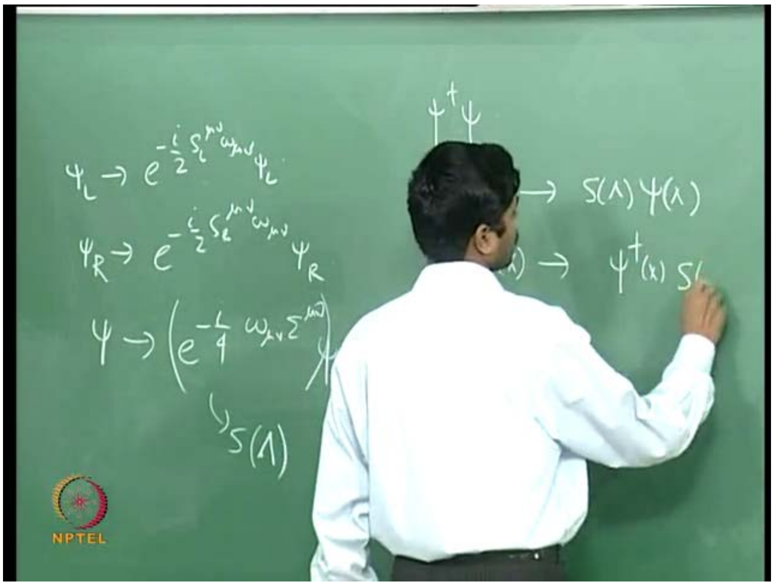

It transforms like psi goes to e to the power minus i over 4 omega mu nu sigma mu nu psi. What we will do now is, we will try to construct various invariants from the Dirac spinor and its derivatives, its partial derivatives.

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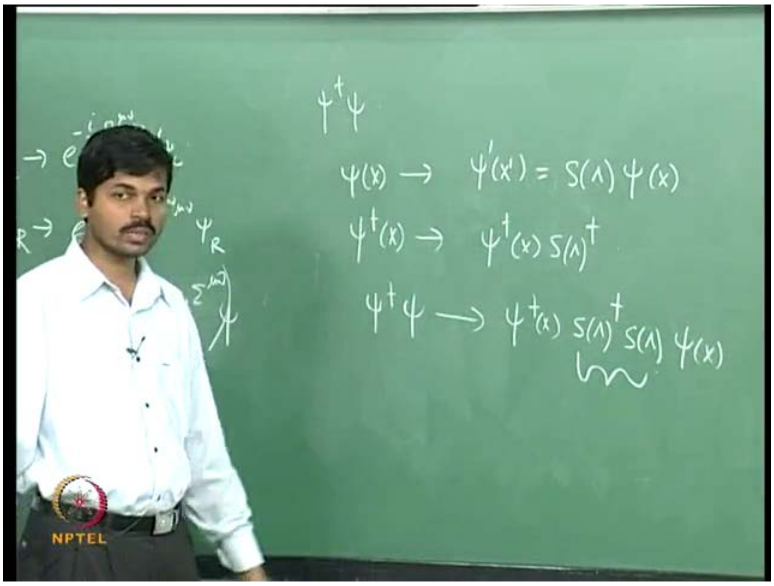

So, first thing the main thing to expect is psi digger psi, usually you might think that this this quantity will be an invariant under Lorentz transformation. However, this is not so we will see why, we will see the, let us call this quantity to be s of lambda so that, it will save me writing this term every time. Then psi digger, then psi of x and Lorentz transformation goes to s lambda psi of x. Whereas psi digger of x from here it follows that, it will transfer like psi digger of x s lambda digger.

Student: X also goes to lambda?

Yeah so what do I mean is that psi prime of x prime this is nothing but s lambda psi of x. So, x prime is lambda x and psi prime is this similarly, here this is psi digger of x s lambda digger. So, you can see that psi digger psi basically goes to psi digger of x s lambda digger s lambda psi of x, what is. So, if this quantity is identity then this would be invariant under Lorentz transformation. However we will soon see that, this is not identities is not a unitary operator therefore, this is not identity. Therefore, psi digger psi is not Lorentz invariant, to do that let us, let us try to see what is this quantity, what is s digger.

