Quantum Field Theory
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Module - 2
Interacting Quantum Field Theory
Lecture – 13
Interacting Field Theory - VI

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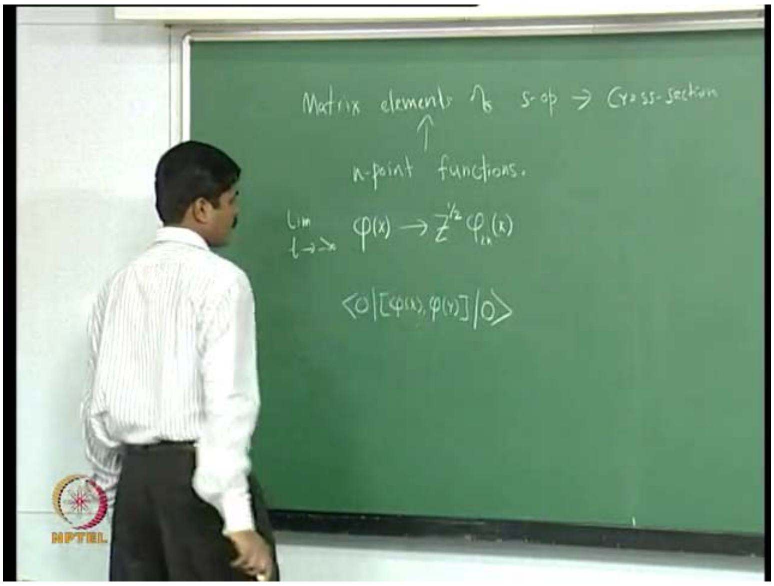

In the last lecture, we saw these scattering cross sections. We have derived the formula for this scattering cross section, we expressed the rate in terms of the matrix elements of the S operator. So, the matrix elements of the S operator give us the cross section, which is an experimentally measurable quantity. We would like to express therefore once we know the matrix elements of this S operator, we know what the cross section is. Similarly, we can derive a formula for the decay rate and so on. All the physical physically measurable quantities are expressed in terms of the matrix elements of the S operator. In other words, if we know the s matrix, then we know physically measurable quantities.

What we can do is even something, which is even better. We can consider the matrix elements of the S operator. We can express them in terms of something, which we will call as the n point correlation functions or the greens functions. So, n point functions give you the S matrix and the S matrix gives you physically measurable quantities. We will see in today's lecture is once we have the S matrix elements or rate transition amplitude, how can we express this transition amplitude in terms of the greens function. So, this is what goes with the name is the LSZ reduction formula, which is what we will do in this lecture.

So, to do this, let us consider a theory, which is self interacting. You consider there are these particles states in the theory at some t goes infinity, the particles are t goes minus infinity, the particles are essentially free. At finite time, they come close to you so that they interact and then again finally, there is some output which you measured. So, what you do usually? What we would like to do is we would like to carry out all our computations in weather in the in Hilbert space or in the out Hilbert space. Essentially, we know everything about the free field theory. So, we would like to express all our computations in terms of the quantities, which are free field quantities.

That is what we would like to do. So, therefore, we have the let us say, for example, we have the free field which is phi in of X. This free field generates the in space the Hilbert space is generate by this field. All the observables in this free theory are essential expressed in term phi and various functions of phi and you compute the matrix elements of those functions. Those are the observables. What you what you have in practice is the interacting theory and you want to compute various things in the interacting theory. What you would like to see is how to express various competitions in the interacting theory in terms of the computations in the free theory. That is the goal here.

So, what you do is you have suppose you are having a self interacting theory. You considered the interaction. You assume that the interaction takes place at some finite time, but as t goes to minus infinity or t goes to plus infinity, the theory is essentially free. So, what you do is you have this interacting terms and you essentially introduce some adiabatic functions in the interacting in the interacting terms or in the equation of motion you have the couplings and you introduce and adiabatic function in couplings, which essential is 1 at finite time. This function as t goes to plus infinity or minus infinity this functions goes to 0.

So, that way at finite time, you have this interacting theory. You study the interaction. I mean the particles come close and interact, but as t goes to plus infinite plus infinity or minus infinity, the theory is essentially free. So, because of this introduction of this adiabatic function, what you essential learnt of is when you suppose phi of X is the interacting field and phi in of X is a free field, then in the limit t goes to minus infinity, this field phi of X goes to phi in of X up to a normalization constant, which I will called it Z. So, Z to the power half for convenience, I will introduce this number here, this coefficient here Z to the power half.

The interacting field goes to this at some h as t goes to infinity. You might think that you can essentially redefine the normalization and observe this coefficient in the definition of the field. However, we will see in a moment that that is not true. In fact, this statement is true matrix element wise. So, let us consider the computation relation of the field phi of X, the interacting field phi of X with phi of Y and compute its vacuum expectation value. We will find a representation for this quantity. Then from there, we will see that this constant Z here will be a number which lies between 0 and 1 for the theory to be interacting.

