Quantum Field Theory

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Module - 3
Free Field Quantization Spinor and Vector Fields
Lecture - 17
Fermion Quantization – I

So, we have learned how to quantize the free electromagnetic field. In today's lecture we will discuss the quantization of free fermions.

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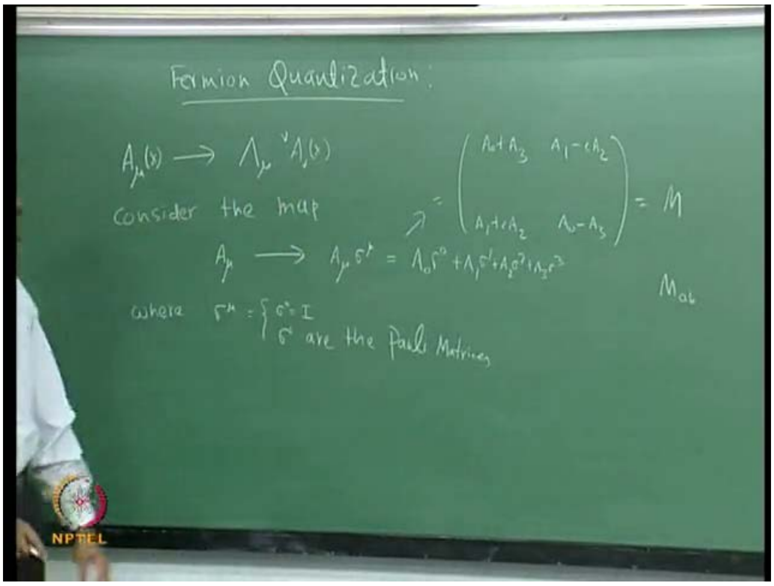

So, let us start discussing fermion quantization, what we need to do first is to construct lagrangian for the fermion fields, which are Lorentz invariant. And then discuss how to quantize them. Before that, let me introduce you how the fermion fields transfer under Lorentz transformation and so on.

You are very much familiar with the vector field, which transforms under Lorentz transformation like this, when you Lorentz transformation on a vector field, it goes like lambda mu nu a mu x. So, let us write down this transformation rule in a fancy way to do that, you consider this myth of vectors in four dimension, to the following quantity a mu sigma mu. Where sigma mu is such that, sigma 0 equal to identity and sigma i are the poly matrices. So, what you have here is a 0 sigma 0 plus a 1 sigma 1 plus a 2 sigma 2 plus a 3 sigma 3.

If you write it in the 2 by 2 matrix form, then what to get is, this quantity is simply a 0 plus a 3 a 1 minus i a 2 a 1 plus i a 2 and here a 0 minus a 3. Let me call this matrix as m so it has components M a b. So, for every vector field in four dimensions you have one such 2 by 2 hermitian matrix and all so for every 2 by 2 hermitian matrix, you can associate a vector in a four vector in four dimensions.

You already know the transformation rule for a vector field, in terms of these components, the four vector components a mu. Now you can ask exactly the same question, how can we phrase the transformation rule of the vector, in terms of the components of these metrics M a b. So, or in other words if you make a Lorentz transformation here, how do the component of the matrix M a b, how the elements of this metrics M a b transform under this Lorentz transformation? So, this is just a fancy way of re-writing this very simple transformation rule for a vector field.

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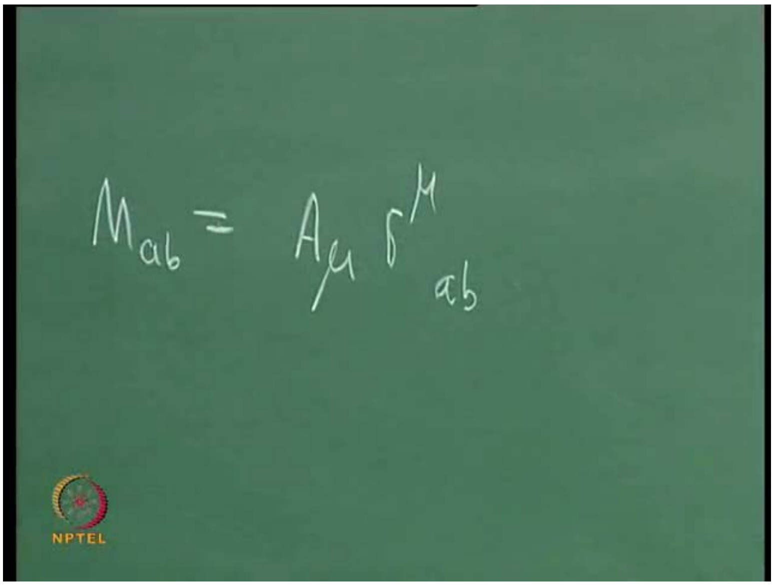

We will work out, it will take as a few more steps. So, M a b is nothing but a mu sigma mu a b, that is what is our M a b. What is delta M a b?

