Select/Special Topics in Atomic Physics
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Lecture - 9
Angular Momentum in Quantum Mechanics
Dimensionality of the Direct-Product (Composite) Vector Space
Angular Momentum Coupling Clebsch-Gordan Coefficients:
Recursion Relations

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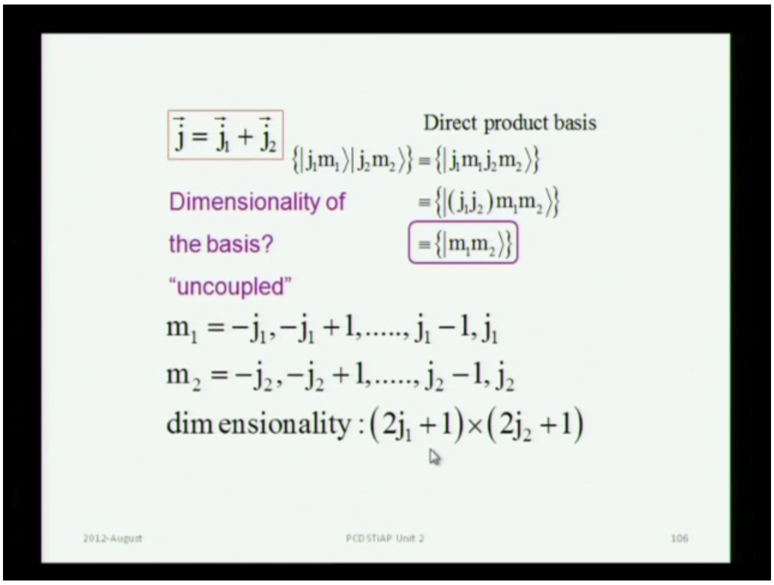

Greetings, we considered the coupling of Angular Momenta j 1 and j 2. And we recognized that each of these has got a vector representation in a basis, which is 2 j plus 1 dimensional, so there is a 2 dime 2 j 1 plus 1 dimensional basis, for j 1 and the 2 j 2 plus 1 dimensional basis for j 2. And the uncoupled basis will have a product dimension of 2 j 1 plus 1 times 2 j 2 plus 1, these spaces are individually completely disjoint, they have no common parts. Because, angular momentum j 1 and j 2 are completely independent of each other, each component of j 1 would commute with each component of j 2. And we are now composing from these individual spaces, the product space and the product space will be the composite space, which is made up of the individual spaces of j 1 and j 2, right.

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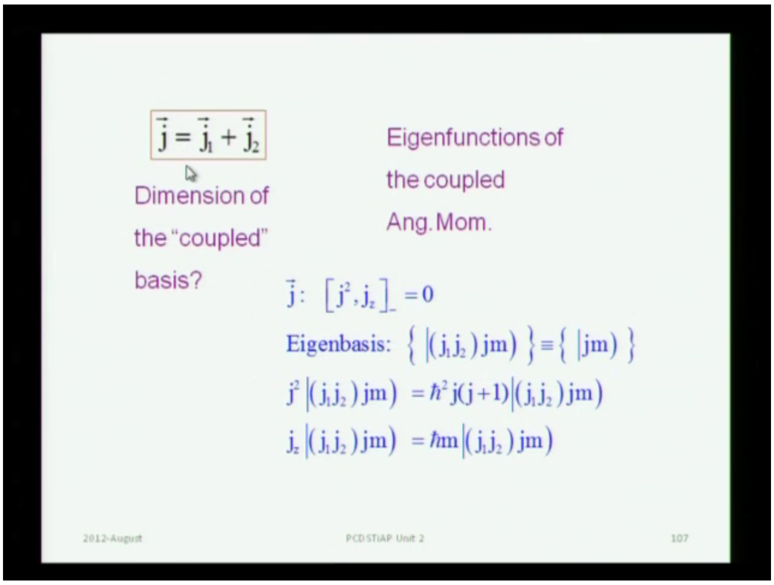

So, now if you look at the left hand side, which is the net angular momentum coming from, the sum of these 2 and this is the new type of sum this is not just the vector addition, because it is an addition of 2 quantum vector operators. We certainly expect that, this space of the coupled angular momentum will have the same dimensionality as the product space, because it is coming from that.

So, we anticipate the answer to be 2 j 1 plus 1 times 2 j 2 plus 1, but that is not obvious from, what we are looking at the only thing, we know from this is that, this j will have 2 j plus 1 degeneracy coming from the isotopic right. So, it will have m quantum number, which will go from minus j to plus j, so there will be a 2 j plus 1 dimensional dimensionality that will come from that, but that will have to be compounded by some additional factor, which is coming from the values that, j can take. Because, we have to find out what kind of values j can take and that will give us the net dimensionality of the coupled angular momentum.

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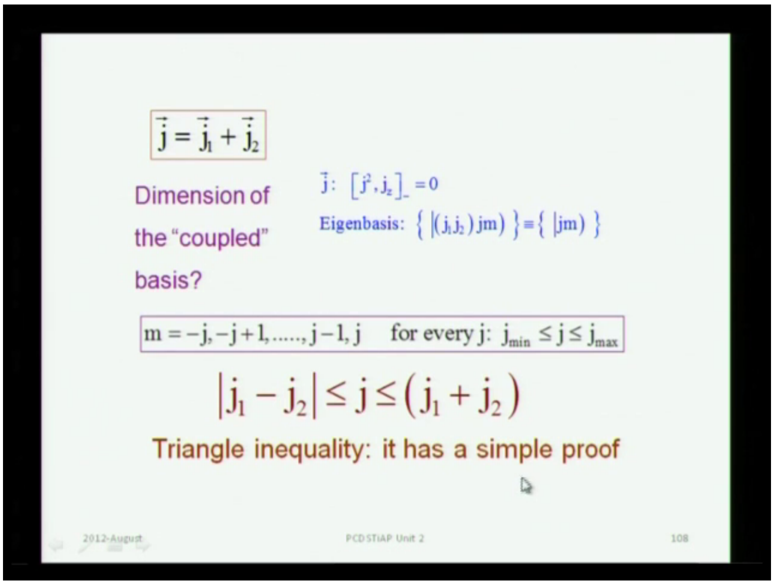

So, the coupled angular momenta, I am representing by these curved brackets and uncoupled by angular brackets, that is just for convenience, now this we know, that m can go from minus j to plus j, so there will be 2 j plus 1 values. J quantum number j, this will have a certain ray from a minimum to a maximum, but we do not know exactly, what this value of min minimum is and what is the maximum values that something, which we probably know from some of the other courses.

And we perhaps know it from some literature that, we have red, but we are going to determine exactly, what this value of j min should be, so that it will not be an assumption in our minds. So, we will find out, what j min can be and what j max can be and it will turn out that the minimum value is modulus of j 1 minus j 2 and the maximum value will be j 1 plus j 2 and we will prove this particular inequality that, we have in front of us.

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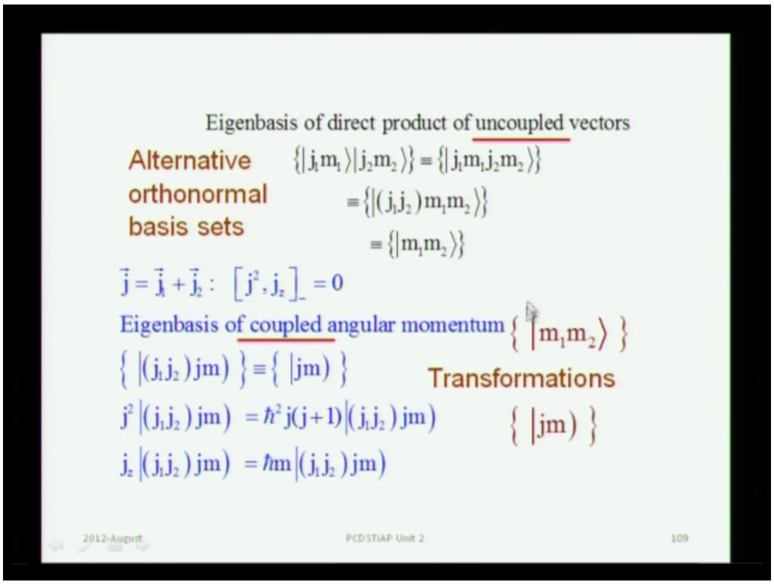

So, essentially we are dealing with alternative orthonormal basis sets, one is the basis of the uncoupled vectors, which is the direct product of uncoupled vectors, so these are the angular kates and you con compose the product basis. In brief you can suppress the notation j 1 j 2 and simply write these as m 1 m 2, so this is the m 1 m 2 basis and then you also have the alternate basis, which is the Eigen basis of the coupled angular momentum, which is j and this is the Eigen basis of j square and j z, these 2 operators, since they commute with each other. These are the 2 simultaneously diagonalizable operators, so you can consider transformation from 1 basis to another, because an arbitrary vector, you can always express, as a Leneous of proposition of any basis that, it does not matter, which it is.

