Physics of Linear and Non-Linear Optical Waveguides

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Module - 03
Modes (Cont.)
Lecture - 23
Modes in Slab Waveguide

Welcome student, to the course of Physics of Linear and Non-Linear Optical Waveguides. So, today we have lecture number 23. In this lecture, we will going to continue with the Mode calculations in Slab Waveguide.

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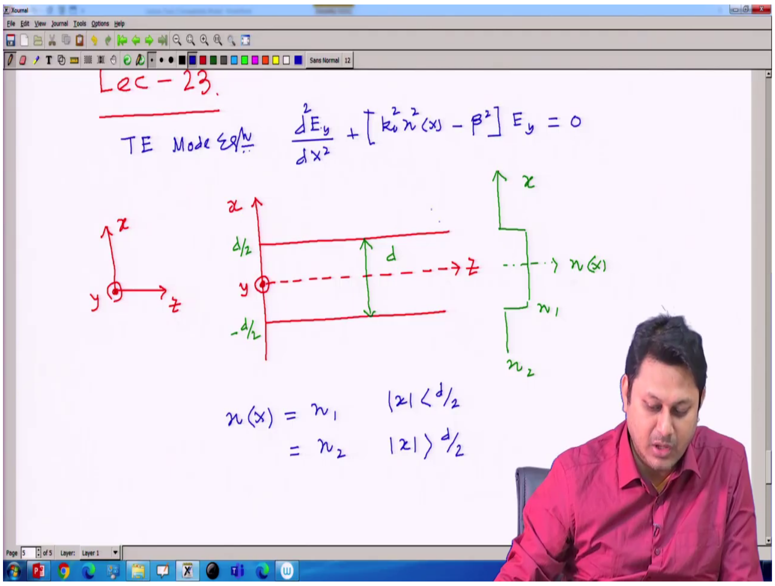

Lecture number 23 today. So, let me remind what we have done so far. So, we calculated the mode equation in this form. So, the mode equation is d 2 E y d x square k 0 square n square as a function of x minus beta square E y equal to 0. This is for T E mode. So, I should write as T mode equation.

Also, we define the slab waveguide and the cross-sectional view of the slab waveguide is defined in this way. This is z this is x and perpendicular to that I have y. So, let me draw it once again x y perpendicular to the plane and z. The refractive index is a step like structure. Along this direction if it is x, this is my n as a function of x this is n 1 this is n 2. This length of the waveguide; if I consider d, then if it is origin, then this is d by 2. And this is minus of d by 2.

After that we define the refractive index. The refractive index is defined at refractive index n x is equal to n 1, when mod of x is less than the value d by 2 that is in the core region and n 2 when mod of x is greater than d by 2. Like a step index fiber, we have the refractive index profile mathematically in this way. Now, we are going to use this information to find out the solutions. We have already the equations; n x is now known to me and now, I can calculate that.

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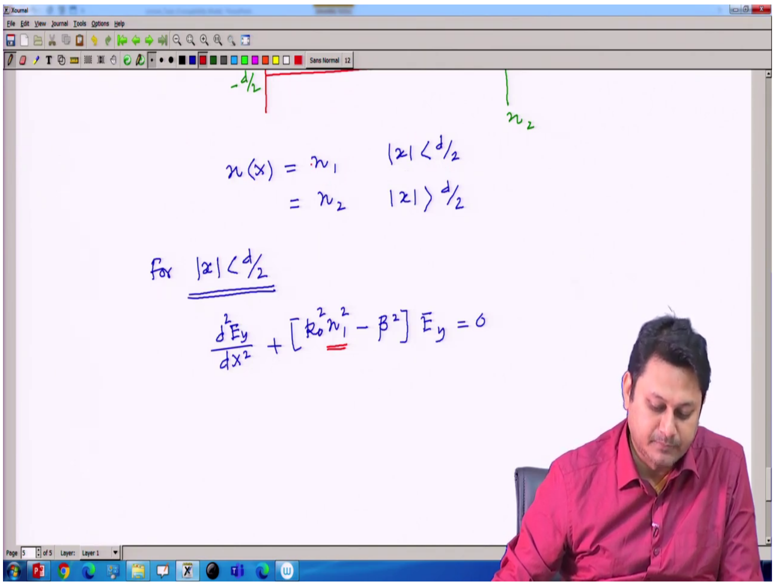