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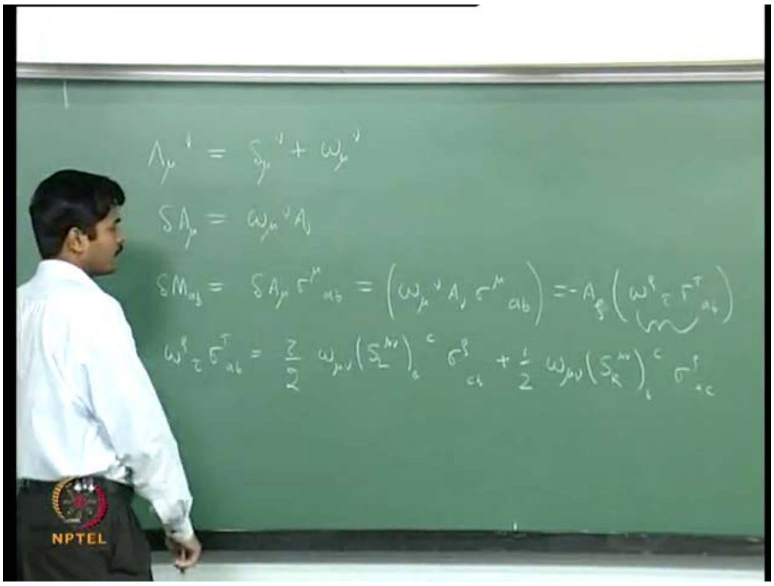

Delta a mu we will consider an infinity small Lorentz transformation lambda mu nu, for an infinity small Lorentz transformation is delta mu nu plus omega nu mu. So, under this infinity small range transformation our gauge field delta A mu transforms like omega mu nu a mu. Whereas, this delta M a b else is nothing but delta A mu sigma mu a b.

This is omega mu nu A nu sigma mu a b. So, this is the transformation rule which is not very complicated of course, but what we will do is that we will re-write this transformation rule in a slightly different notation. So, what I claim is this quantity is nothing but a nu or I will call as a rho omega rho tau sigma tau of a b. I want, I want to re-write this, what it is inside the bracket in a slightly different notation. What I claim is omega rho tau sigma tau of a b is equal to i divided by 2 omega mu nu S L mu nu of a c sigma rho c b plus i over 2 omega mu nu S R mu nu b c sigma rho of a c. I made totally what are these S L and S R mu nu.

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So, S L mu nu is SL i j is nothing but half epsilon i j k sigma k and S L k 0 is half i sigma k S R mu nu star s minus S L mu nu. So, I told you what are these S L mu nu's and S R mu nu's and I claim that if this is how the s l and s r are defined, then I can write this transformation rule for the matrix elements in this fancy way.

This simple transformation rule can be written in such a complicated manner and then we will see why we are interested to write such a transformation rule. But before that, let me just check, verify that this relation in fact holds, I will do it for one of the cases. So, the point here is that you can look at various coefficient here omega 0 1 0 2 0 3 etcetera. So, you can check whether of omega 0 i or omega i j. They individually match on both sides, both left and right hand sides. So, you can just compare the coefficients what I will do is, I will compare the coefficients. I will compare this coefficients for omega 1, 2 for you and then coefficients of omega 1 2 and I will leave the rest for you to workout yourselves.

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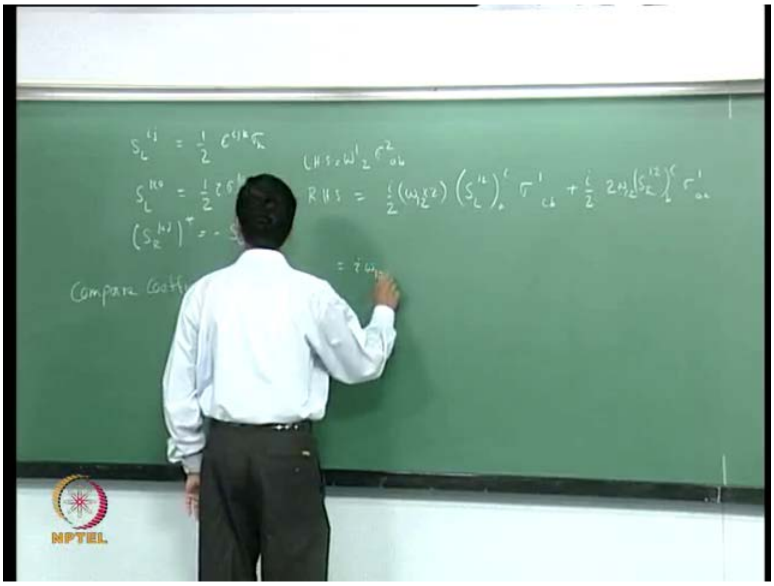

So, to do that what you need to do, you just take rho equal to 1 and so omega 1 2 will contribute here in the following omega, 1 2 sigma 2 of a b in the left hand side. This is a right hand side is i over 2 and omega 1 2 will appear 2 here, because of the symmetry omega mu nu and S L mu nu that will both. If mu equal to 1 nu can be 2 if mu equal to 2 nu can be 1 so both this term contributes of omega 1 2 times 2 S L 1 2 of a c sigma rho equal to 1, that is what we have taken here sigma 1 of c b. And then you have plus i over 2 and again 2 omega 1 2 S R 1 2 of b c sigma 1 a c, it is absolutely straight forward.