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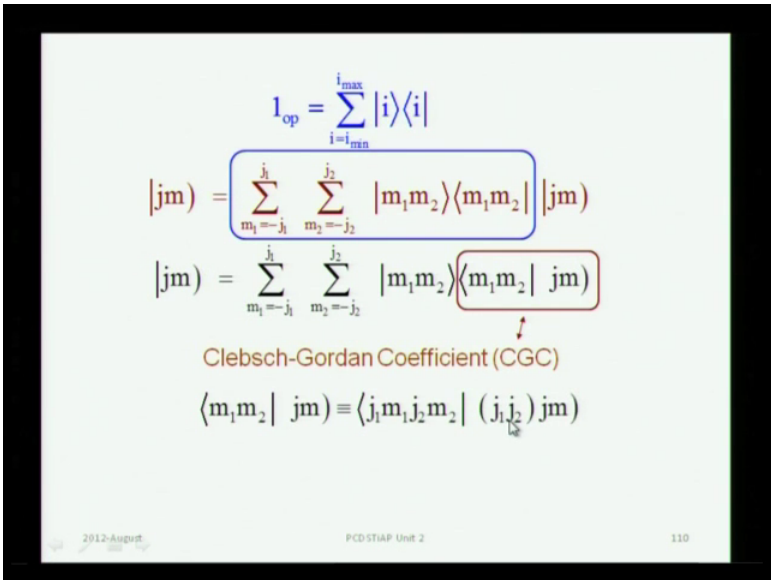

So, let us represent this coupled vector, which is written here as well, but it is operated upon by the unit operator and I have decomposed the unit operator, this is the resolution of unity, as we call it. In the basis of the product of uncoupled vectors, m 1 m 2 are the product vectors from the uncoupled basis, so this is the resolution of unity and you find that this expression amounts to a linear superposition of these m 1 m 2 vectors, scaled by these scalars that, you find in these bracket.

These brackets has got an angular bracket on 1 side and a curved bracket, on the other, but essentially it is a bracket, it is a scalar and whether it is angular or circular is just a matter of convenient notation, there is no big physics in it. It is just a Scaler and these Scalers are known as the Clebsch Gordan coefficients, in general these can be some complex numbers, but they turn out to be real, as we will see. And these are the Clebsch Gordan coefficients, you can write them in the complete notation inclusive of j 1 j 2 and this is the coupled vector j m coming from j 1 j 2, and you can insert the j 1 and j 2 quantum numbers on this side, as well just for completeness.

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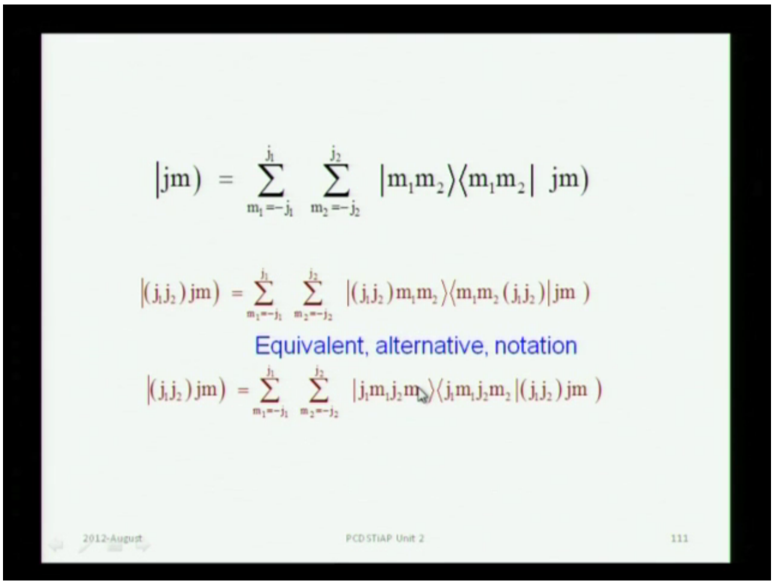

So, here we have this expression along with the Clebsch Gordan coefficients, either in the brief notation or in the complete comprehensive notation, again some books write j 1 j 2 over here and then m 1 m 2, some other books write j 1 m 1 and then j 2 m 2. So, this is just a matter of choice, the actual physical content of both of these notations is exactly the same. So, sometimes you know different books on quantum mechanics, you know Merzbacher Sharif and so on, they use slightly different notations, so no big deal about it, just keep track of what is the physical content, which is going into the labels.

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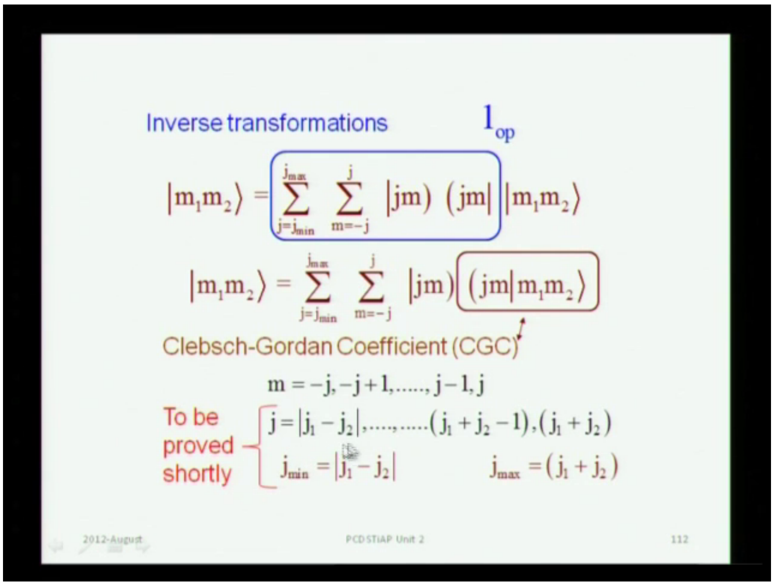

So, you know construct the inverse transformations, we had expressed the coupled vectors in terms of the uncoupled product vectors, now we do the inverse transformations and on the left hand side, we have got the product of uncoupled vectors, which is now expressed, in terms of the coupled vector, because you can simply pre multiply the same vector m 1 m 2 by the resolution of the unit operator, but this resolution is now carried out in the coupled basis.

And therefore, what you find over here is an expansion in terms of the vectors of the coupled, angular momentum and then again you have coefficients, which play a similar role as the coefficients that, you saw earlier, so they are again the Clebsch Gordan coefficients. And you know that in this basis at m will go from minus j to plus j that is coming from isotropy of space, because axis of quantization can be anywhere right.

So, you have a 2 j plus 1 for degeneracy coming from the m quantum number and the j itself will go from a certain minimum value to a maximum value and what we are going to prove now that, this minimum value is the modulus of j 1 minus j 2 and the maximum value is j 1 plus j 2. So, that is what we are about to prove now.

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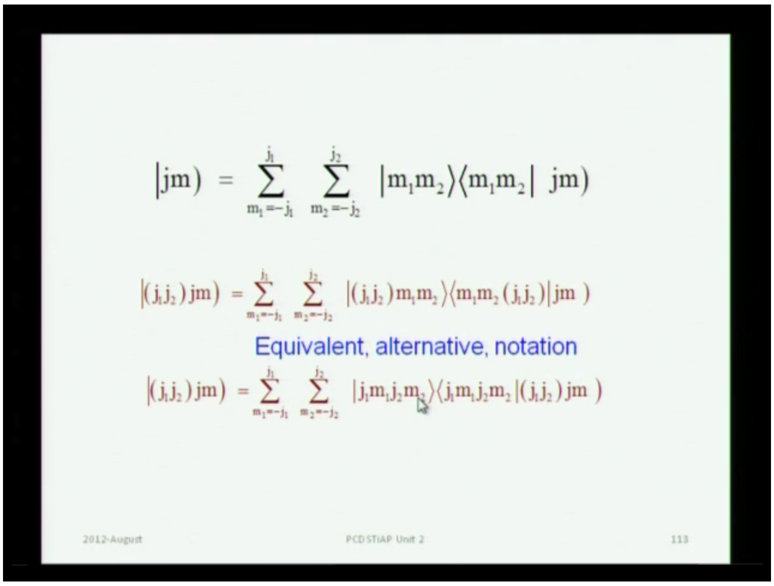

So, let us see how the proof goes, so this is the coupled angular momentum and this again, you can write in alternative complete notation, either by writing j 1 j 2 in a parenthesis over here or spread it out fully in this product vector. So, this is just a matter of notation and I like to highlight this, because you use different sources of books on quantum mechanics, so you are going to find all of these notations.

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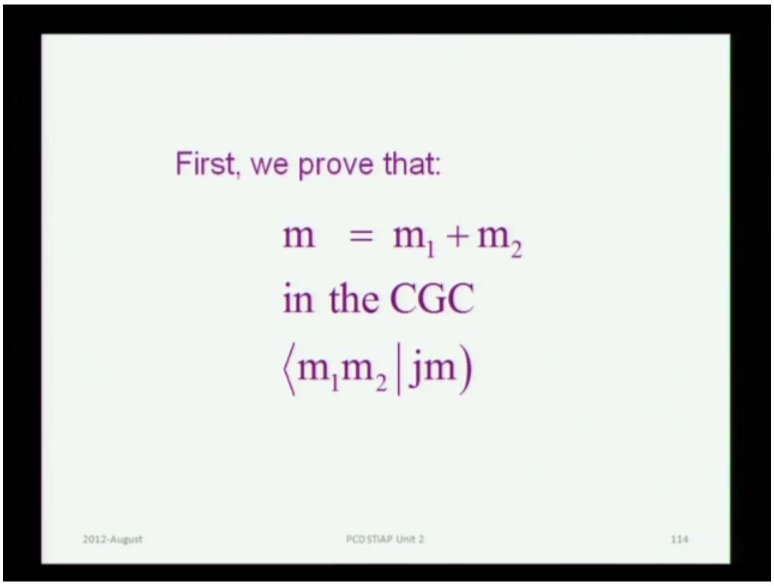

So, our proof is going to be based on a few factors and the first thing, we do is to prove that the Clebsch Gordan coefficient must have this m quantum number to be equal to the sum of these 2 m quantum numbers, that is a necessary condition, for the Clebsch Gordan coefficient to be non zero.