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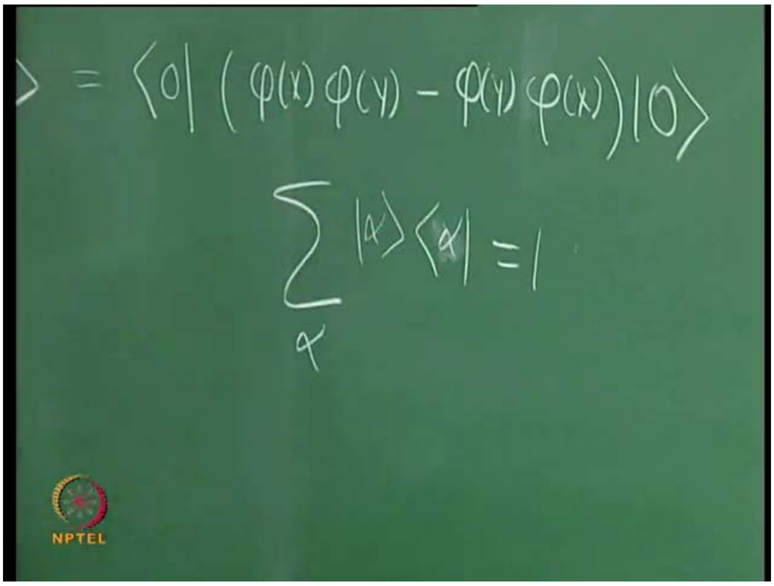

So, let us consider this commutator here. This commutator is essentially phi of X phi of Y minus phi Y phi of X. What I will do is that I will introduce a complete set of positive energy states here. So, if alpha represents a positive state, then I will use the complete relation. This expression, this is the identity and I will insert it here as well as here. When I do that, what I will get is that this gives two terms.

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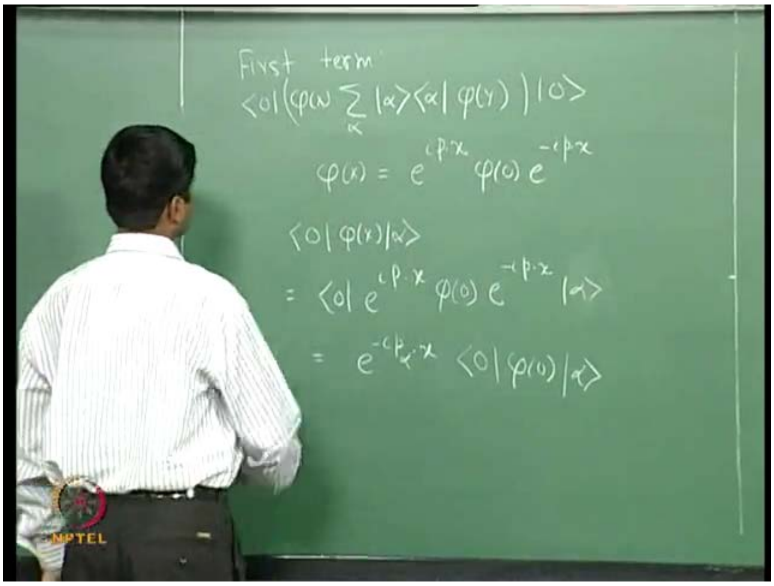

The first term is written as the first term becomes vacuum expectation value of phi of X sum over alpha. I can take the sum outside. Also, I will use this relation phi of X equal to e to the power i p dot x phi 0 e to the power minus phi p dot x. Then what I have here is this matrix element phi X alpha, this quantity will become 0 e to the power i p dot x phi of 0 e to the power minus phi p dot x. Here this acting on the vacuum will keep the vacuum in variant, whereas this will give a factor of e to the power minus i p alpha dot x. So, what you get is this phi 0. Here you have an alpha. So, I will use this relation here. You will get a similar expression for the second matrix element.

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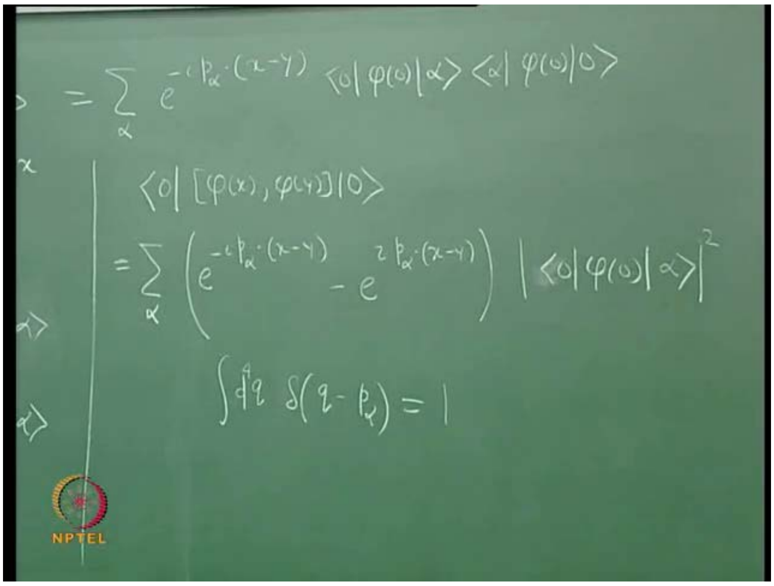

Hence, this quantity here becomes sum over alpha e to the power minus i p alpha dot x minus y times 0 phi 0 alpha and alpha phi 0 0. So, this is the first term in the commutator. Therefore, the vacuum expectation where live up the commutator becomes there is a sum over alpha e to the power minus i p alpha dot x minus y. Then because of the commutator, you have minus e to the power i p alpha dot x minus y times this quantity here with this mode 0 phi 0 alpha square. This is what we get for the vacuum expectation value of commutator of two fields in the interacting theory.

What we would like to do is we would like to express this quantity in terms of the vacuum expectation value of commutators in the free field theory. That is what we will try to do. Let us do. To do that, let us write it in a form that will be more convenient for us. Let us introduce this identity operator, which is integration d 4 q e to the power delta of q minus p alpha before dimensional delta function is equal to 1 of course. This is a trivial identity. It follows from the definition of delta function. I will use this is proof for any value of p alpha. So, I will use this here.