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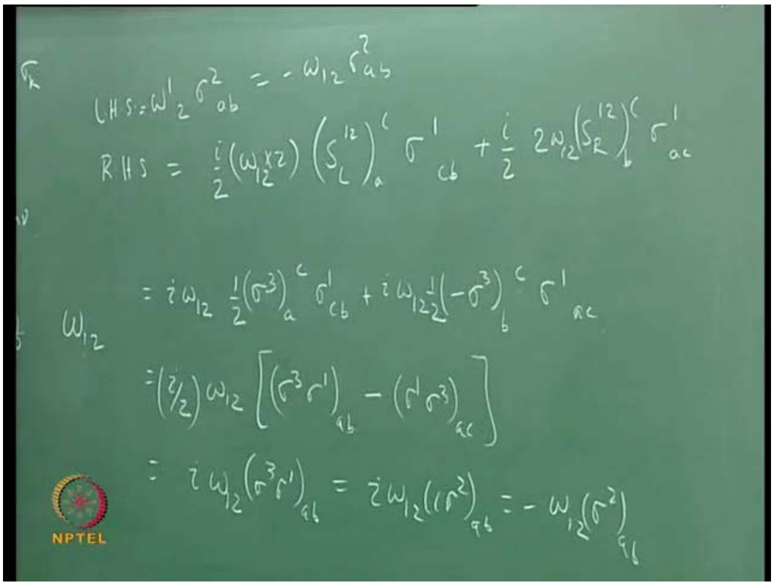

So, what you have is i omega 1 2 but S L 1 2 here you can see is just half sigma 3. So, this is half sigma 3 of a c sigma 1 of c b and here similarly, plus i omega 1 2 S R 1 2 is just minus S L 1 to star. So, this is minus sigma 3 of b c sigma 1 of a c there is a half so what you get is i over 2 omega 1 2 and this is nothing but sigma 3 sigma 1 of a b in the first term and in the second term you have minus. And you can see that, the matrix elements are contracted in such a way that, it is sigma 1 sigma 3 of a c the number 3 here comes. So, it is sigma 1 sigma 3 sigma 1 and sigma 3 anti-commute, therefore this is nothing but 2 sigma 3 sigma 1.

So, this is i omega 1 2 sigma 3 sigma 1 of a b which is nothing but sigma 3 sigma 1 is i times sigma 2. So, this is just i omega 1 2 times i sigma 2 a b which is minus omega 1 2 sigma 2 of a b. This is what we have here omega 1 2, which is nothing but minus omega 1 2 sigma 2 a b. So, see that both left hand and right hand side match.

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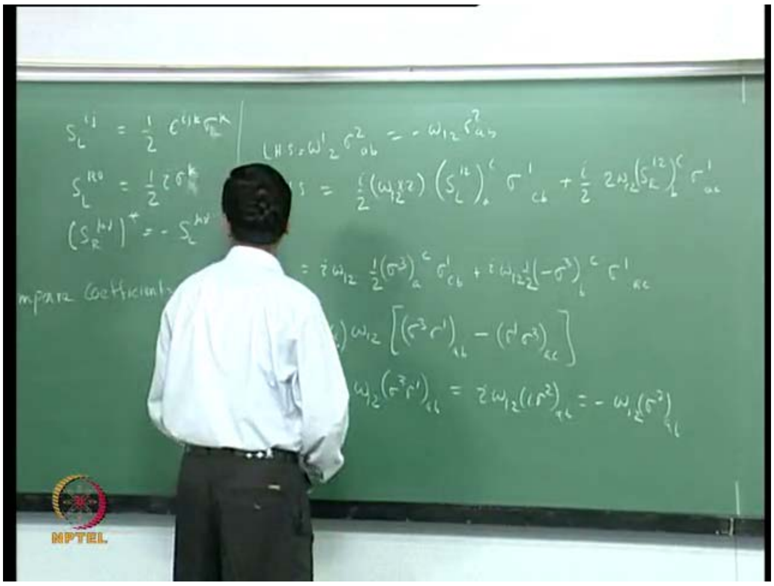

I have to be a little bit careful this sigma k here, this is sigma k here and we are considering a metric, where eta 0 0 h plus 1 eta i j is minus del i j. So, you have minus sign here, minus sign here.