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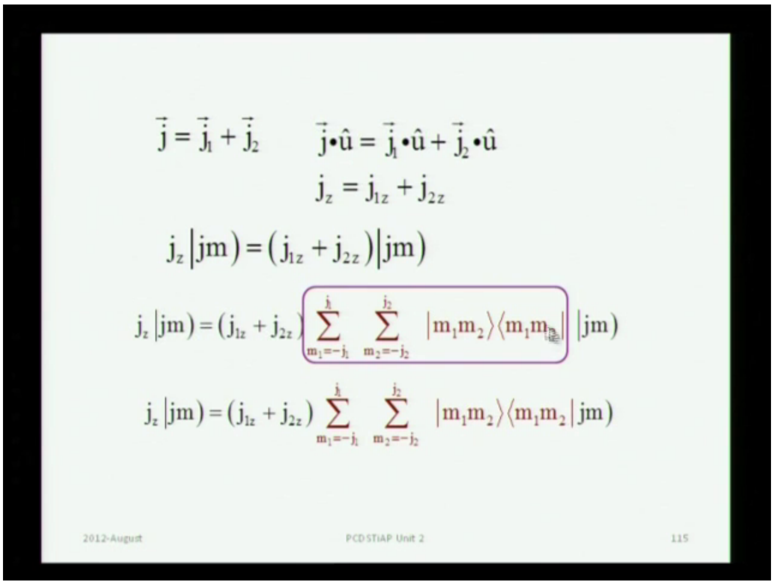

Now, this proof is a very simple 1, it comes from the simple consideration or the fact that you simultaneously diagonalize j and j square and j square and j z or j dot u in some direction. And you can therefore, construct an Eigen value equation of j z and now over here in between you can sandwich, the unit operator that is one thing that, you know, theorists can do very freely, which is always insert a unit operator and resolve it, that is the resolution of unity, so you sandwich a unit operator. So, that you express, it as a linear superposition of base vectors in the uncoupled basis, this is a product of uncoupled vectors, uncoupled angular momentum Eigen states right, so this is one expression that, you have for the result j z operating on j m.

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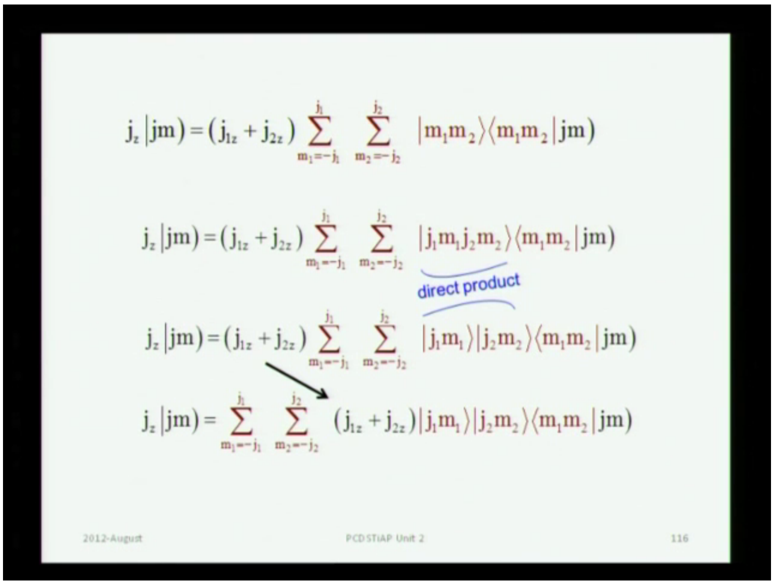

Now, you can expand these terms further, I have written the j 1 and j 2 more explicitly over here, this is a direct product of these 2 vectors, which is j 1 m 1 and j 2 m 2, so essentially, you have got these 2 operators j 1 z and j 2 z operating on this direct product. And there are 2 terms 1 coming from j 1 z and the other coming from j 2 z, so you can get these 2 terms by explicitly carrying you t the operation by j 1 z plus j 2 z on this product j 1 m 1 j 2 m 2.

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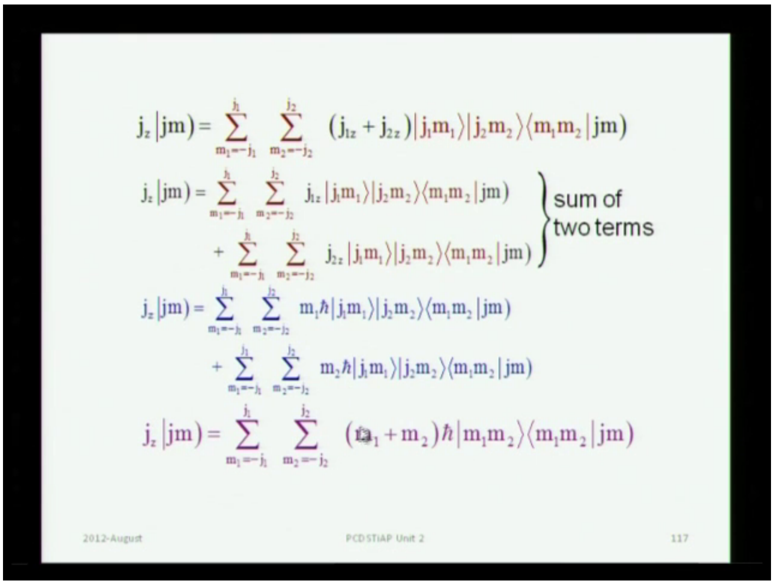

So, that result is quite simple, you will have j 1 z operating on j 1 m 1 and j 2 z operating on j 2 m 2, so these are the 2 terms, so you have written this entire double sum as a double sum of 2 terms. So, this is a fairly simple simplification of this expression, now what is j 1 z operating on j 1 m 1, it is of course, m 1 h cross times j 1 m 1 right and j 2 z are placed on j 2 m 2, which is this vector over here, it does nothing to j 1 m 1. Because, j 1 and j 2 are completely independent angular momentum that is what, I mean, when I said that the 2 individual spaces are completely disjoint, they have nothing in common.

So, j 2 z will do nothing to j 1 m 1, but it will operate on j 1 m 2 giving you the Eigen value m 2 h cross and now you can, sum these 2 terms, because both are summations over the same base vectors m 1 m 2, 1 scaled by m 1 h cross and the other scaled by m 2 h cross. So, extract h cross as common and you have got a factor m 1 plus m 2 times h cross, so this is a linear superposition of these base vectors, scaled by appropriate coefficients and these coefficients are h cross times m 1 plus m 2 and do not forget the Clebsch Gordan coefficient, which is also there, so that is also involved in the scaling.

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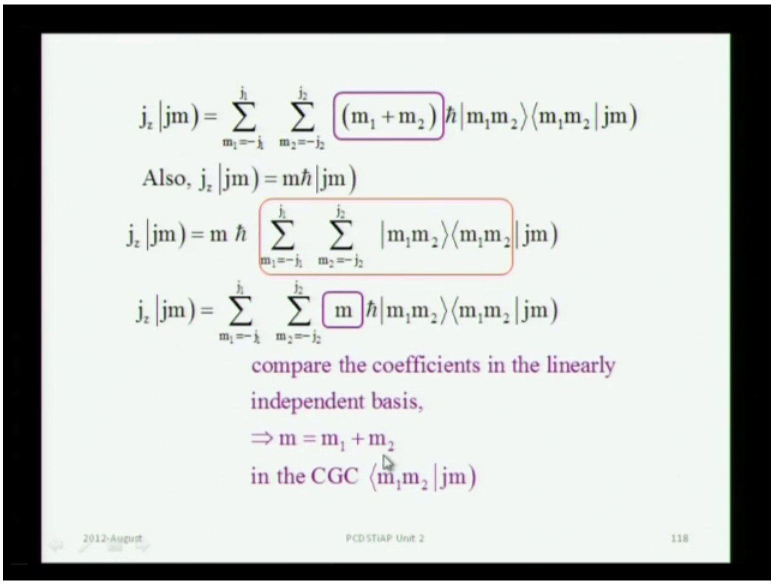

So, that is the right hand side, left hand side is very simple to resolve, because j z at operating on j m gives you an Eigen value equation and you can express this term once again by inserting a similar unit operator. So, that both on the left hand side and right hand side you have got linear superposition and basis as which are exactly the same then you can always extract the coefficients of corresponding terms that is the idea.

So, let us do that, so you have got j z operating on j m, which gives you m h cross times this whole sum and now if you compare the coefficients, you have got 2 expressions on the right hand side, both being equal to the left hand side, which is j z operating on j m. Both this term, as well as this term are expansions of the same basis sets, which is m 1 m 2 basis, they have got the same factor h cross, which is common in both this Scalar comp the Clebsch Gordan coefficient is also common in both.

And now you have got m 1 plus m 2 is 1 factor here, which must correspond to the factor m over here, because whenever you have expansions in linearly independent basis and if you have 2, such expansions, which are exactly equal to each other in the same orthonormal in the linearly independent basis, that the coefficients of the corresponding base vectors must be necessarily equal. So, that is a fundamental theorem that, we make use of and we find that m must be equal to m 1 plus m 2 in every Clebsch Gordan coefficient, that is good.