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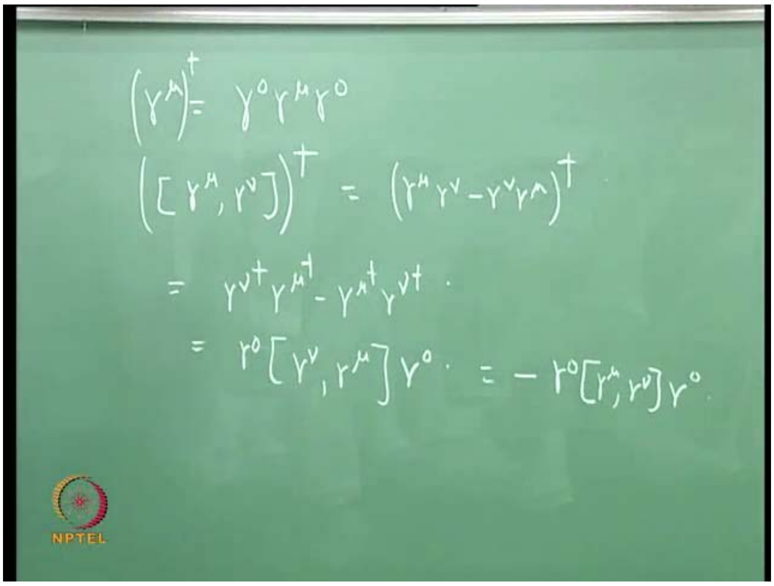

To do that what I will first, what I need to know is, what is gamma mu digger. You can see that gamma mu digger is gamma 0 gamma mu gamma 0, you can, you can write down the expression for gamma mu's in any representation. And then you can explicitly check both the sides and then you can show that they are equal. So, gamma mu digger is gamma 0 gamma mu gamma 0 now, what is this quantity in s mu nu I have this sigma mu nu i times sigma mu nu. So, it is sufficient to know what is the Hermitian consequent of i times sigma mu nu. i times sigma mu nu is just equal to gamma mu gamma nu commutator, after some real coefficient or else I will not worry.

So, the Hermitian conjugate here is gamma mu gamma nu minus gamma nu gamma mu digger, but this is gamma nu digger gamma mu minus gamma mu digger gamma nu digger. So, this is gamma 0 commutator of gamma nu gamma mu gamma 0 or so this is equal to minus gamma 0 gamma mu gamma nu gamma 0. This is not therefore, this is not simply minus of gamma mu gamma nu. If this gamma 0's were not there, then it would say that s lambda digger is simply minus of s lambda.

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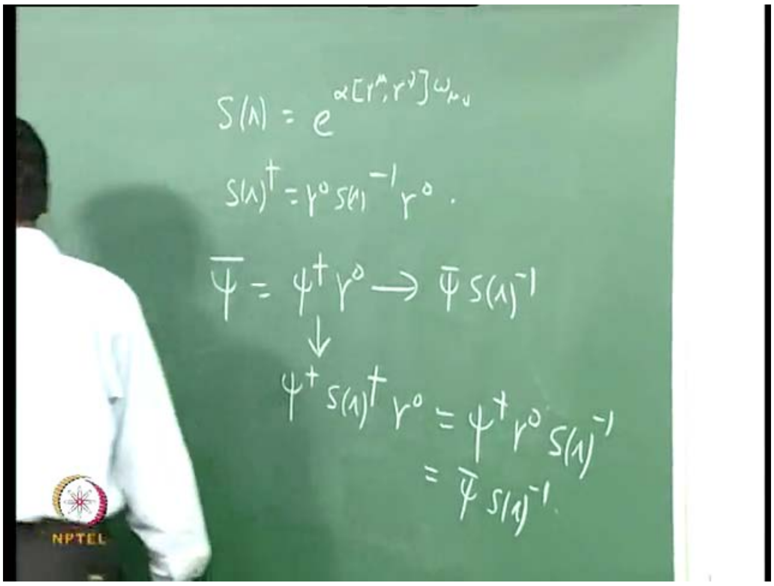

S lambda is e to the power some real coefficients which have I will call alpha times gamma mu gamma nu omega mu nu.