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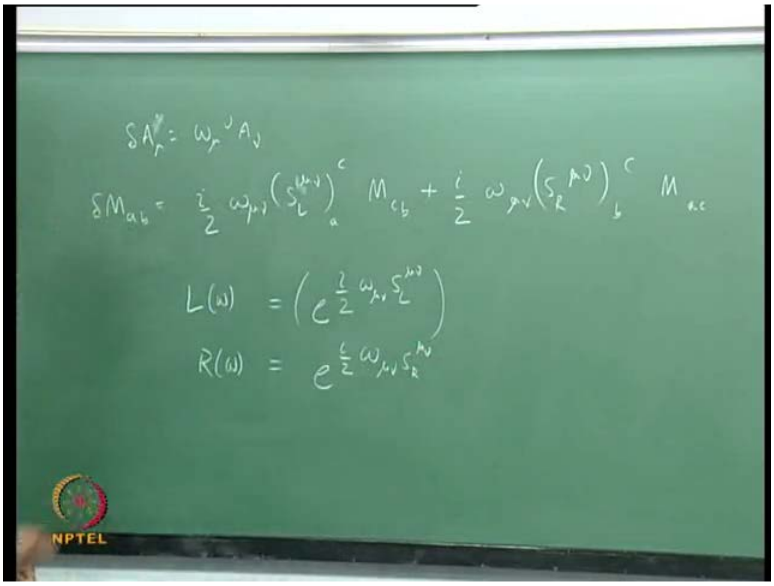

So, what you get for the transformation rule here is that, under infinity small transformation, the vector field transforms like delta a mu equal to omega mu nu a nu. Or the same transformation rule you can face it in a different way saying that, delta m a b is equal to i over 2 omega mu nu s l. Because, our m is nothing but a mu sigma mu so a mu sigma c b is m c b here and a mu sigma a c is m a c, the rest all are as it is. So, the finite form for this transformation will be the following, if I define this matrix lambda or I will call it L of omega a c is nothing but e to the power i over 2 omega mu nu s l mu nu of. And I will call r of omega to be e to the power i over 2 omega mu nu s r mu nu, then the vector field m a b transforms like m goes to L of omega M R transpose of omega.

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This is how a vector field a transforms under Lorentz transformation or in other words this simple transformation a goes to lambda a or a mu going to lambda mu nu a nu can be rephrased in this from. Now, that we, we, we have Lorentz, this we can ask individually what do these transformations mean or in other words do we have fields, which transform under Lorentz transformation in the following manner.

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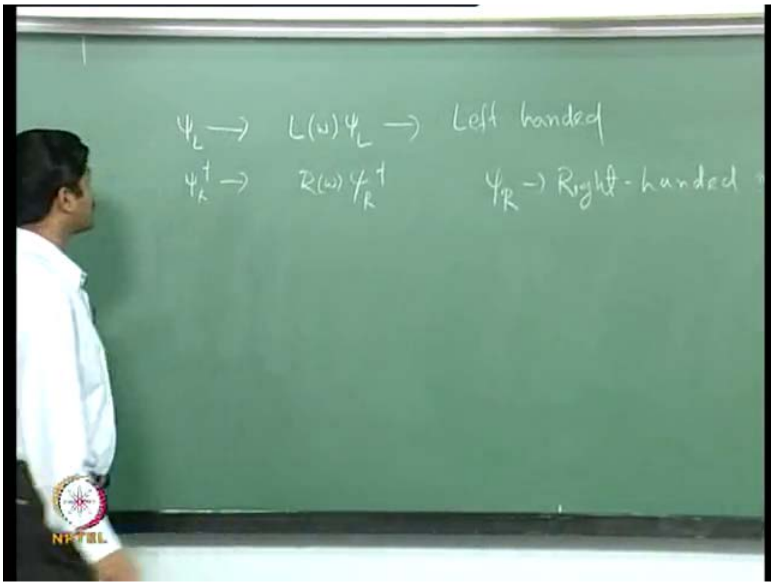

Do we have fields, which goes under Lorentz transformation like, what is notation L of omega psi L or under Lorentz transformation psi r dagger equal to r omega psi r dagger. And what are these fields called?

Student: ((Refer time: 19:43))

Yeah that is why I took this dagger here so psi R goes like psi R, R omega transpose. This is known as the left handed spinor and psi R is the right handed spinor or in other words, yes indeed there exist fields which under Lorentz transformation transfers in these manners. What are these fields? These are fermionic fields and we will see why these are fermionic fields. Because, let us say you consider the left handed field, which under Lorentz transformation transforms like this.