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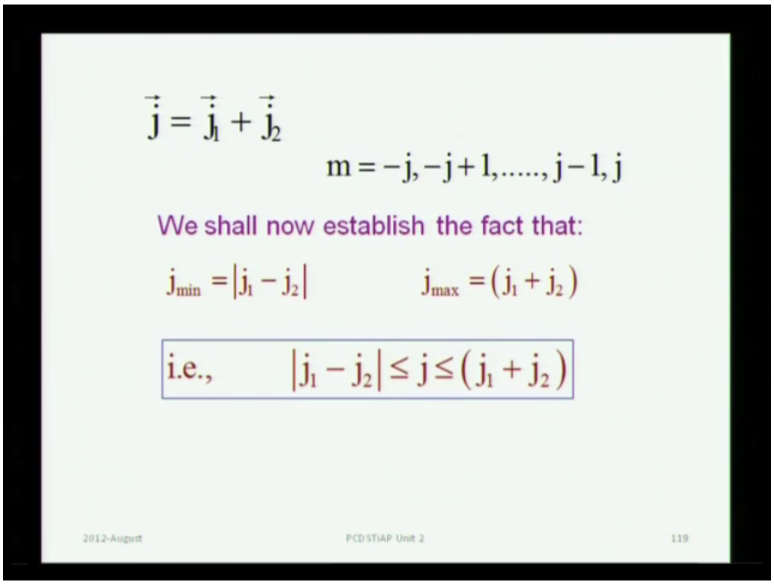

Now, we will proceed to establish, what the minimum and maximum values can be like, now this is what the result will turn out to be.

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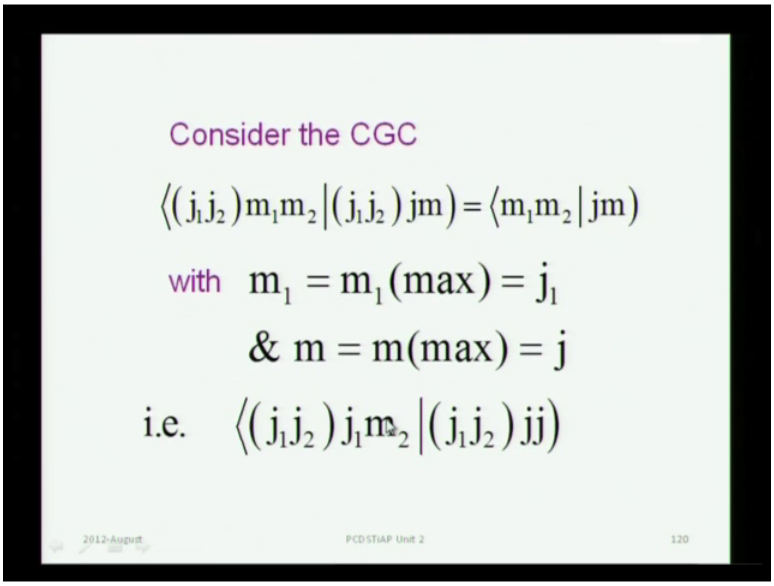

And to do that, we now have a look at the Clebsch Gordan coefficient and just see, it is terms, so this is the full notation for the Clebsch Gordan coefficient, this is the brief notation in which, I have suppliers j 1 j 2. What we do know in this is m must be equal to m 1 plus m 2 that is something, we have already established and we will use that result, now we use a particular Clebsch Gordan coefficient now. A particular 1, in which m 1 takes, it is maximum value and there is no ambiguity, there we know that the maximum value that m 1 can take is j 1.

We also pick the maximum value of m, which must be j, there is no ambiguity about, this either, we know that the maximum value, m can take is j, we do not know, what is maximum value of j, but we do know what is a maximum value of m. The maximum value of j is what we are yet to determine, which we are about to determine, we do not know that as yet. But, the maximum value for each value of j is known that is m, so that is what we take, so now, we consider such a Clebsch Gordan coefficient, for which this m 1 is equal to j 1, this m is equal to j. So, this is this j m becomes j, because m is equal to j, m 2 is whatever it can be, but can only be m minus m 1, so we do not put any constraint on m 2 externally, but it does have it is own constraint that, it has to be m minus m 1.

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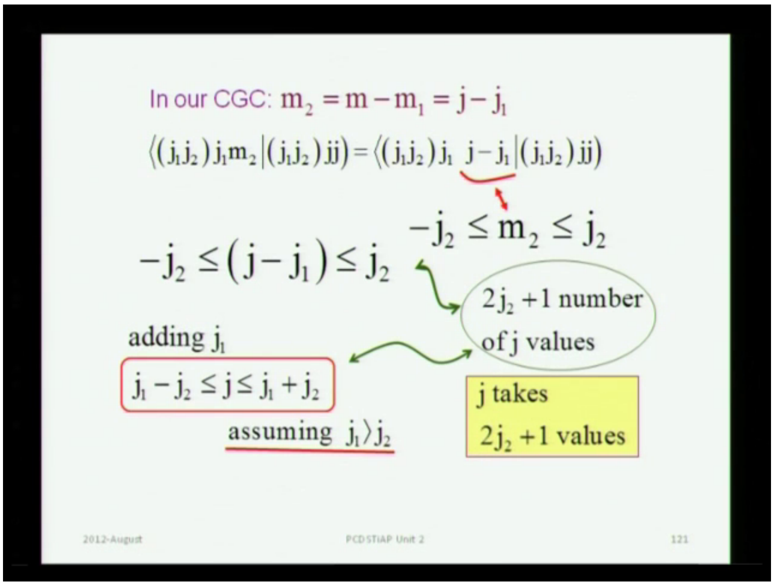

So, we will exploit that, so m 2 can be only m minus m 1, otherwise the Clebsch Gordan coefficient does not exist, so now, we know that this m 2 will be m minus m 1, but m has take it is maximum value, which is j and m 1 has taken it is maximum value, which is j 1. So, this value m 2 is j minus j 1, now that is good because this is a value, which m 2 takes, but you know, you already know that, the number of values that m 2 can take will be 2 j 2 plus 1 and they will range from minus j 2 to plus j 2, that is known.

So, this m 2, which can belong to this range minus j 2 to plus j 2, this m 2 being, what j minus j 1 is you know have j minus j 1 to fall in a certain range, which will have the lowest value, to which it can be equal, so this will be the minimum value, which is minus j 2 and the maximum value will be j 2. Now, all you do is to add j 1 to every term in this inequality, so now, you have got j 1 minus j 2 here, you add j 1, so j minus j 1 plus j 1 will give you j and now what you find is that j belongs to this range.

Now, what is special about, 1 and 2, what we called as 1, we could have called as 2 and what we called as 2, we could have called as 1 right or you could begin with the other way around. So, any way what you essentially have is that, you have got a certain number of values, which m 2 can take and that number, we know very well must be 2 j 2 plus 1 value. And these are the values that can be taken by whatever appears in the middle term of this inequality, which is j right, the middle term can take 2 j 2 plus 1 value. Likewise we have now found that j can take 2 j 2 plus 1 values, so remember this result and this is something that, we are going to use again in a very important, but simple step.

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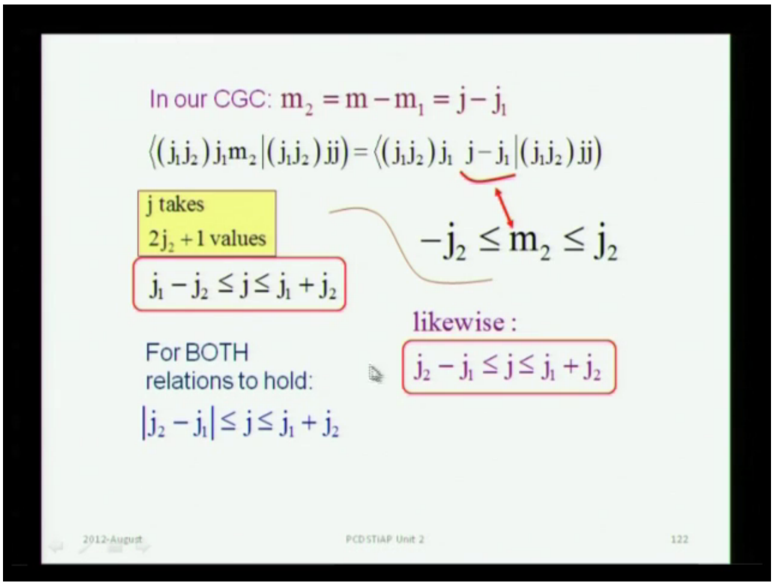

If we had called 1 as 2 and 2 as 1 or we could have begun first with the maximum value of m 2 and the maximum value of m and then find, what would be the maximum value of m 1 right, what are the values of m 1 that could be taken. We would get an exactly identical result that j 2 minus j 1 would be less than or equal to j less than or equal j 1 plus j 2. So, these 2 expressions both are completely equivalent and both must be satisfied.

So, j 1 minus j 2 must be less than or equal to j less than or equal to j 1 plus j 2 and j 2 minus j 1 should also be less than or equal to j less than or equal to j 1 plus j 2, how can both be true, they can both be true, if and only, if you take the modulus of j 2 minus j 1. So, your expression will have to have this constraint, you inequality will have to be that the minimum value will not be j 1 minus j 2 or j 2 minus j 1, it will depend on, which 1 is larger.