Student: ((Refer time: 36:41))

Yeah there are two i's right? In sigma mu nu there is one i, there is another i here. So, that that will make it real so s lambda digger if this was not there, if this gamma 0's were not there, then we would have got this to be s lambda inverse. But here because of the presence of this is gamma 0, this is gamma 0 s lambda 0 Gamma 0, that is the conclusion that we got from here. Therefore, this quantity here is not identity and hence this is not a Lorentz in variant object.

However, you can define something which I will call as psi bar, which is psi digger gamma 0, then this quantity under Lorentz transformation will go to, psi digger will go to psi digger s lambda digger and gamma 0 digger is gamma 0. So, this is simply gamma 0 here s lambda digger gamma 0.

It is gamma 0 s lambda inverse so this is equal to psi digger gamma 0 s lambda inverse which is psi bar s lambda inverse. So, psi bar under Lorentz transformation transfers to psi bar s lambda inverse. Therefore, if you consider psi bar psi, then this is a bilinear entrance of the Dirac fermions, which is invariant under Lorentz transformation.

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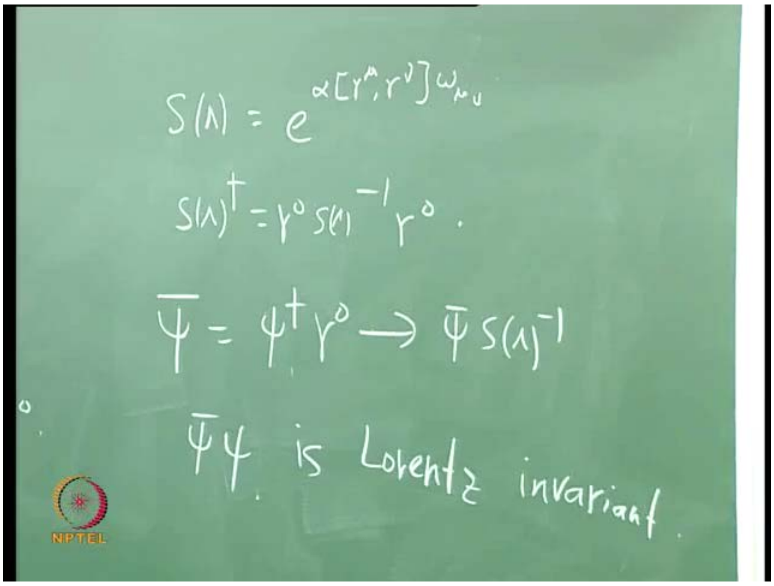

So, what we conclude from this exercise is that, psi bar psi is Lorentz invariant.

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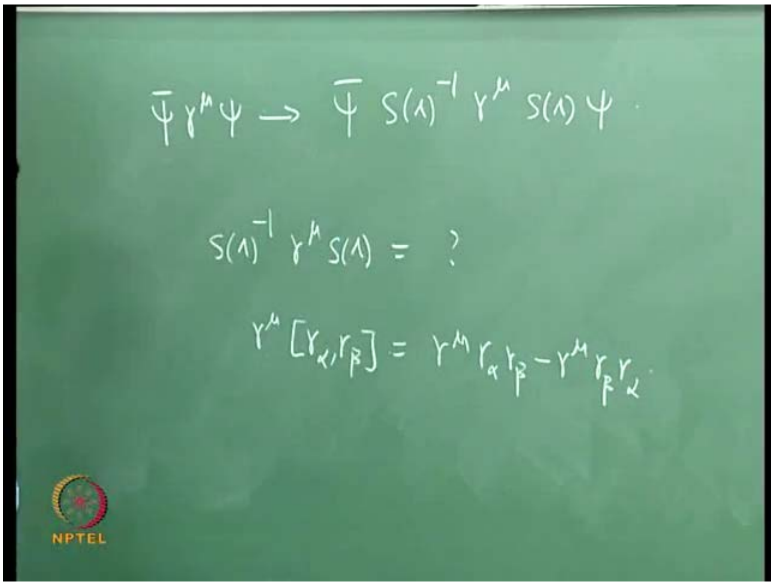