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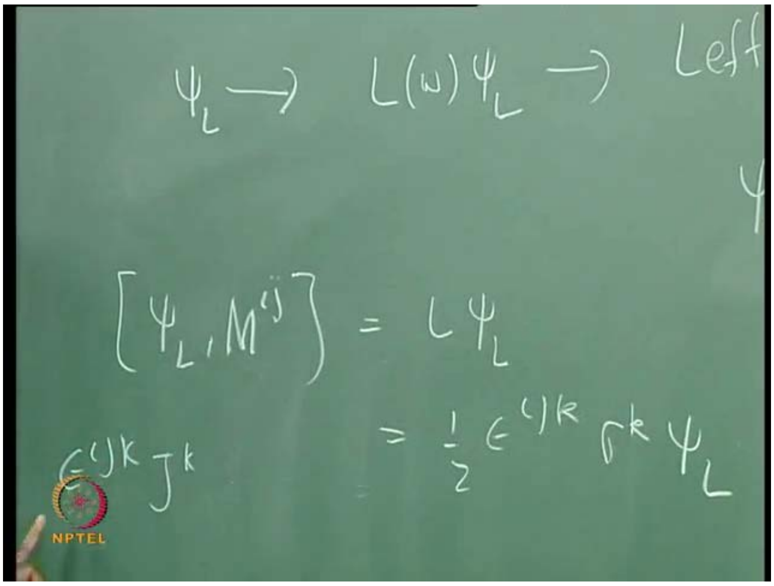

And then you consider the commutation relation here psi L M i j, what is psi l m i j? If you just forget about the orbital part, then this is nothing but for an infinity small transformation, it is just a L acting on psi L, which is half epsilon i j k sigma k acting on psi L. And this m i j is, I mean it is just epsilon i j k j k. Therefore, this, this, this is just a spin half field. So, these, that is the reason I call them spinors.

Student: Sir

Just a minute

Student: psi R, are you writing it as a column vector?

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Psi r all these are two component vectors, psi R psi L both are column vectors.

Student: Then psi R dagger should have R on the right.

Psi R dagger should have R on the right, why? The action here comes this way, the, the right action if you look at these, psi R psi L act form left and then psi R act from the right, but this is. So, therefore the vector field here a mu actually transforms like a weyl spinor, it is because of the presence of the both L as well R, it transforms as a weyl spinor. This is not a surprise because you can look at the commutation relations.

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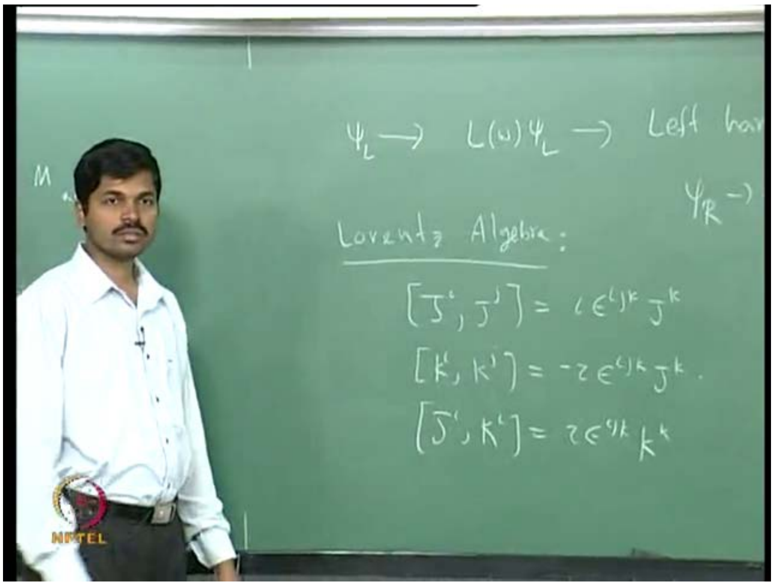

So, the Lorentz algebra is just J i J j is i epsilon i j k J k and k i k j as minus epsilonj minus i epsilon i j k J k, J i K k as i epsilon i j k k k or so these j s are the rotation generators and k s are the boost generators.

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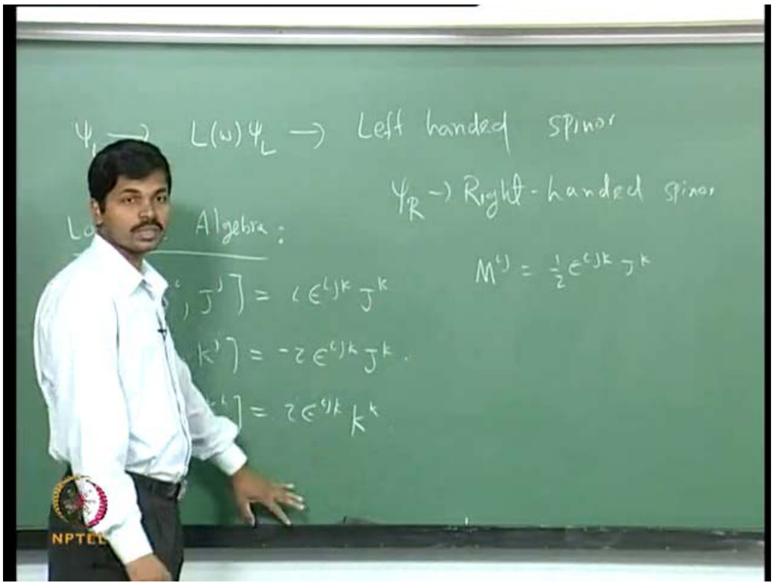

Or in other word M i j equal to half epsilon i j k j k and the 0 i th component of the generators M are identified with these case the boost generator. Then you can see that, the Lorentz algebra just reduces to this.