So, it has to be the modulus of this difference, because if one of them is larger than the other you will get into trouble and you must have both of these inequalities to hold, so you can satisfy both the inequalities by requiring that the modulus of this must be less than or equal to j less than or equal to j 1 plus j 2.

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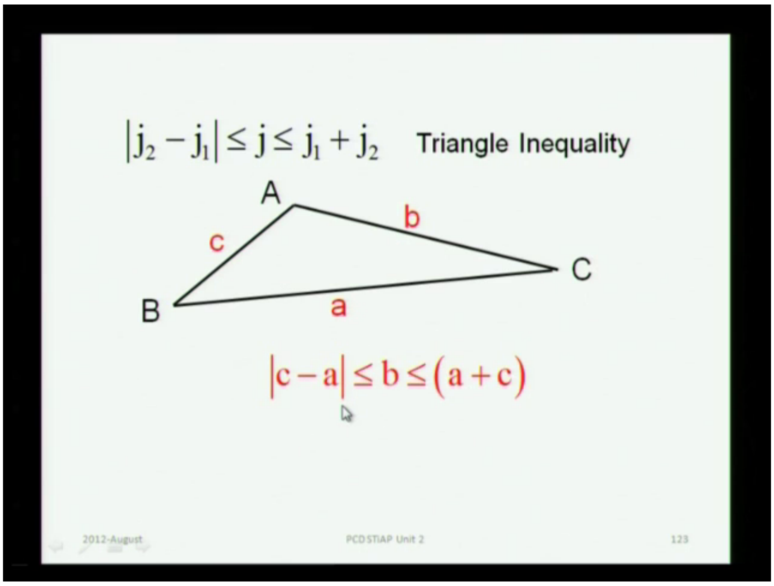

So, this is a very nice result and it is called as triangular inequality, because if you look at the sides of a triangle A B and C then you have this inequality between the 3 sides. The modulus of the difference between these 2 sides will be less than or equal to the third side, which will be less than or equal to the sum of the other 2. So, that is the reason this is called as a triangle inequality and it is exactly the same kind of relation that, you get for the angular momentum, when you couple that, so we know, what is the range of j.

So, far so good, but we still have to get the dimensionality of the coupled space, we ant have anticipated the result, but we have not proved it, so we might as well prove it. So now that, we know that, the result holds good no matter, which is larger of the 2 and without any loss of generality.

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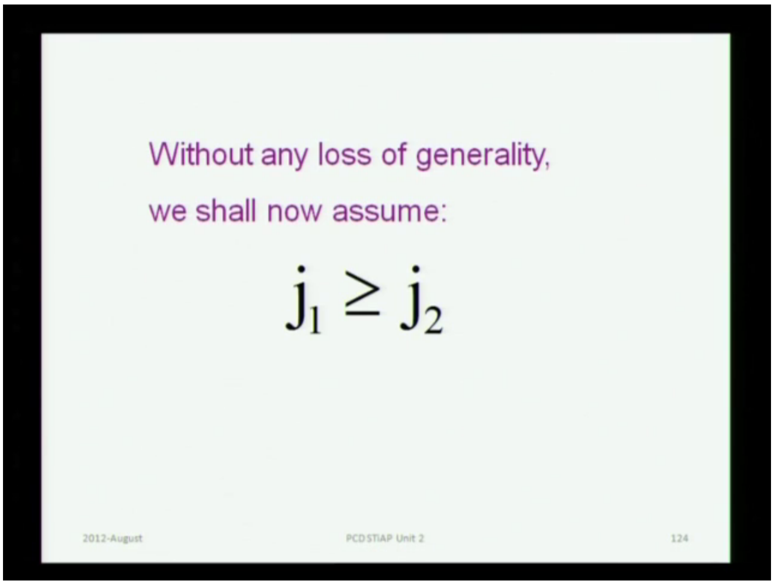

Just for our discussion, we will assume that j 1 is greater than or equal to j2.

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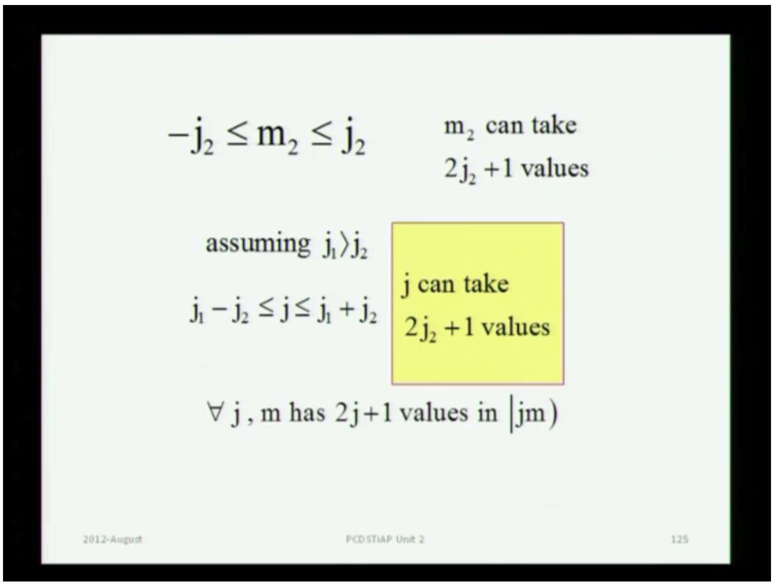

We will use the result from here that m 2 can take 2 j 2 plus 1 values and this is these are the number of values, which the j quantum number can take, so we know that this quantum number j can take 2 j 2 plus 1 values. So, we will use this result and that is the reason this is highlighted in this yellow background, we are going to use it, in a very simple, but important step, so for every j m can take 2 j plus 1 values, this is already known.

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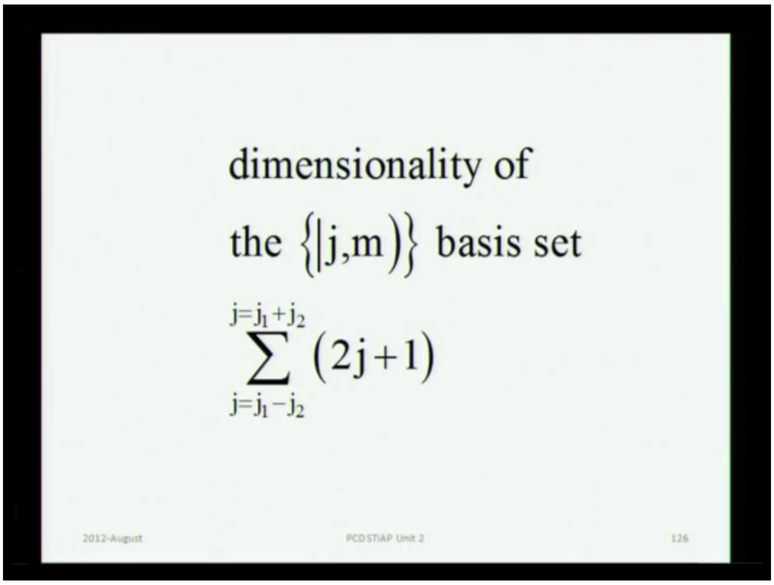

So, what is the dimensionality of this space, it is 2 j plus 1 for every j value m is going to take 2 j plus 1 values, j will take all values from a minimum to a maximum the minimum is modulus of j 1 minus j 2. But, we have assumed for our discussion that j 1 is larger than j 2 that is the reason, I have written it as j 1 minus j 2 to plus j 1 plus j 2, there is no loss of generality in this, we have already established that, what does this sum add up to.

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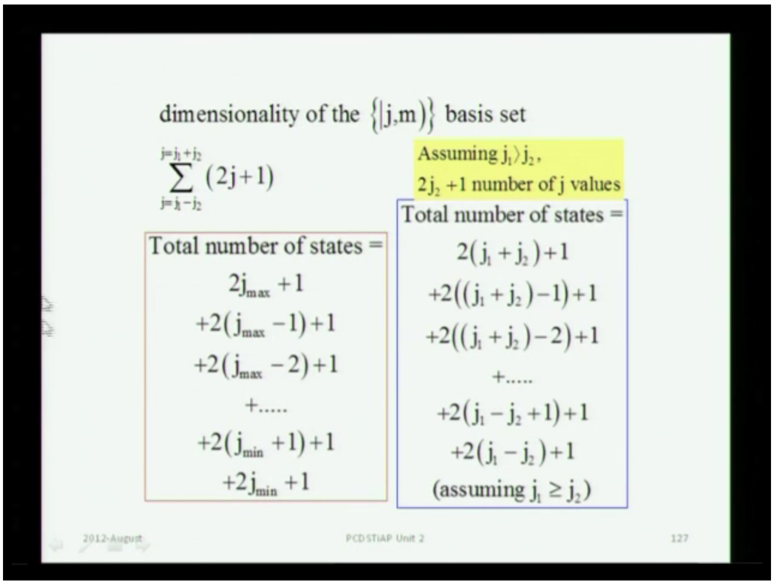

Now, let us carry out the sum explicitly and this sum can be carried out in a number of different ways and there are well know techniques, in you know, how you sum a series that, you can use to work out this sum. But, I will show you a very simple way of doing it, because it also gives you a little bit of physical insight into the whole story, you have the total number of states, which will be for each value of j, you know that, there will be 2 j plus 1 values going from minimum to a maximum.

So, I have written all of these total number of states should be given by 2 j max plus 1 then max reduce by 1 plus 1 then reduced further by 1 plus 1, over here right and the lowest 1, lowest value of j will be j min and it will be 2 j min plus 1. So, this is the total number of states right, now this is the total number of states and you can now write this, j max, we know as j 1 plus j 2, j min is j 1 minus j 2 right and in between the differences go in steps of 1.