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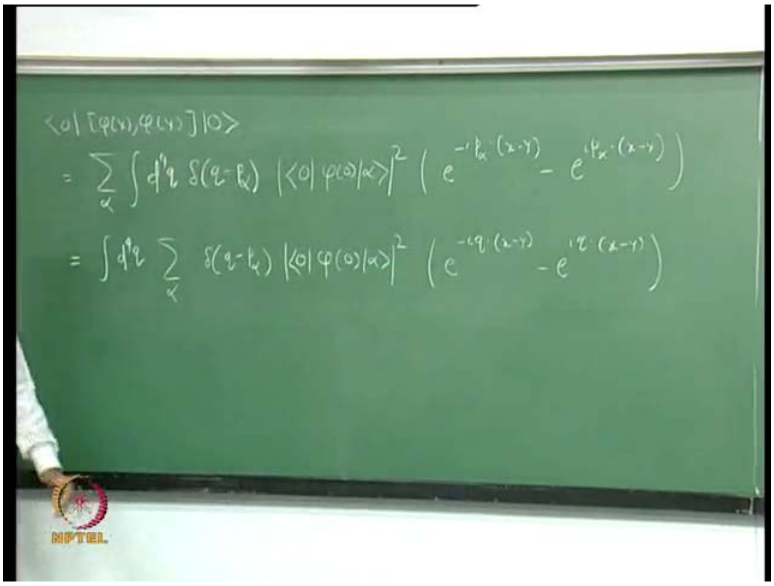

When I do that, what I will get for the commutator is this is equal to integration d 4 q delta. I can introduce it in each term. So, this is sum over alpha integration d 4 q delta of q minus p alpha mod of phi of 0 alpha mod square times e to the power minus i p alpha dot x minus y minus e to the power i p alpha dot x minus y. You will see in a moment why I did that. This quantity does not depend on q. So, I can just pull out of this integration. So, this is on the other hand, here because of the delta function, what I can do is I can replace here; instead of p alpha, you can write here as q.

So, what I get here is integration d 4 q sum over alpha delta q minus p alpha phi 0 alpha mod square. Here you have e to the power minus i q dot x minus y minus e to the power i q dot x minus y. You can see that trivially if you start from this step because of the delta function here, you get this step. Now, the only alpha dependence comes in this term here. So, it is already summed over alpha.

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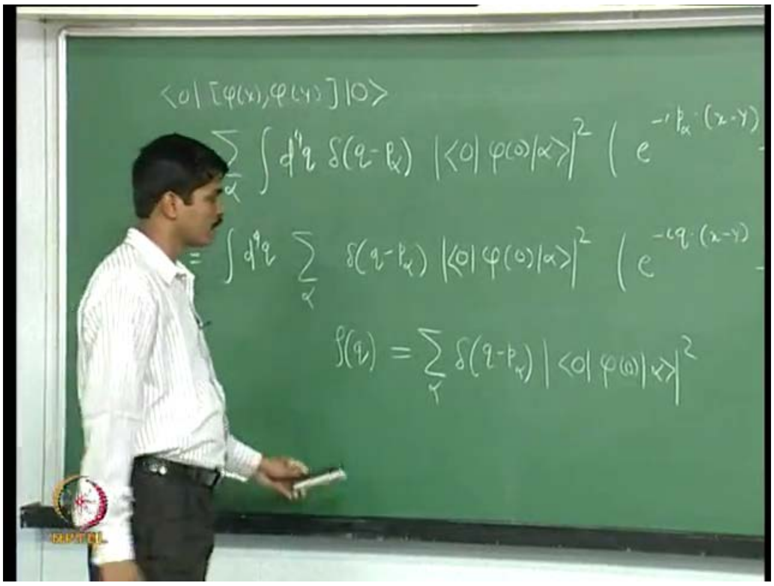

I will introduce a function rho of q. This is known as the spectral function rho of q is sum over alpha delta q minus p alpha vacuum expectation sum, the expectation value of phi of 0 alpha mod square. So, once I introduced that, I can see that the commutator vacuum expectation value of the commutator can be written in the following form.

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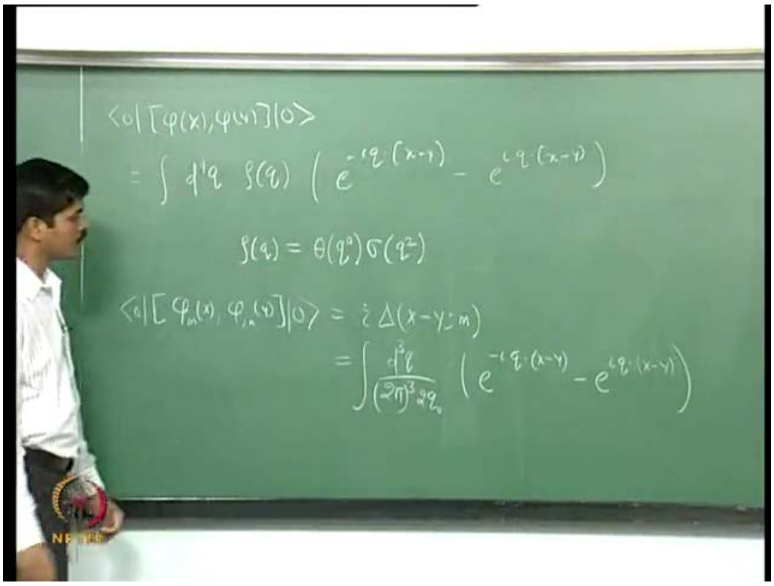

So, this quantity is equal to integration d 4 q rho of q e to the power minus i q dot x minus y minus e to the power i q dot x minus y. Now, you can see why I wanted to write it in this form. This is quite similar to the commutator of two free fields except this function rho of q here. Instead of rho of q, you had d cube q over 2 pi cube q 0. That was the commutator of vacuum, vacuum expectation value of commutator of two free fields. Now, let us look at the the function rho of q. This rho of q, first of all it is because the function p alphas here have positive, these are positive Eigen values.