Now what about psi bar gamma mu psi? So, this quantity is nothing but psi bar we already have seen that it goes to psi bar s lambda inverse and when gamma mu and psi goes to s lambda psi. So, we need to know what is this quantity here s lambda inverse gamma mu s lambda, this equal to what?

So you can of course, find it out very straightforward, you can write it as s lambda inverse. I will do it the hard way, let us consider gamma mu s lambda or you just consider gamma mu times gamma alpha gamma beta. This quantity here is a gamma mu gamma alpha gamma beta minus gamma mu gamma beta gamma alpha. What I can do here is, I can, I would like to try to take this gamma mu to the other side of the commutator.

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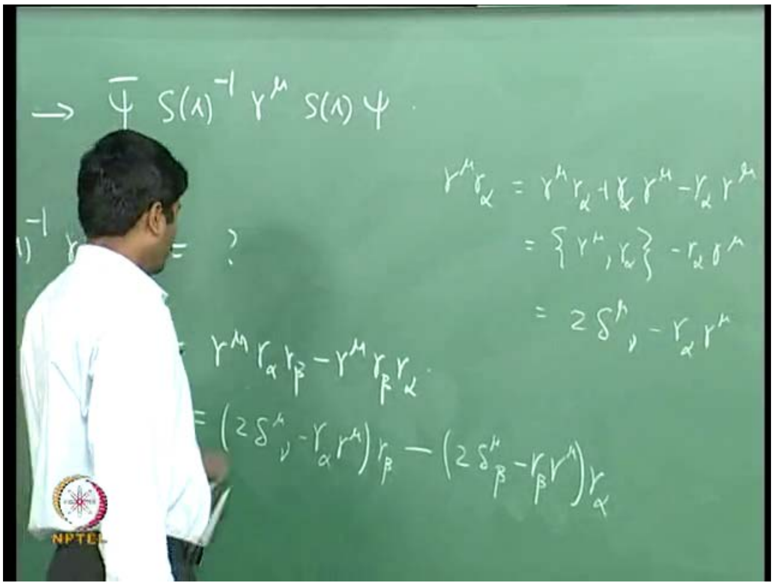

So, here I can flip this will give me gamma mu gamma alpha is equal to gamma mu gamma alpha plus gamma alpha gamma mu minus gamma alpha gamma mu. This is very straight forward I have added and subtracted this term here, but now I can see that this at a commutator of gamma mu gamma alpha minus gamma alpha gamma mu. This anti commutator is nothing but twice delta mu nu minus gamma alpha gamma mu. So, I will repeatedly use this identity once I use, I will get 2 delta mu nu minus gamma alpha gamma mu times gamma beta here and here minus 2 delta mu beta minus gamma beta gamma mu times gamma alpha. So, let us see what we get.