So, I can write for, x d by 2 in this region that is in the core region, the solution the differential equation will be into d 2 d E y d x square plus k 0 square n 1 square minus beta square E y equal to 0. You should know that here, I directly replace in n 1, because I am working in the region x less than d by 2 and I know at the region x less than d by 2 mod of x less than d by 2 a the value of n x is n 1.

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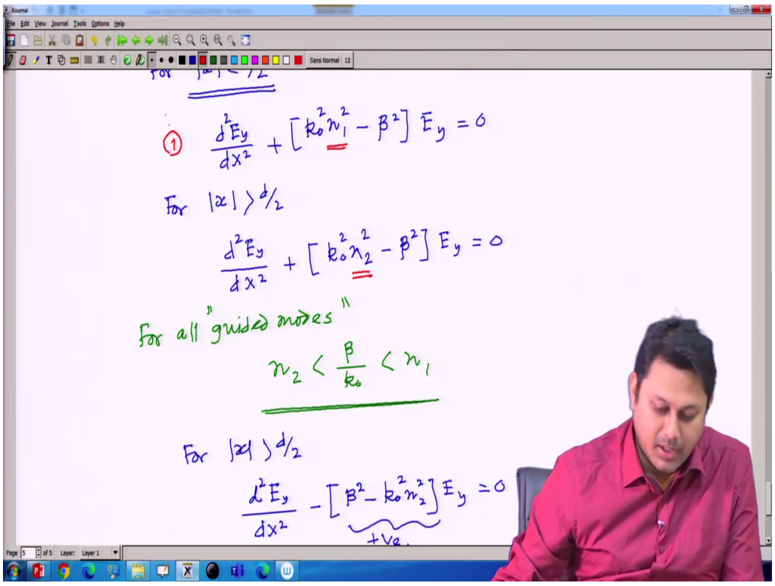

Similarly, for mod of x greater than d by 2 the equation I can write in this form; d 2 E y d x square plus k square n 2 square minus beta square E y is equal to 0. Again, I replace this value here, because I know that at the region x greater than d by 2, the value of n x is n 2.

Now, for all guided modes this is important. For all guided modes, we have a restriction of beta and this restriction we already mentioned in the previous earlier classes that beta divided by k 0 should be in between the two refractive index. So, this is a restriction over beta we have for all guided modes.

For all guided modes this restriction we have. Since, we have this restriction, if I look to this equation for x greater than d by 2 mod of x greater than d by 2. So, k 0 square n 2 square this value is less than beta it has to be less than beta. So, I can rewrite this. So, that this whatever we write here, in this region is positive.

So, again I rewrite for mod of x greater than d by 2, d 2 E y d x square minus beta square minus k 0 square n 2 square E y equal to 0, because now, this quantity become positive ok. So, I have one equation here, let me write it equation 1 and equation 2 here. So, equation 1 and 2, we can write in a more convenient way, because now we know that whatever we have here beta and k 0 n square is constant it is not a function of x anymore. So, I can write it as a constant.