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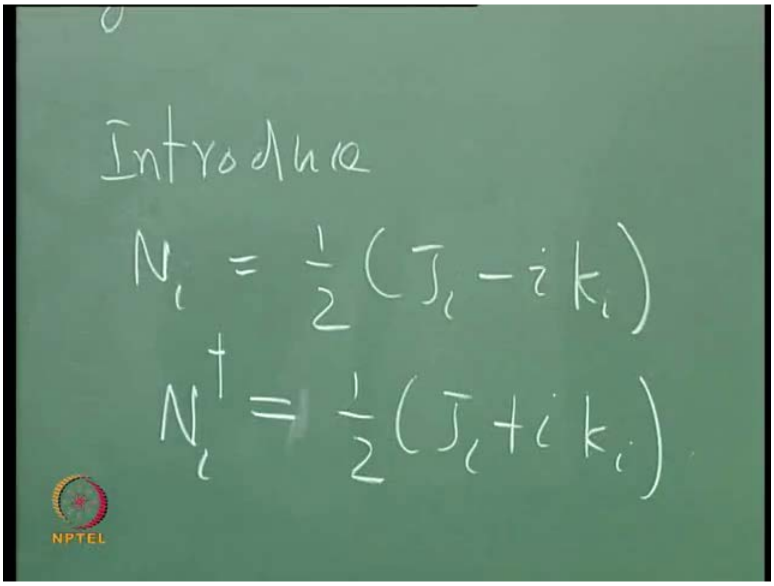

Now what I will do is that, I will introduce N i which is half J i minus i k i and its hermitian conjugate in i dagger, which is given by half J i plus i k i. Then I can look at the Lorentz algebra and then I can ask what is this algebra in terms of the N i s and N i dagger. So, let us work it out. So, what is this spin half field, I mean in quantum mechanics, a spin half field, under I mean rotation it just transforms in the following way.

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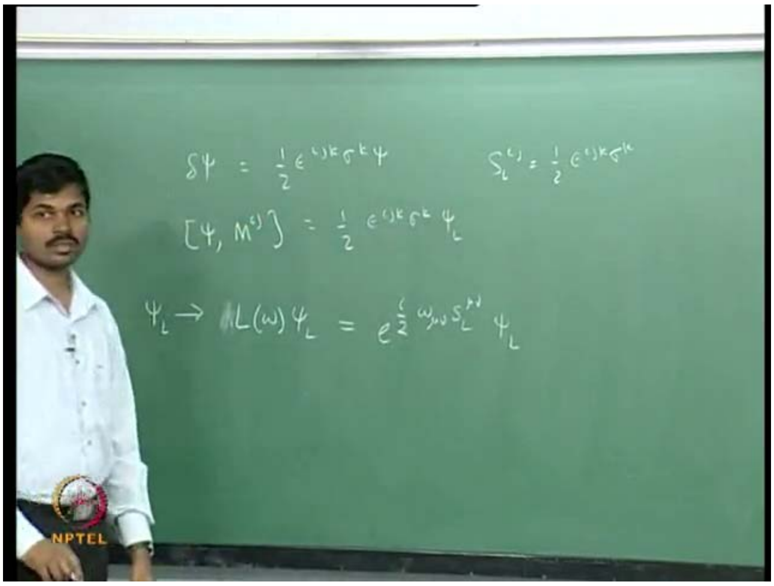

Delta psi for spin half field is just half epsilon i j k sigma k psi, when this happens it is just a spin half field. Here delta psi is nothing but it is the commutator of psi, with the generator of the transformation. So, you look at the rotation generators so the rotation generators are just M i j, M i j. Now what is my transformation? My transformation is psi L, is just L of omega psi L, L of omega is nothing but what is this e to the power i over 2 omega mu nu s L mu nu acting on psi L.

So, you look at the infinitesimal transformation here and because of the definition of the psi L if you remember, s L i j is just half epsilon i j k sigma k. This is how we have defined, yeah absolutely. So, this is the definition. So, if you look at the omega i j component of this, it is just half epsilon i j k sigma k and psi L. That is the reason its spin half field. So, let us write down the algebra the Lorentz algebra this so 3 comma 1 algebra enters of this n i s and n i daggers.