Therefore, this quantity can be expressed as theta of q 0 times sum function sigma of q square here. You can substitute this value of rho of q here. Then you can compare this with the commutator of two free fields which is phi in of X phi in of Y. This commutator here which I will also or the vacuum expectation under that which is also denoted as i delta of X minus Y, I will use, I will write, I will label it by m also to make sure that this is a commutator of some free field whose mass is m. This commutator we already know is given by d cube q over 2 pi cube 2 q 0 times e to the power minus i q dot X minus Y minus e to the power i q dot X minus Y.

So, therefore, it is clear that this commutator here will essentially be super position of a this delta X minus Y, m, the super position of commutator of this for various masses. You know the single particles states are stable. So, you have because of the interaction in the theory, you can have I mean the field can create multi particle states and so on.

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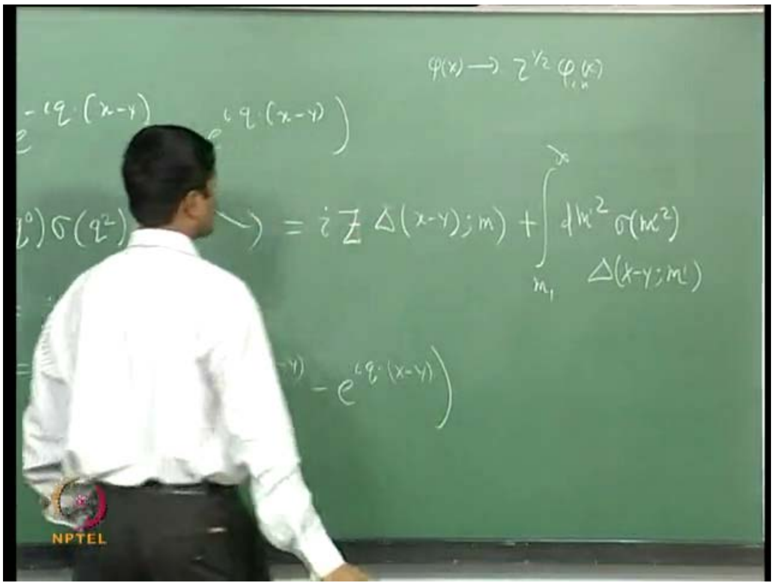

Let us assume that m1 is some energy scale above which multi particle states are created. Then you can see that this commutator here will sincerely be the super position of these deltas except that the multi particles states are created at some energy which is m1. So, this will be i delta. This will be bunch of some of this deltas of various masses except that the lowest one has delta X minus Y, m. Then you will have because I am denoting m1 to be the threshold energy for creation of multi particle states, so you can write it as something.

You can represent this as integration over d m prime square sigma of m prime square times delta of X minus Y m prime except that because of the normalization condition because we have introduced this factor Z here, phi of X goes to Z to the power half phi in of X because of that, you will have a factor of Z here. So, you have i Z times this. This is what you will get for the commutator of these two interacting fields from this relation. It is clear. You can express this as integration over d 4 X and you can compare this these terms to conclude that the vacuum expectation value of the commutator must have a form, which is like this.

Now, you look at this here. You can see that if you differentiate this with respect to time and then if you set t equal to 0, then if you differentiate it with respect to t, you will see that this q 0 factor here will cancel in both these terms. If you set t equal to 0, then this will become a three dimensional integration. Both these terms will be the same.

Therefore, that will also cancel the factor 2 here. So, you will get a three dimensional delta function.

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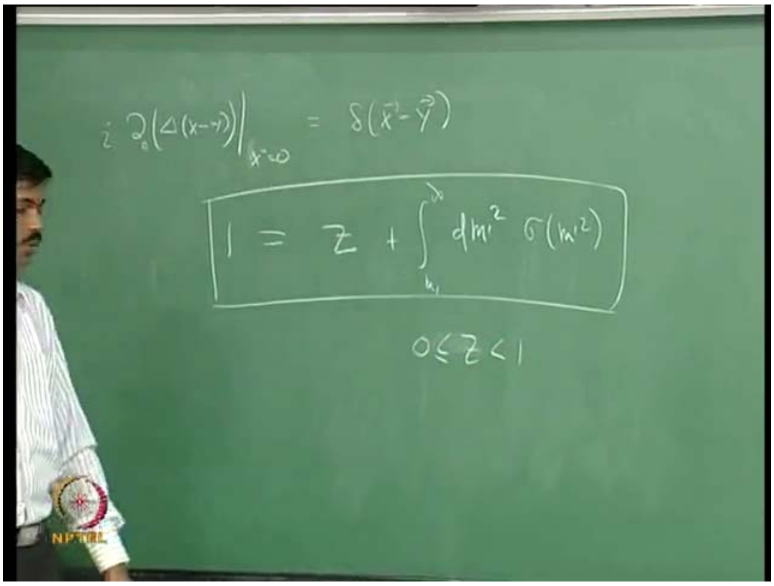

So, i del 0 delta X minus Y at X 0 equal to 0 is delta X minus Y. This is clear because this differentiation here, this because of this minus sign here, you will get 1 minus sign, both this terms adopt and that cancels this factor to the differentiation gives you q 0 that cancel this q 0. You can make a change of variable here from q to minus q in the second term. All those things make sure that you get a delta function here. So, you can do that. You can consider this term here. Differentiate it again with respect to t. Then you will get a delta X minus Y here, three dimensional delta here, here, and here in all this terms.