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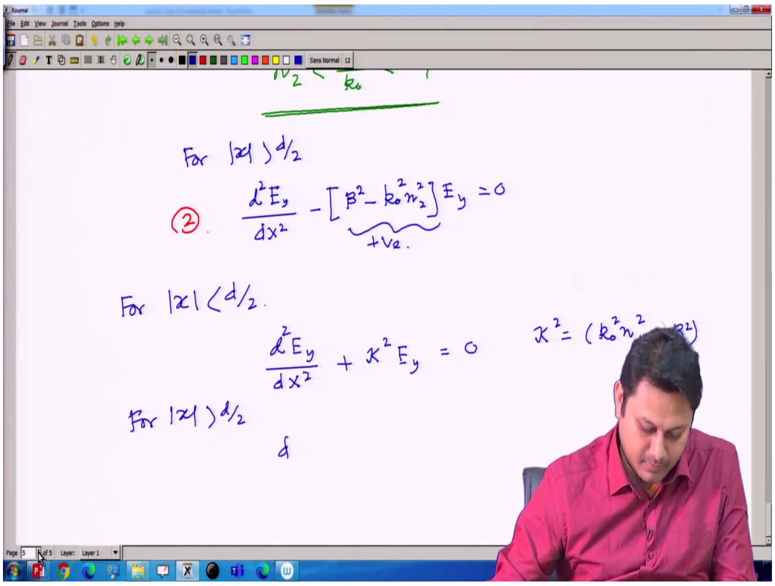

So, for mod of x less than d by 2, I write d 2 E y d x square plus kappa square E y is equal to 0. I write I put a new parameter kappa where kappa square is k 0 square n 1 square minus beta square. In the similar way, for x greater than d by 2, I can have another equation.

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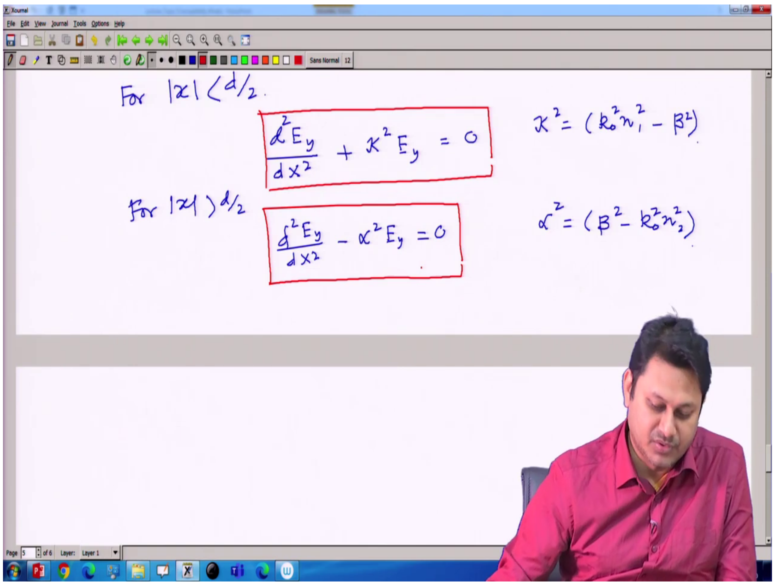

Which is d 2 E y d x square minus of alpha square E y equal to 0, where alpha square is beta square minus k square n 2 square; alpha and beta now, I define and this equations are now more compact as well. So, this is a well known equation. So, the solution is well known for both these two equation that is a good thing for us, because we do not need to calculate any more straight way I can write down the solutions.

So, the first case we know this is say some sort of sinusoidal solution. One can expect with these differential equation. On the other hand, in the second case, we should have some kind of exponentially decaying solution that is all.