So, the maximum value of written explicitly as j 1 plus j 2 the minimum value, I have written explicitly as j 1 minus j 2. So, it is the same box, which I have now written over here, which gives me the total number of states and I am going to bring this box in the right hand side to the next slide.

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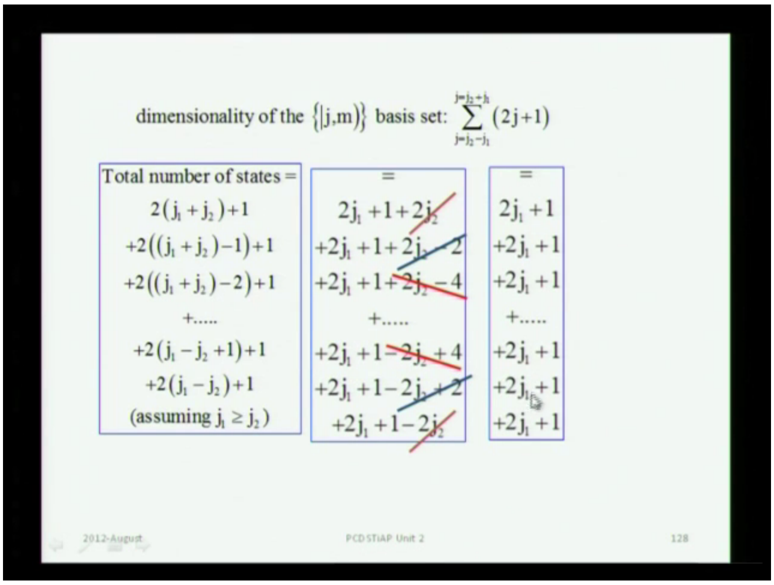

And continue the analysis, so this is the same box, we have got 2 j max plus 1, which is now 2 j 1 plus j 2 plus 1, the slowest 1 was 2 j min plus 1, which without any loss of generality is 2 j 1 minus j 2 plus 1 right. And now let us analyze this further, because you expand this, this is 2 j 1 plus 2 j 2 plus 1, so I have written this as 2 j 1 plus 1 plus 2 j 2, this is you have 2 j 1 again 2 j 2 then you have 2 into minus 1, which will give you minus 2, so that minus 2 comes here.

So, this entire expression is completely equivalent to this row over here, like wise if you look at the next one, in which this number is reduced by 1, so this becomes minus 2 and now you have 2 j 1 plus 2 j 2. So, you have 2 j 1 here, 2 j 2 is here you have 2 into minus 2, which is minus 4 minus 4 comes here and then you have got this 1, which is sitting over here, so all the terms are taken care of.

And you see that every term or set of terms is completely equivalent to the set of terms in the next box, the lowest 1 is 2 j 1 minus j 2 plus 1, which is 2, j 1 minus 2 j 2, this plus 1 comes here and this 2, it is already there right, this is 1, this is 1. So, this is 2 j 1 and this plus 1 is coming from here and this minus 2 j 2 is coming from this 2 into minus j 2, so this is the corresponding term good. Now, let us have a look at this, this 2 j 2 plus 2 j 2 cancels this minus 2 j 2.

They are all in the same box, they are all adding to each other, but this one comes with a plus sign, this one comes with a minus sign, so they kill each other then the 2 j 2 minus

2, this one comes with a plus sign and a minus sign over here, these are with opposite signs over. So, these 2 terms kill each other and likewise, this 2 j 2 minus 4 will cancel minus 2 j 2 plus 4 and they will all cancel out likewise.

Every term and what you are going to be left with is just 2 j 1 plus 1 coming from here, 2 j 1 plus 1 coming from here, 2 j 1 plus 1 from here 2 j 1 plus 1 coming from here and all you will be doing is to add up 2 j 1 plus 1 to 2 j 1 plus 1 plus 2 2 j 1 plus. And how many times, you do it 2 j 2 plus 1 times, you will do it exactly 2 j 2 plus 1 times, because those are the values that j can take, so the result is that when you carry out this series summation, it turns out to be a product of 2 j 1 plus 2 times 2 j 2 plus 1.

So, it is a very simple proof and you know that the final dimensionality of the space turns out to be, what we expect it to be and it is not a mystery and it is not coming just from our expectation from your intuitive expectation that, it has to have the same dimensionality. But, you can actually show it by carrying out the summations explicitly and looking at the dimensionality of the space.

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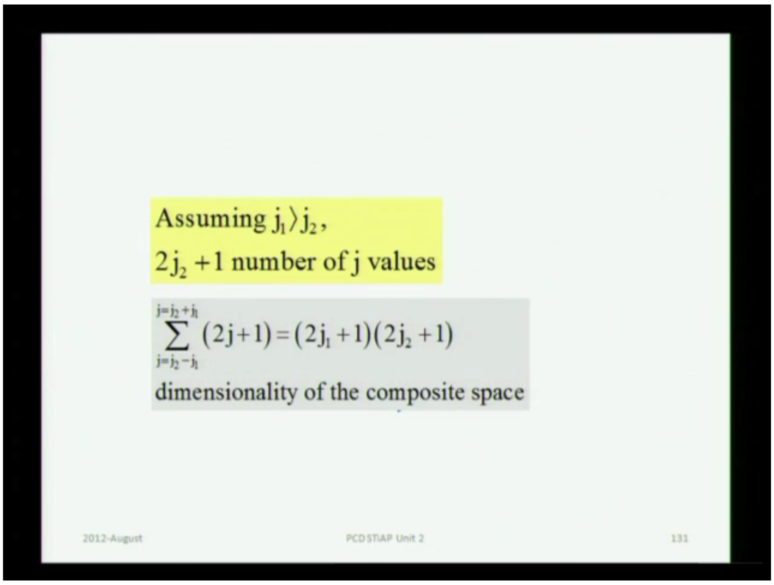

So, now we have got the final expression, for the dimensionality of the space.

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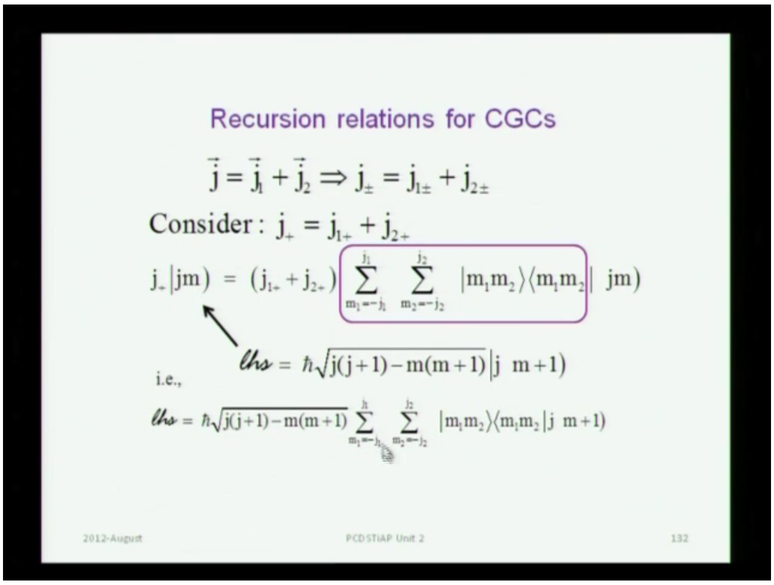

And we will proceed to work with the Clebsch Gordan coefficients and I will develop the recursion relations for the Clebsch Gordan coefficients, because it is very nice that, you can get all the Clebsch Gordan coefficients, just from one number. And you begin with that number, which happens to be 1, which all of us know, what it is and from that, you can get every other coefficient, but we have to see, how and why this is true.

So, let us look at the summation of the angular momenta and this guarantees that, the if you add the corresponding step up and step down ladder operators, they will give you the step up and step down, for the coupled angular momentum. And we consider the 1 with plus sign, there is a similar expression for the minus sign, for the step down operator as the well, but I will consider the step up operator.

And operate on the coupled angular momentum by the step up operator, on the right hand side, I decompose the coupled step up operator in terms of these components j 1 plus and j 2 plus and insert the resolution of the unit operator. So, left hand side we know from, what we have done earlier is given by this square root term, we have done this number of times already.

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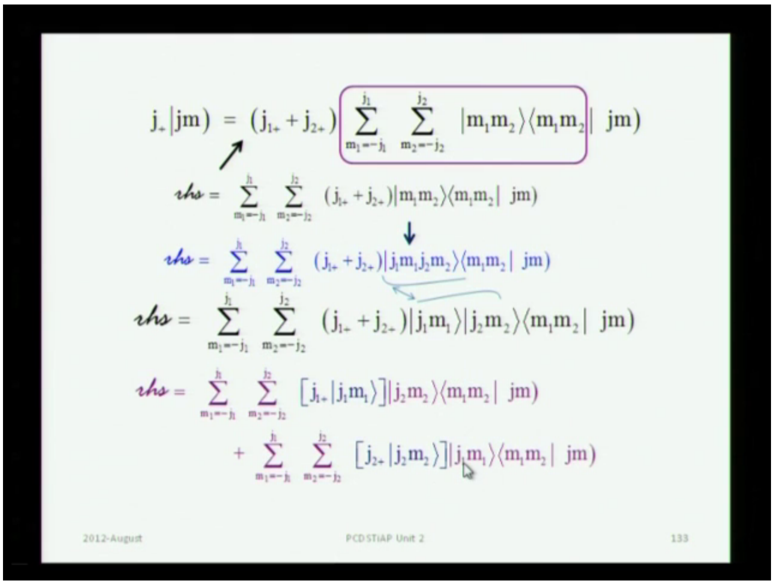

So, this is your left hand side and you can again insert a unit operator here and then take this coefficient inside, so we know look at the right hand side, now the right hand side is obtained from the operation by a sum of 2 operators. So, you can separate the 2 terms by operating by these 2 terms individually on this and j 1 plus will operate on the m 1 pod and j 2 plus will operate on the m 2 pod, because the 2 vector spaces are completely disjoined.