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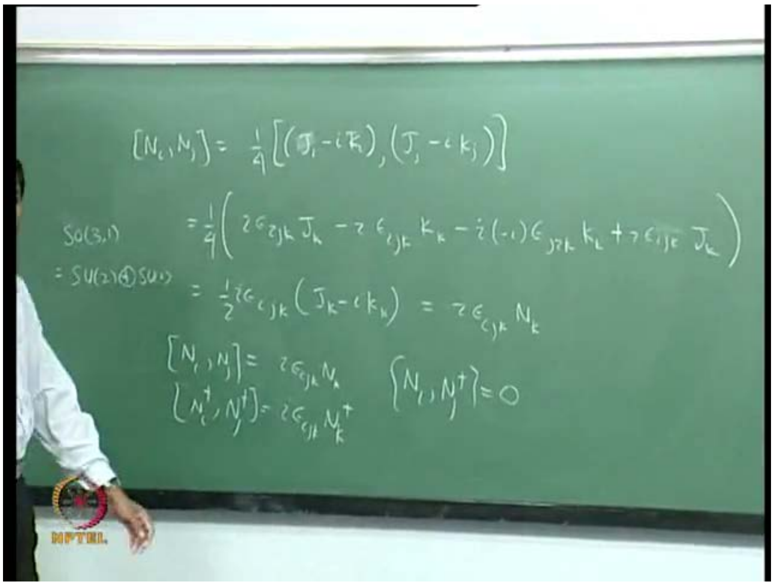

What is n i n j commutator? It just one-fourth times commutator of J i minus i k i and J j minus i k j, J i J j commutator is nothing but i epsilon i j k j k. And the second term j i k j as minus i epsilon i j k j i k j is k k and then this is k i j j this minus i it has k i j j commutator is minus i epsilon j i k k k. And then finally, minus, minus plus phi square is minus, so minus k i k j commutator as minus i epsilon i j k j k.

This is very straight forward and then you can see that, this is nothing but half epsilon i j k j k minus i k k. So, is there a i also there is on i. So, this is nothing but i epsilon i j k n k so what we saw is that the commutator of n i and n j just gives you the s u 2 li algebra i epsilon i j k N k.

You can check this far the N i daggers also, N i dagger M j dagger will also give you i epsilon i j k and k dagger. More importantly, the N i N j dagger they commute with each other. So, what we have seen by this very simple calculation is that, the s o 3 algebra s o 3 comma 1 algebra is nothing but it is just two copies of s u 2 algebra, it is just direct sum of s u 2, s u 2.

One of them I will call as the s u 2 left and then the other one I will call it as s u 2 right and you know all about irreducible representations of s u 2. I can level the erupts of s u 2 enters of the dimension of their representation. So, s u 2 scalar will simply be denoted as 1, s u 2 doublator as s u 2 spinor is 2 and so on. So, then accordingly I can have the s o 3 comma 1 fields or the irreducible representation of the Lorentz algebra 1 comma 1. So, because of there are two copies of s u 2, therefore I will have two numbers, two integers to level an irreducible representation for the Lorentz algebra.

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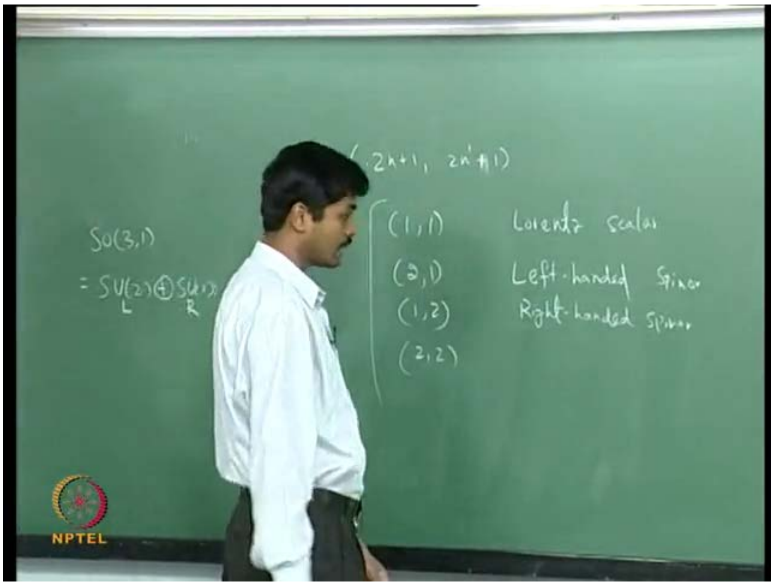

So, I will call this as 2 n plus 1, 2 n prime plus 1 and an irreducible representation of Lorentz algebra can be level like this. Just as an example 1 comma 1 will simply be a Lorentz scalar. And 2 comma 1 is a left handed spinor, so I will call this to be s u 2 L s u 2 R and this quantum numbers denote the transformation under the s u 2 L and these under the s u 2 R. So, this is just left handed spinor and these are all two components spinors right and 1 comma 2 is just right handed, you can have 2 comma 2 and so on.