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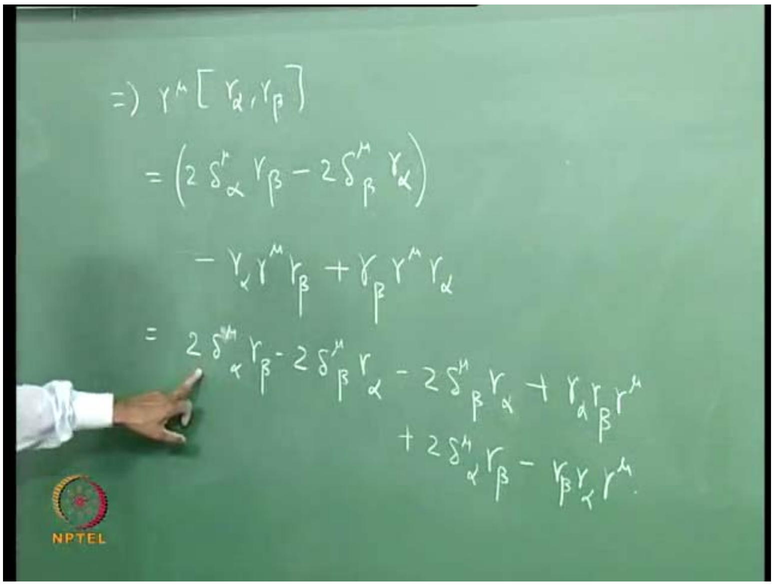

This implies gamma mu gamma alpha gamma beta is equal to 2 delta mu alpha here right, mu alpha gamma beta and minus 2 delta mu beta gamma alpha. And then I have minus gamma alpha gamma mu gamma beta plus gamma beta gamma mu gamma alpha. Now, I will take this further so what I get by doing that is 2 delta alpha mu alpha gamma beta minus 2 delta mu beta gamma alpha. This will give me minus 2 delta mu beta gamma alpha plus gamma alpha gamma beta gamma mu. And this one here will give me, plus 2 delta mu alpha gamma beta minus gamma beta gamma alpha gamma mu. So, this term adds these two terms here so similarly, here these two add.

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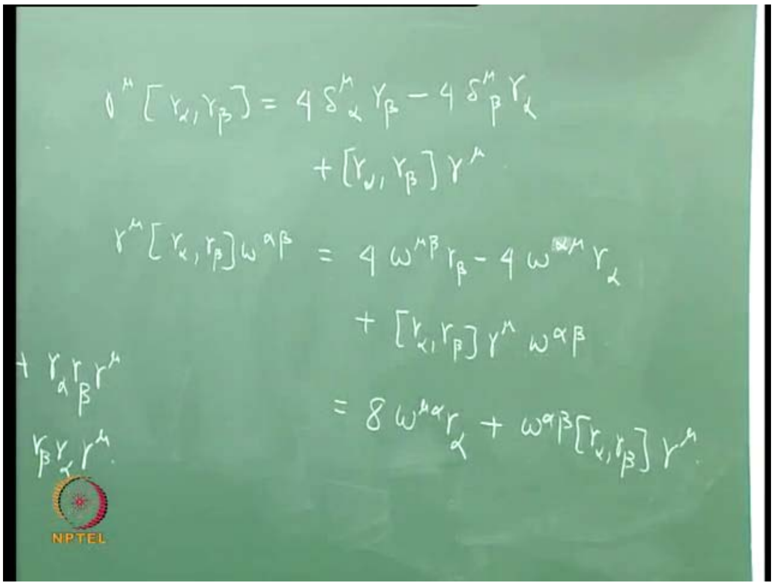

So, what I find finally, is that gamma mu times gamma alpha gamma beta is just equal to 4 delta mu alpha gamma beta minus 4 delta mu beta gamma alpha plus gamma alpha gamma beta gamma mu. Now, if I consider gamma mu gamma alpha gamma beta omega alpha beta, then this quantity will give me 4 omega alpha equal to mu beta gamma beta minus 4 omega mu omega alpha mu gamma alpha. And then I will get plus gamma alpha gamma beta gamma mu omega alpha beta, because of anti-symmetry of omega these two terms will add. So, I will get here 8 omega mu alpha gamma alpha plus omega alpha beta gamma alpha gamma beta gamma mu. So, ultimately so when I use that what I will get here is this quantity is a, is lambda.