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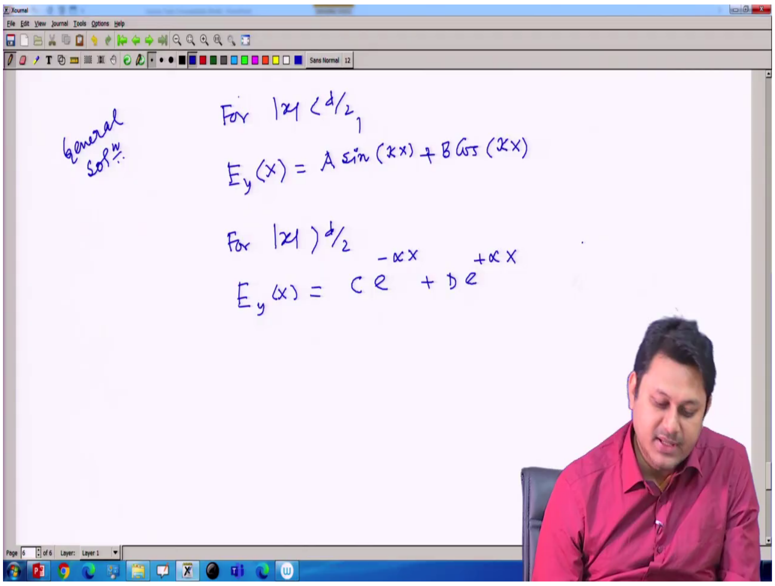

So, the general solution if I write, general solution for mod of x less than d by 2 I can readily have a general solution sinusoidal in nature as A; let us put as sin plus B cos of kappa X. So, this is the general solution we have for x less than d by 2. For mod of x greater than d by 2, I have a solution of the form E y x. As another constant c e to the power of minus of alpha x exponentially decaying or amplifying solution depending on the boundary condition plus alpha x ok.

So, I have a sinusoidal solution in the region when x is less than d by 2 on the other hand if x is greater than d by 2, I have an exponential decaying solution which is which is very much expected which is very much expected. So, this is another solution we have ok. So, let me erase this, because it looks ugly actually.

So, let me erase this, this is only to highlight that these are the solutions one can expect now, looks better. So, I have a solution here in the core region and I have a solution in the cladding region. So, let us visualize how the solution is going to look like.

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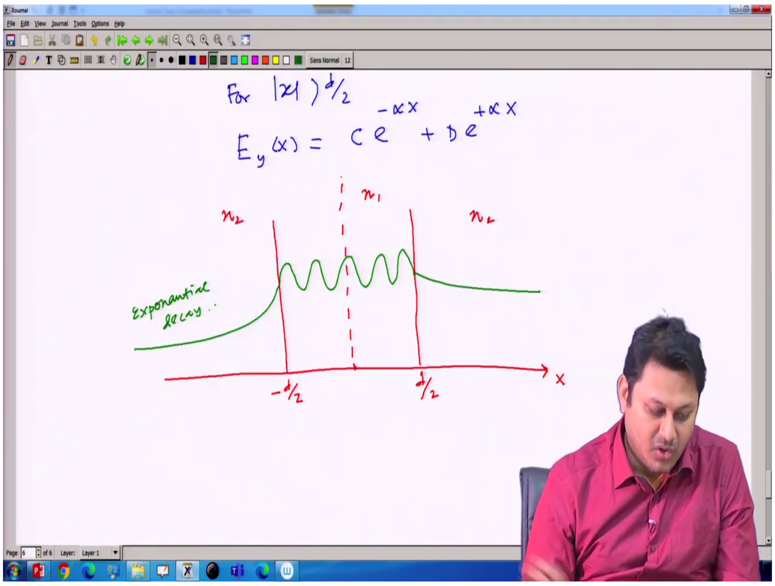

So, my structure wave guide structure is something like this. I have a core region and there is a cladding region. So, if this is my origin, then this point is d by 2 this point is minus of d by 2 along this disk direction I have x, if the solution distribution of the. So, this is my refractive index by the refractive index n 1 this is refractive index n 2 and this is refractive index n 2. So, in the core region I have a sinusoidal kind of solution.

So, the solution is something like this whereas, in the cladding region I have a solution which should exponentially decay something like that. So, this is a exponential decay. So, I have an exponential decay solution in the region in the which is in the cladding region. In the core region, I have a sinusoidally oscillating region. So; that means, since it is exponentially decaying in both the side.