When you do that you see that, this integration here does not, the m dependence goes away. So, if you differentiate this with respect to t and set t equal to 0 and collect the coefficient of delta X minus Y, what you will get here is there is identity here, 1 is equal to Z plus m1 to infinity dm prime square sigma of m prime square. The sigma of m prime square is positive. You can see this from the expression for rho here, which basically contains delta function and Mod Square of some quantity for this is a non negative quantity. Therefore, this quantity is positive. Hence, you can see that this Z here is essentially a number, which last between 0 and 1.

Student: Sir, how do we get that identity, 1?

This identity? This 1 is you can see that this commutator here…

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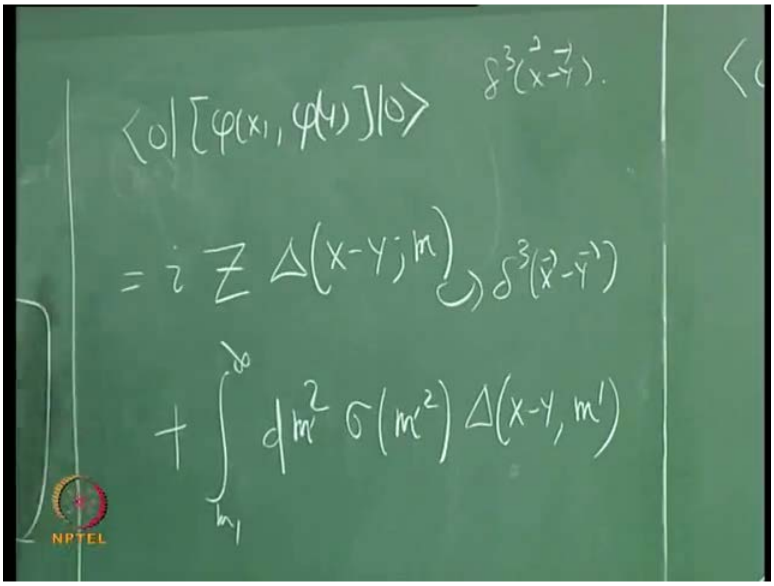

So, you start with this relation. Now, you differentiate it with respect to time. Then what you will get is in each of this term, you will get a delta function, three dimensional delta function. You set this time t to some value, let us say you set t equal to 0. It need not be 0. Only the difference has to be 0 here, X minus Y here. If you set that to be 0, you get the left hand side here simply becomes delta cube of X minus Y. Here again, you will get a delta cube X minus Y, this also because of this relation and same about here. This basically gives you that the identity holds.

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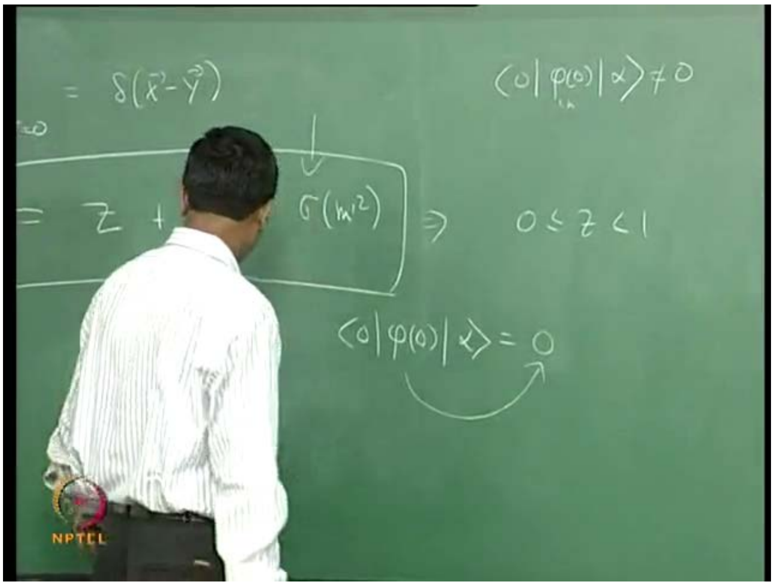

So, both the quantities are positive here. This is positive. This is positive. Hence, you get this identity, this relation that the constant Z lies within 0 to 1. Z IS equal to 1 means this one is 0 here. Therefore, the theory has to be essentially free theory. The field has to be free field free theory. No multi particle states are created here, this quantity here for example, 0 to phi. So, this is 0 means what? This quantity has to be 0 for all multi particles states here, but you do not want in an interacting theory, you do not want this to be 0 for all multi particle states. Only in a free theory, you have this quantity here. The field phi creates a one particle state out of vacuum.

So, in a free theory, this is non zero when this is a one particle state. This quantity is 0 when alpha is multi particle state for free theory. On the other hand, in an interacting theory, you have also; you want the interaction to create multi particle states. Therefore, this is also non zero when alpha is a multi particle state. Here, I have separated out this single particle state contribution and the multi particle state contribution.

So, when alpha is a single particle state, you get the first term here, when alpha is a multi particle state, you get all the terms here. This quantity Z is equal to 1 essentially means that this has to be 0 here. Thus, that means that this is 0 when this alpha is a multi particle state. So, you essentially led in free theory. So, to have an interacting theory, you have this Z, which is a number, which is less than 1. Any question after this? Now, you we would like to consider this.