So, this product of the uncoupled vectors, I have written explicitly as a direct product of these 2 vectors, we know what a direct product is and now, we will let j 1 plus operate on j 1 m 1 and j 2 plus operate on j 2 m 2 and we know, that these are step up operators. So, it will raise the corresponding Azimuthal quantum number respectively by 1, so j 1 plus will operate on j 1 m 1, j 2 plus will operate on j 2 m 2, over here j 2 m 2, simply comes out as a multiplier whereas, over here j 1 m 1 comes out as a multiplier. But, of course, there is a residual multiplier, which is coming from the Clebsch Gordan coefficients, so that will be carried forward.

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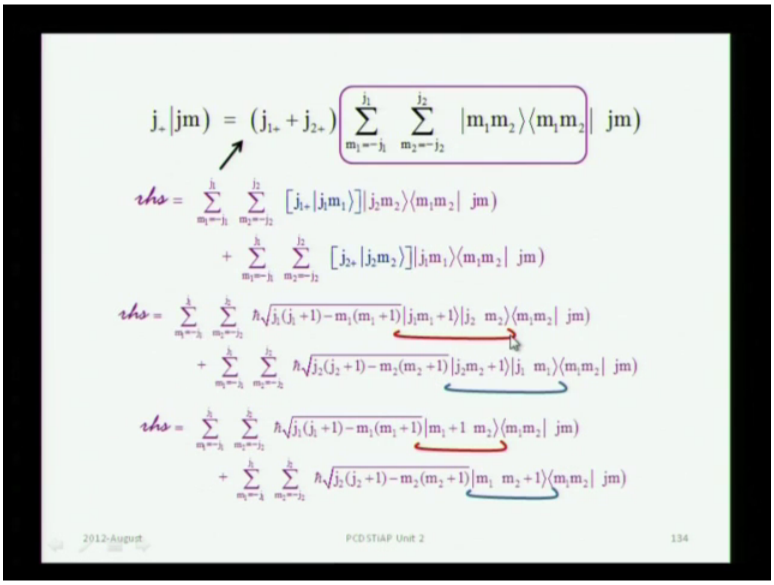

So, we have this right hand side and j 1 plus when it operates on j 1 m 1 will raise the m 1 index to m 1 plus 1 and then it will scale the resultant vector by this x cross times, this square root factor, which we have handled earlier. You are going to have the same thing happen to the second term, in which you have the j 2 plus operate on the j 2 m 2, so the m 2 index will go up by 1, so here you have the m 2 index go up to m 2 plus 1. And then you have the h cross sign, the square root factor and you do have the Clebsch Gordan coefficient factor.

Now, this is rather interesting both of these are summations in the m 1 m 2 basis, but you cannot combine them, because in one the indexes m 1 plus 1 m 2 and in the second term here, this index is m 2 plus 1. So, you will like to combine these 2 terms and you can certainly do that, because both of them are complete summations here m 1 goes from minus j 1 to plus j 1, m 2 goes from minus j 2 to plus j 2 same thing happens in the second term that m 1 goes from minus j 1 to plus j 1 and m 2 goes from minus j 2 to plus j 2.

So, both are summations in the complete set of bases, it is going to pick every term, so instead of counting from here and then take the next term and then take this term and then go all over and come back, you can begin counting from here, take the next term and come back, I will pick all the terms in the complete set of bases right. So, it is like counting the number of students in the class and I can begin with Arati and go around or

I can begin over here and then go around. So, it does not matter, where you begin the counting, it is exactly that.

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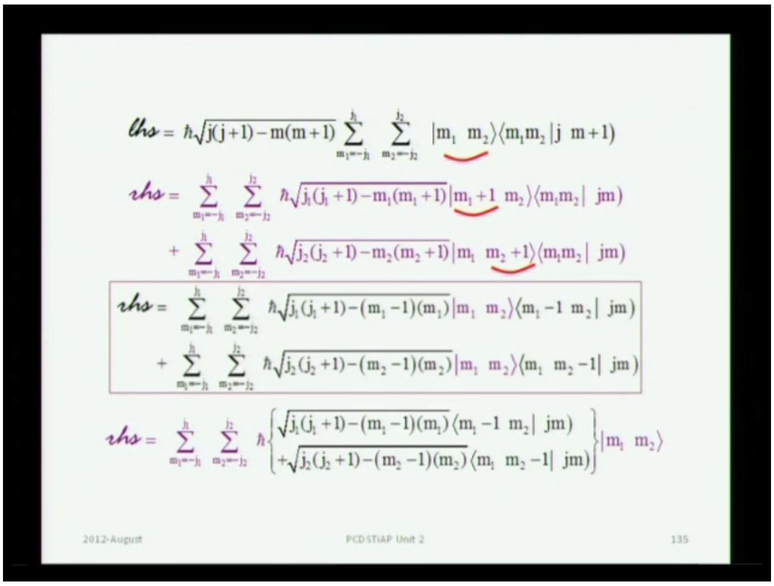

So, let us do that by shifting the index, I can get both the terms expressed as superposition over the same base vectors, but the indices must be shifted, because here the index is m 1 m 2, in this first term, you have m 1 plus on in the second term, you have m 2 plus 1. So, if you want to make the summations correspond to each other, you must drop this index m 1 plus on to m 1, but then you must drop every m 1 by 1. So, this m 1 must be reduced by 1, this will become m 1 minus 1, this m 1 will also become m 1 minus 1, this m 1 plus 1 must be dropped by 1, so it must become m 1.

So, by dropping that index that quantum number by unity, you make all the terms, such that the linear superposition is over the dummy index m 2 rather than, m 2 plus 1 or m 1 plus 1. And it does not matter, whether the dummy index is m 1 plus 1 or m 1 itself, it is exactly like counting the number of students in the class, it does not matter, where you begin the counting, you can count with the next student or the 1 after him or the 1 after him and then go back count everybody and then go back to the previous 1.

So, that is exactly, what you are doing and now having shifted the indices, but in shifting must be done in a very consistent manner, so if you change this index from m 1 plus 1 to m 1, then in the corresponding term, every m 1 must be dropped by 1. You do the same in the second term and now you have got expansions in both the terms, which are in the same running index as such, which is m 1 m 2, but the coefficients of course, are different and you can combine these coefficients now.

So, you have now combined the coefficients h cross comes out as a common factor, the 2 remaining co factors are combined together, which includes the square root term and the corresponding Clebsch Gordan coefficient, this one has got m 1 minus 1 over here, this 1 has got m 2 minus 1 here. Because, how did we get that, because of shifting the indices in a consistent manner, so now we have got the right hand side expressed as a linear superposition of base vectors m 1 m 2.

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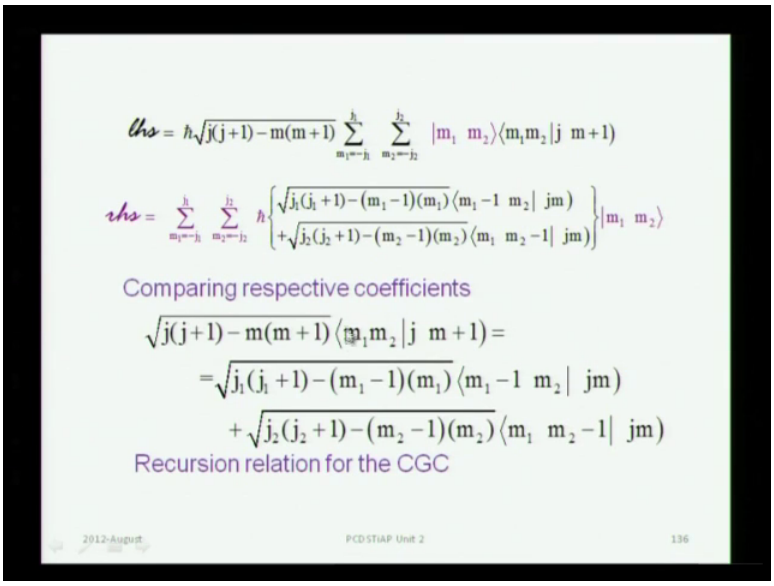

Left hand side is also expressible as a linear superposition of the same base vectors m 1 m 2 and now you can compare the corresponding coefficients h cross cancels. And now you have got a relation, which x connects this Clebsch Gordan coefficient with this and this, in which the indices are shifted, now this is the recursion relation.

You get it means, that Clebsch Gordan coefficients are not completely independent of each other, they have some relationship with the neighboring Clebsch Gordan coefficients. And you can find, what these exact values are because there is another recursion relation, that you will get by using the step down operator.