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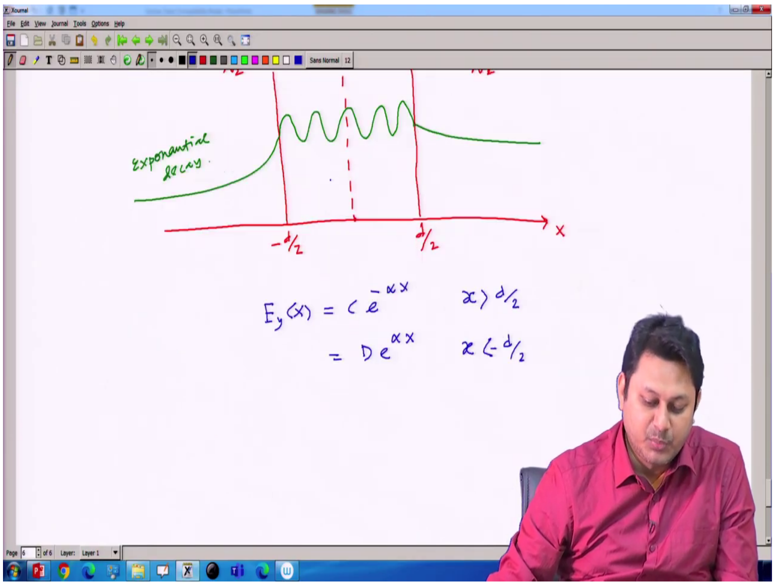

So, I can write down my E y in the region for. So, E y as a function of x is equal to c e to the power of minus alpha of x, when x is greater than d by 2. However, when x is less than d by 2, I should write this solution as D e to the power of alpha x, when x is itself negative. So, it is less than minus d by 2.

Now; that means, x is itself negative. So, these are the two solution in the two boundary two claddings which is situated at x greater than d by 2 and x less than d less than d by 2. Now, I will put another restriction to find out the solution, because I need to put certain boundary condition to find this A B C D all these coefficients. So, I put some kind of restriction over the solution.

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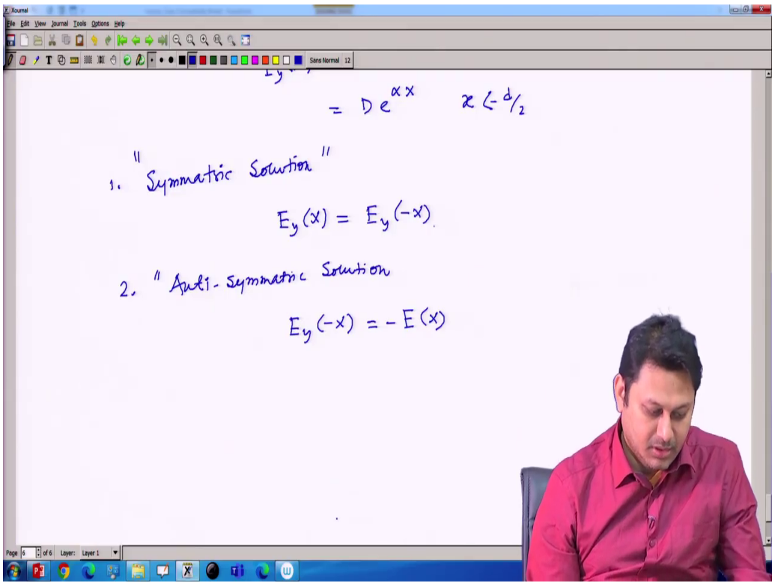

So, first I will try to find out something called symmetric solution this let me erase this. Symmetric solution: in a symmetric solution, we have E y function of x is equal to, because it is symmetric. So, if I go to minus x, then the solution looks same.

So, E y x should be equal to E y of minus x; this is the condition for a symmetric solution. Another kind of solution one can also find which is the anti-symmetric solution where anti-symmetric solution if I go to minus x, then I will have whatever we have in the x with a negative sign that is why it is a symmetric or anti-symmetric.