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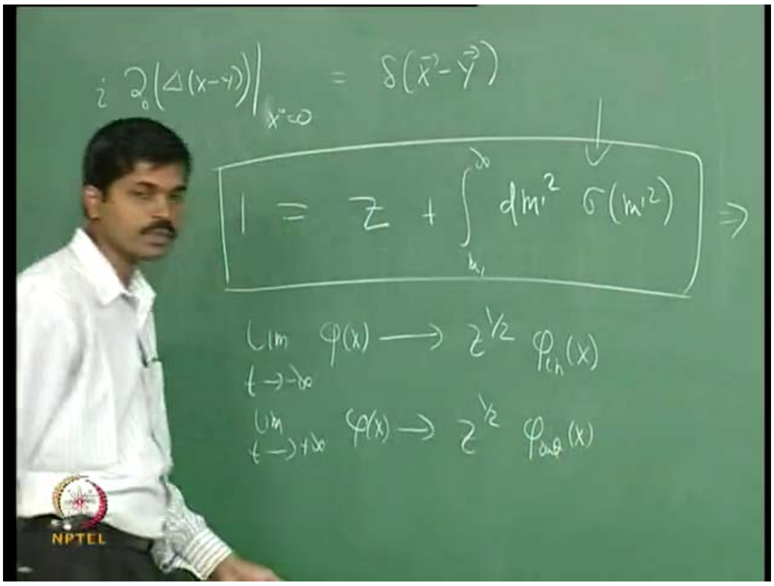

So, what we have shown so far is phi in the limit t goes to minus infinity, this goes to Z to the power half phi in of X. What happens, when t goes to plus infinity? Again, you can use the same argument here because the out state also, I mean the field phi out X is also a free field. So, all the arguments will hold except that you do not know whether you get the same coefficient here or you get a different coefficient. So, what I claim here is that again, when you consider t goes to infinity, phi of X goes to Z to the power half phi out of X with the same Z. The reason is the following. When you quantize, of course, the ground state is unique.

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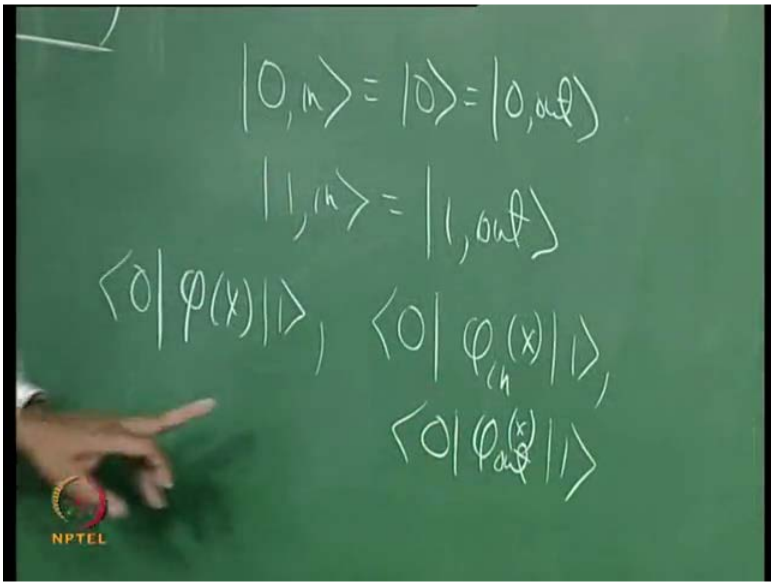

So, in vacuum is same as the out vacuum. The single particle states are again stable. So, the single particle states are also the same up to overall phase vector, they will be the same. If you consider quantity like this, the X dependence here and the X dependence in phi in of X 0 phi out 1, so they have the same functional for here. So, the only thing that can differ here is the overall constant, but because both the vacuum here and the single particle states are the same for both in as well as out vacuum, therefore the multiplication factor here also has to be the same. So, you get the same quantity. Now, with this, what we can do is we can now consider the transition amplitudes.

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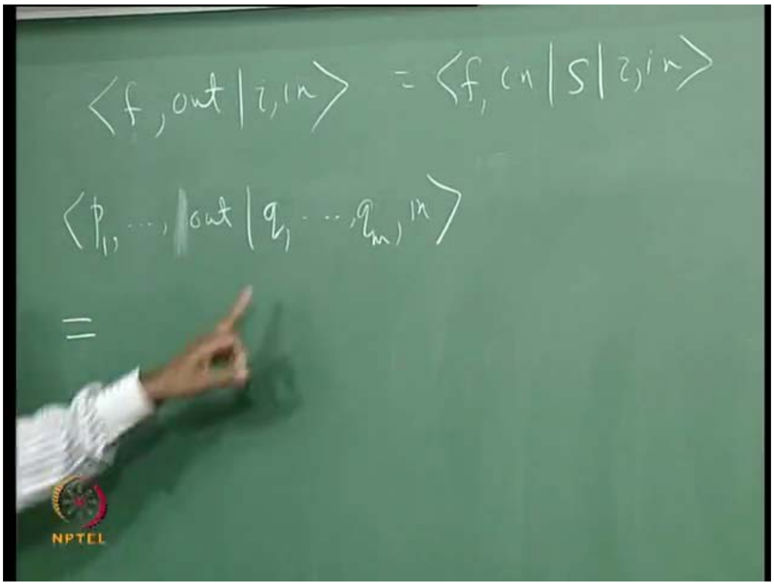

So, let us say some transition amplitude, which is also some matrix element of the S matrix. We will do some manipulation and we will express this quantity in terms of the n point correlation functions. So, consider some final state which is p1. When we did this scattering cross section calculation, we have seen that even if we consider wave packets, ultimately the cross section is expressed in terms of some quantity where both these quantities are Eigen states of the momentum. So, you consider p1 up to some pn, out and q1 up to some qm, in. This is what this is the amplitude that we are interested in. So, let us start with this amplitude.