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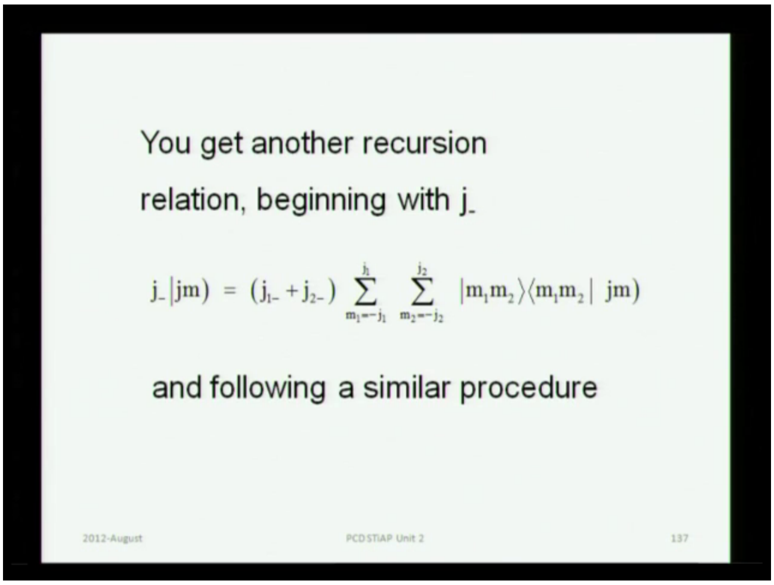

And using exactly the same procedure, so there is more help available than, what we obtained in the previous run the other relation is the recursion relation, that you will get, which I leave as homework, it is the same procedure, you can do it quite easily. So, this is these are the 2 recursion relation that, you get.

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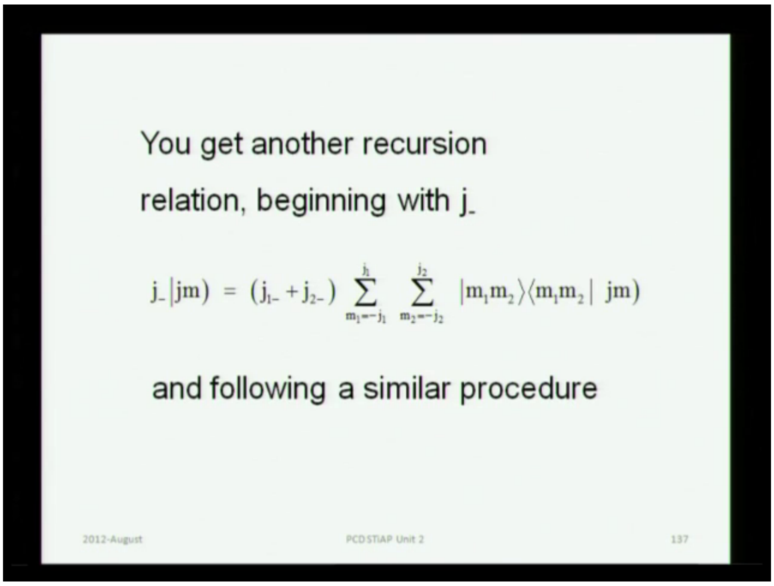

And I have combined the 2 recursion relations by writing, this expression inclusive of a plus and minus sign over here, and there are corresponding plus and minus signs, there is a minus and plus signs. So, you get the top signs to correspond to each other lower signs to correspond to each other and that is where the 2 relations are written together in this expression.

So, these are the recursion relations for the Clebsch Gordan coefficients, you already know that, the term under the square root can be written, as a product of 2 factors that is just a matter of notation again and these recursion relations are extremely useful in getting the Clebsch Gordan coefficients and doing angular momentum algebra. So, you can write them, in terms in 1 or the other notation and different books make use of you know different kinds of notations, that is just a matter of detail.

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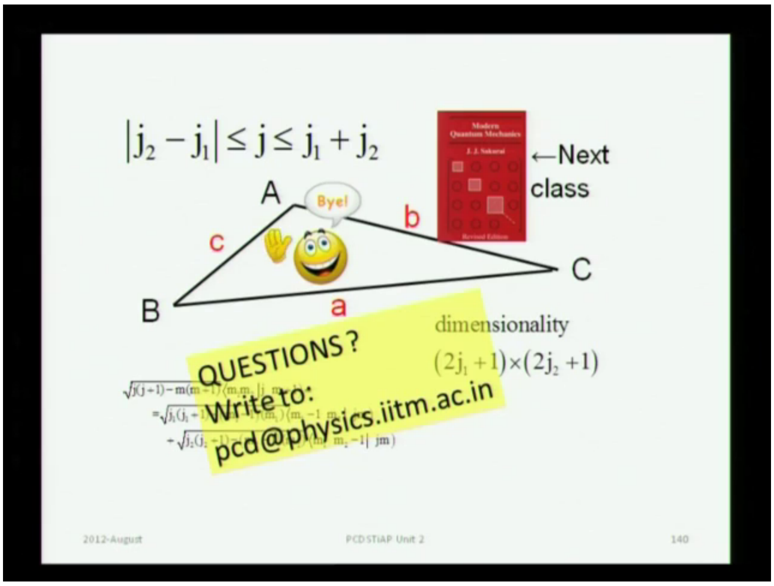

So, this is the summary of what we have got, so far and we established the triangle inequality, we know exactly, what the angular momentum j can be, it can take only these values, it can take a minimum value, which can be the modulus of j 1 minus j 2 or modulus of j 2 minus j 1, however, you would like to say that. And this is called as a triangle inequality, you also learned today that, the dimensionality of the basis must be 2 j 1 plus 1 times 2 j 2 plus 1, as we expected already. From our reasoning that, it is the same product space, which is being expressed in a different basis set, which is the Eigen basis of the coupled angular momentum.

So, the other thing that, we did today was to obtain the recursion relations between the Clebsch Gordan coefficients and these are very important steps or important elements of the angular momentum algebra, you are going to find them very useful. And we will continue to use the wiggler d matrices, which are quite important, but now we are much better equipped to do that. So, if there are any questions, I will be happy to take otherwise, I will stop here, today and we will continue from this point, in the next class, questions.

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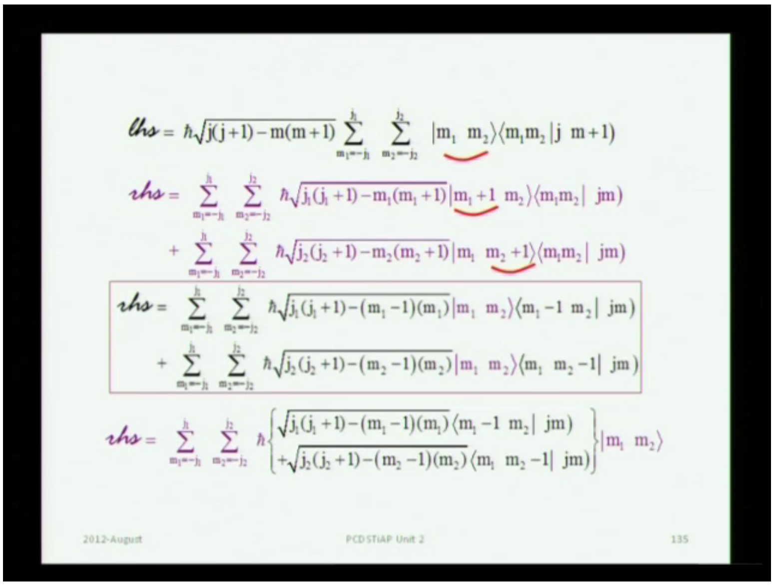

Here this is the part of the expression, but the sums is here these 2 sums are completely independent to each other, these are 2 separate terms, what happens in the first term, does not affect anything in the second term and vice versa. So, in the first term, we are dropping the m 1 index and we do it consistently for everything in the first term, in the second term, we drop the m 2 index, but then we do not touch the first the other index, which is m 1, we have to do it independently.

It is not that, we can we have no choice, we have to do it independently, because the 2 terms are completely independent to each other, you do the counting the counting is within each term. You have a double sum, you decompose this double sum into these 2 terms right. So, this what I have over here, if you are able to see the locus of this arrow, what is inside this is 1 term, this is what I call as t 1 first term, this is now the second term, which is t 2.

And that summation in t 1 has a value, which is quite independent of what is happening in t 2, you have decomposed it and then each term has it is own significance, it has its own merit, it has it is own strength and the indices have got an impact within that term. A dummy sum in term 1, a dummy index in term 1, does nothing to a dummy index in term 2, but a dummy index in term 1, has to be handled consistently within the term 1.

And you must do the same with the second term that, whatever is the dummy index, if you shift the dummy index, within the second term then within the second all the corresponding indices must be shifted consistently right, any other question. But a good point it is need to know that, because you know, you will be doing fairly complex algebra with angular momentum terms and then it is important to do this carefully, because it is very easy to make a mistake, let me assure you that.

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It does not matter, the summation symbol is a dummy symbol, all it is telling us is that go pick every term, you can call, you can define m 1 plus 1 as m 1 prime and then have a dummy index m 1 prime equal to m 1 plus 1, you can add 1 more step, if you like, it is not going to change the result. All you have to remember is that, it is a dummy index, which must pick every value in the range of m 1 and every value in the range of m 1 is minus j 1 to plus j 1.

And the example of counting the number of students in this class, I really a very appropriate 1, because you have to count every single term and it absolutely does not matter, where you begin the counting from, I can begin counting with Arati or begin counting with Ankur although cosmic may complain, why not me. So, we can start counting from cosmic and then go back till there right, but you are going to get the total number of students will be exactly the same. So, you are picking every term in the basis, any other question.

So thank you very much.