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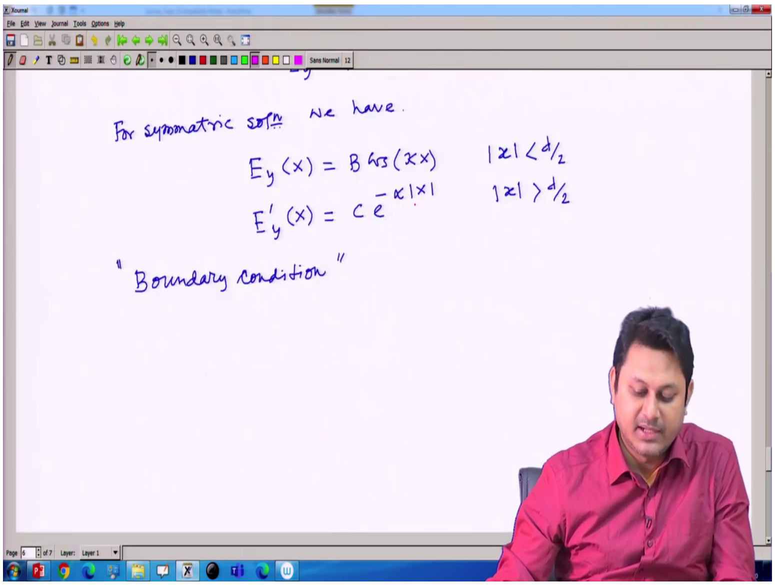

So, this symmetric solution as soon as I put the condition for symmetric solution in the region in the core region, so for symmetric solution, we have E y x is equal to B of cos of kappa x, because initially, it was A plus the general solution was A sin kappa x plus B cos kappa x.

Now, if I if I put a condition that this solution is symmetric in nature; that means, if I replace E y to minus of E y x to E y minus of x, then it has to be same. And from that restriction we can readily find that this coefficient a has to be 0 for symmetric solution. So, that I can have this condition valid E y x is equal to E y of minus x and for the cos solution which is very much valid. So, this is in the region when x is less than d by 2.

Well, I can also have the solution. So, this is in the core region. So, in cladding region, I also have a solution. So, let us put this solution as E prime. So, E prime is a solution in the region x is greater than d by 2. So, it should be c e to the power of minus alpha I can put a mod of x.

So that, whenever I put the value of x plus or x minus, then this solution will remain same. This is for mod of x is greater than d by 2.

Well now in order to execute B and C, because I now already by putting the symmetric condition I already reduce the number of, number of unknown constant. In first case, I just remove A and in second case, I just remove another constant, but still, I have two constant B and C. So, I should have some kind of boundary condition to evaluate something or at least get some relationship with B and C. So, the boundary condition next, the boundary condition.

The boundary conditions suggest that since, I draw the field here you can see at this boundary if I look carefully to this figure here, at this point it has to be a smooth transition this point and this point so; obviously, the value of the electric field here, at the region x less than d by 2 and x greater than d by 2 at the junction that is at x equal to d by 2 or x equal to minus of d by 2; the value of the field at core and cladding should be same and also the derivative has to be same to make this smooth transition.

So, there should be any kind of jump here, in this boundary and that is obvious condition. So, mathematically if I put the boundary condition, then I should write it here like.