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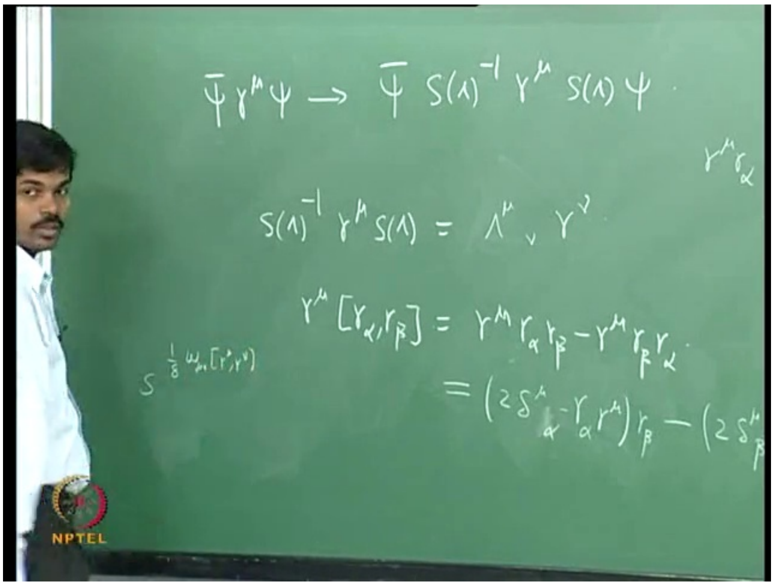

This will simply give me lambda mu nu gamma nu. The reason is the following, this simply says that the commutator of gamma mu with gamma alpha gamma beta is nothing but 8 omega alpha beta gamma alpha. And there is an, a so there is there will be a factor of eight here in s, in the s if you just or we will work it out in the next class, but in the expression for S there is some 1 over 8 omega mu nu commutator of gamma mu gamma nu. So, when you use that finally, what we can see is that, when you use that the commutation relation that we have derived.

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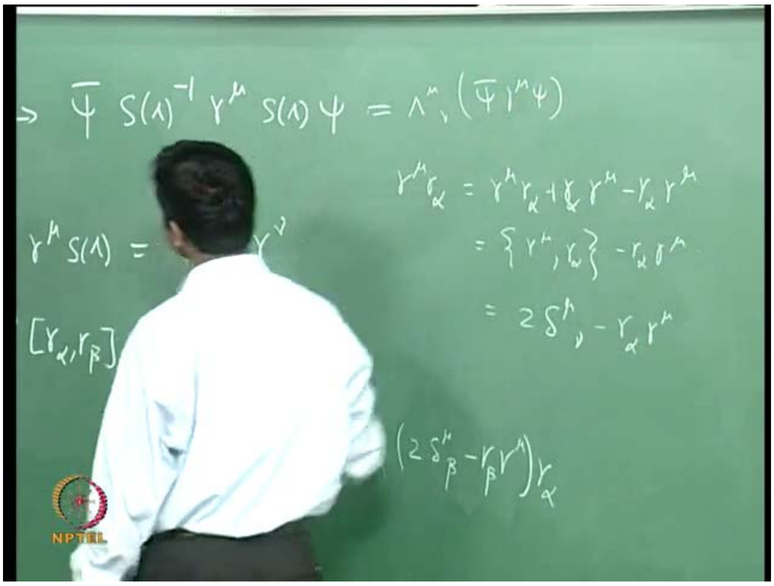

You finally, can get that this is actually lambda mu nu gamma mu. Therefore, this simply implies that, this quantity here is equal to lambda mu nu times psi bar gamma mu psi. Therefore, this combination here actually transforms like a vector under Lorentz transformation. So, what we will do in the next class is that we will complete this calculation here and then we will discuss the remaining bilinears of psi, which transforms variously under Lorentz transformation. And then we will construct the Dirac Lagrangian and quantise it.