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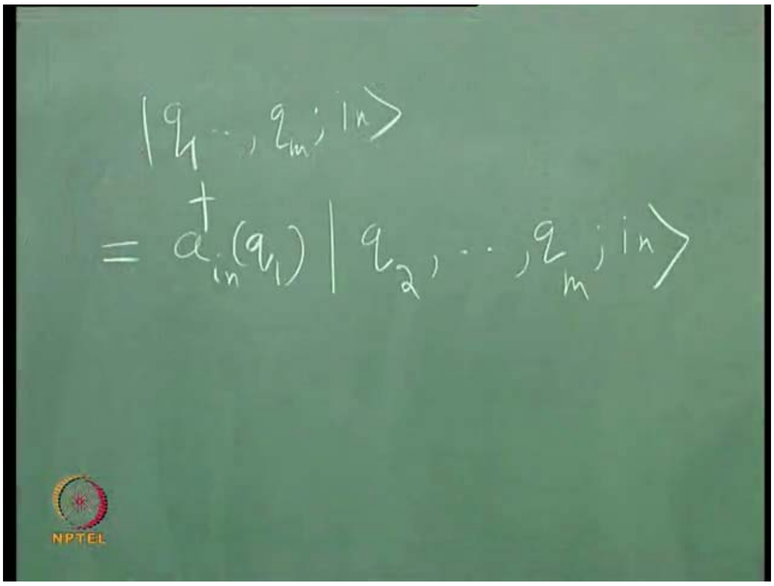

Let us consider this state q1 to qm; in. This can be written as a in q1 dagger. This will create one particle state of momentum q1. Therefore, this state is equal to q2 up to qm; in. So, I will use this formula and I will use it m times iteratively. Then finally, you will see what we get.

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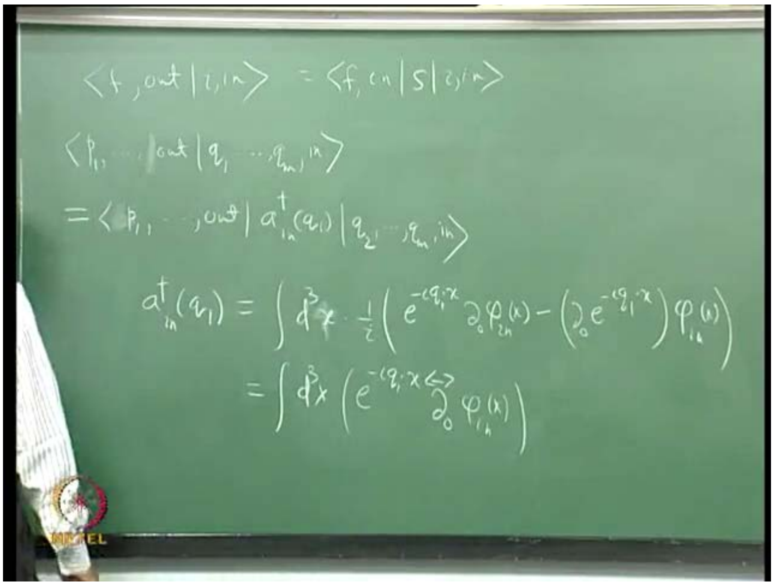

So, let us consider the first step which is p1 here, a in dagger q1, q2 up to qm, in. This is what we have. Now, this a in dagger, this I can express in terms of the field phi in of X. So, if you go back and check your notes, you can see that this quantity phi a dagger in of q1 integration d cube x times 1 over i e to the power minus i q dot x del 0 phi in of X minus del 0 e to the power minus i q1 x phi in of X.

So, you can substitute this expression here. Remember that although here, you have, each of this term here depends on t, the t dependence goes away. This is some quantity which is independent of t. So, you can evaluate this integration at any given time, at whatever time you like, you can evaluate that. I will write this quantity as d cube x e to the power minus i q1 dot x del 0 phi in of X. So, I am introducing this notation here to write this derivative here both ways with a minus sign. No. Thank you. There is a 1 over i. So, this is what you have. So, I will introduce this. I will use this.

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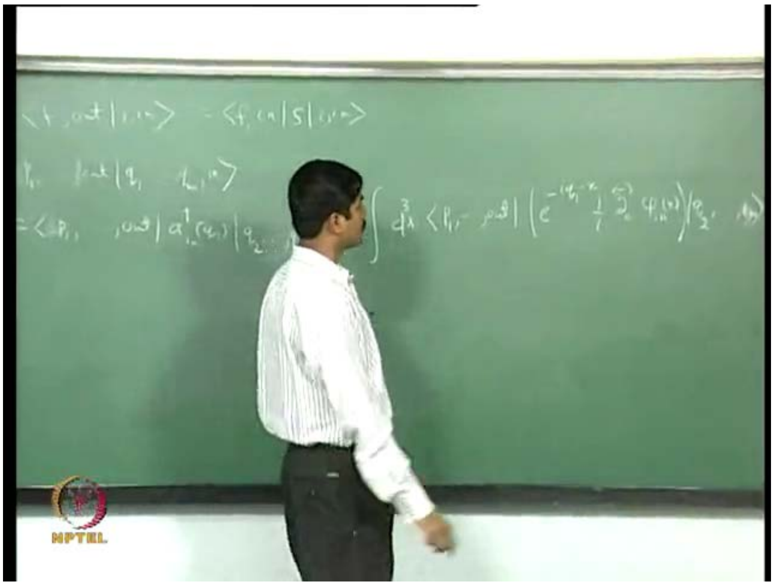

Then, what I get here is integration d cube X p1 e to the power minus i q1 dot x 1 over i del 0 phi in of X and then q2 up to phi out, this is in. This derivative here effects only on this phi. It does not affect any of these things and this is a number here. So, I can just pull all these things.