This is just what you have said in the start, this is vector representation and then so on, so this how you can characterize the irreducible representation of the s u 2 of the Lorentz algebra. And what you have seen in the particular is that, we can have I mean this algebra admits spinor representations. There are these wild spinors, which are I mean which transforms under s u 2 L as well as s u 2 R.

Now what can we do is, we can ask the following question. Now that we know what are these spinors, can we construct a Lorentz invariant lagrangian, which involves these spinors. These are by the way also called as the wild spinor. So, we can have these left handed as well as right handed weyl spinors and then we can construct a Lorentz invariant lagrangian or a Lorentz invariant action from these wild spinors. What we need to do is that, we need to keep track of which one is left handed spinor and which one is right handed spinor and so on.

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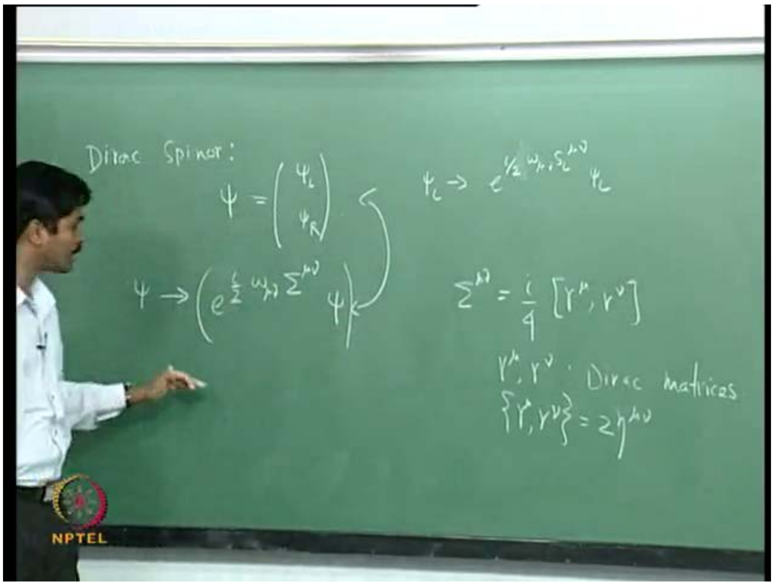

So, we will, what we will do is that we will introduce something which is known as Dirac spinor, which unlike these left handed and right handed spinor, that we have introduced just now. These are four component spinors and they do not belong to irreducible representation of the Lorentz algebra, the. These spinors are I mean they are some reducible representation and the Dirac spinor, I will call this as psi, a four component Dirac spinor is just psi L and psi R two copies of this wild spinor is just one Dirac spinor.

So, what we will do is that we will consider the Dirac spinor and then we will see how this Dirac spinor transforms under Lorentz transformation, you already know how psi L and psi R transforms under Lorentz transformation. Therefore, it is not so hard to write down the transformation rule for the Dirac spinor, under Lorentz transformation.

We will write down the transformation property for the Dirac spinor under Lorentz transformation and then I will ask, can given this Dirac spinor, can we write down a Lorentz invariant action involving this Dirac spinor, which gives us physically meaningful solution. That is the question, that we would like to ask.

So, this so I will just tell you how this Dirac spinor transforms under Lorentz transformation and then I will, I will close the lecture here and then tomorrow we will we will write down a Lorentz invariant lagrangian involving the Dirac spinor. And then its solutions in quantization of Dirac field. So, under Lorentz transformation how psi L and psi R transforms, psi L goes to some e to the power i s l mu nu.

Student: It should be psi R psi L?

No just fine, it is just a four column, its I will how do you level is up to you and the then Lorentz transformation here will, I mean you have to, you can re-write the Lorentz transformation, if you want to write the Dirac spinor is psi R and psi L you are free to do that. The transformation property of the Dirac spinor will be written accordingly. So, this here is e to the power i over 2 omega mu nu I will call this as sigma mu nu and psi. The sigma mu nu here is nothing but i over 4 gamma mu gamma nu gamma mu and gamma nu are known as Dirac matrices.

And the commutator of gamma mu and gamma nu is nothing but the generator of the Lorentz transformation for the Dirac spinors, gamma mu gamma nu etcetera are the Dirac. They have the following property if you look at the anti-commutator, gamma mu gamma nu it is just twice eta mu nu.

You can find a suitable representation for the Dirac matrices and then you can show that this transformation is indeed equivalent to this transformation here, which we will do in next lecture. And then we will show that, a transformation of the Dirac spinor under Lorentz transformation like this, a most saying that the left hand and the right handed components of the Dirac spinors under Lorentz transformation transforms like this. And hence, you know that this I mean, first of all this is a reducible representation and this reducible representation decomposes into two irreducible representations. These two erupts transform in the following manner, that is what we will do and then we will write down a Lorentz invariant lagrangian for the Dirac field and then quantize it.