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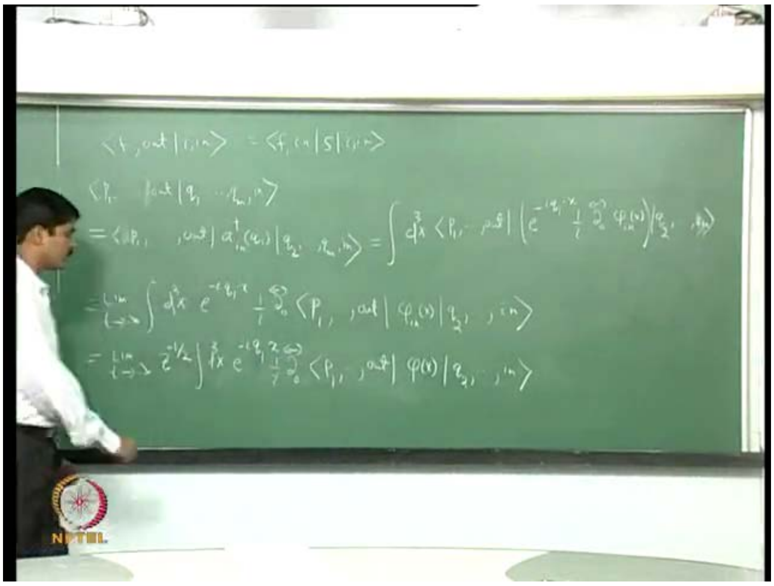

Then, what I get here is d cube X e to the power minus i q1 dot x 1 over i del 0. Now, since I can evaluate this integration here at any given time, I can evaluate it as t equals to minus infinity. Therefore, this quantity here again is same as itself in the limit t goes to minus infinity. But, now what I can do is now I know in the limit t goes to minus infinity, this phi in is related to the interacting field phi by a factor of Z to the power minus half. So, this quantity now is equal to limit t goes to minus infinity Z to the power minus half integration d cube X e to the power minus i q1 dot x del 0 here, p1 up to pn out phi of X q2. There is a 1 over i.

Now, what you can do is you can consider, quite independently, you can consider this quantity here and take the limit t goes to plus infinity. If you take the limit t goes to plus infinity, what you get here is instead of phi in, you get phi out. Hence, here instead of a in dagger q1, you get a out dagger q1. So, this is a out dagger. Now, what it will act is it will act on this state here and then it will annihilate one state. So, you start with, but that is the quantity which is taken in the limit t goes to plus infinity.

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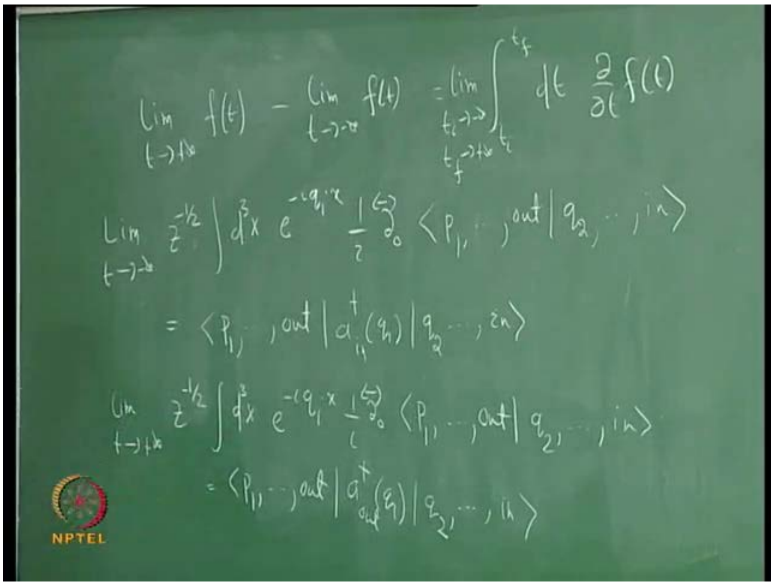

So, you consider some arbitrary function of t and you consider limit t goes to plus infinity f of t. You consider limit t goes to minus infinity f of t. You subtract these two terms. What you get here is, integration some t i to t f d t del over del t f of t in the limit t i goes to minus infinity, t f goes to plus infinity. So, what I will do here is I will consider this quantity here. What I have shown here is if I take this quantity, take the limit t goes to minus infinity. Then I obtain t goes to minus infinity Z to the power minus half this is p1 a in dagger.

Now, what I would like to do is that I would like to consider the same quantity limit t goes to plus infinity Z to the power minus half d cube X e to the power minus i q1 dot x 1 over i del 0 p1. So, when I take the limit t goes to plus infinity, what I have here is p1 out a out dagger q1, q2 up to qm in. But, now if I subtract this term from this term, this limit just goes away. What I get here is integration over d 4 X, time derivative of this whole quantity here. So, this integration d 4 X time derivative of this whole quantity here is just the difference of these two terms.

So, we will start with this relation in the next lecture. Then we will see that we will get a reduction formula. You can do that. Now, we can start acting this on this state and we will keep doing that. Then we will get a reduction formula, which essentially will express this matrix element that we have started with in terms of the n point correlation functions.