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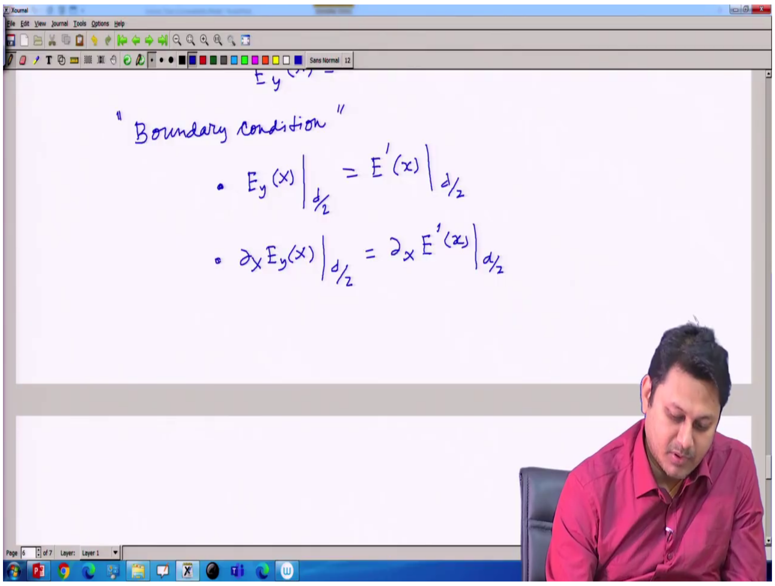

E of y x executed at d by 2; whatever the value I will get it should be equal to E prime which is the solution in the cladding region x d by 2. And the second condition the smoothness condition, the derivative of this field at d by 2 at the boundary is same. So, I know the boundary conditions here, this is one boundary condition and this is another boundary condition.

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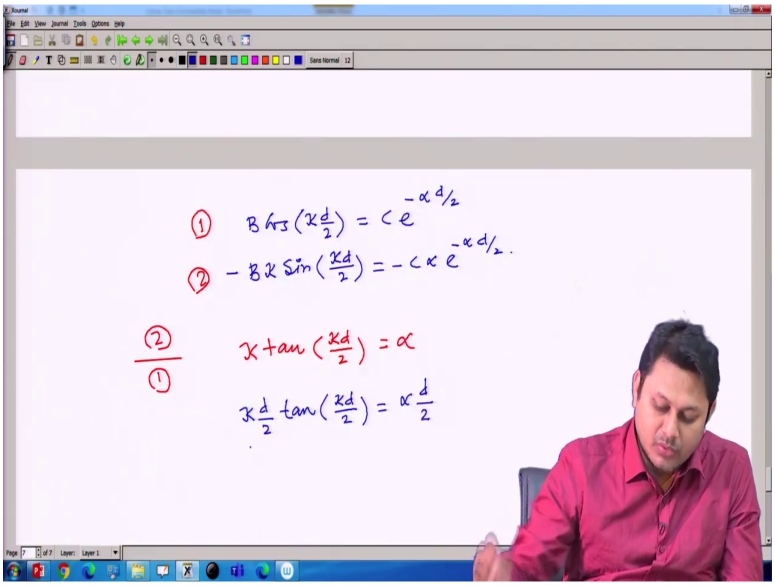

So, once we have the two boundary condition in our hand, then I will just put this thing. So, the first boundary condition if I put the first boundary condition, I simply have B of cos of kappa d by 2 is equal to c e to the power of minus of alpha d by 2; that is one. Second boundary condition is, if I make a derivative over that. So, minus of B kappa sin of kappa d divided by 2 will be equal to minus of c alpha e to the power of minus alpha d by 2.

Now, from this equation if I put a name here, say 1 and 2. I can readily eliminate the two variable B and C and I have an equation in my hand through which I can have the value of kappa or from this value of kappa, I can find the value of beta which is the propagation constant.

So, that is the one of the major goal to find out that for a given field, so, how the propagation constant can be calculated. And this propagation constant can be calculated by simply using the boundary conditions.

So, if I make if I make equation 2 divided by equation 1, I simply have equation like kappa tan of kappa d by 2 is equal to alpha. I just make this divided by this equation, e to the power alpha minus alpha d by 2 will going to cancel out cos and sin. So, sin divided by cos it will be tan. So, I have something like this.

Now, I will going to manipulate this equation. I will do like this way. I put multiply d by 2 and, then I have tan of kappa d over 2 equal to alpha d over 2. So, now, after multiplying d by 2, I can have the same value here before the tan and also the argument I have same value and in the right-hand side I have alpha d by 2. So, this kappa d by 2, I can write as a new variable.

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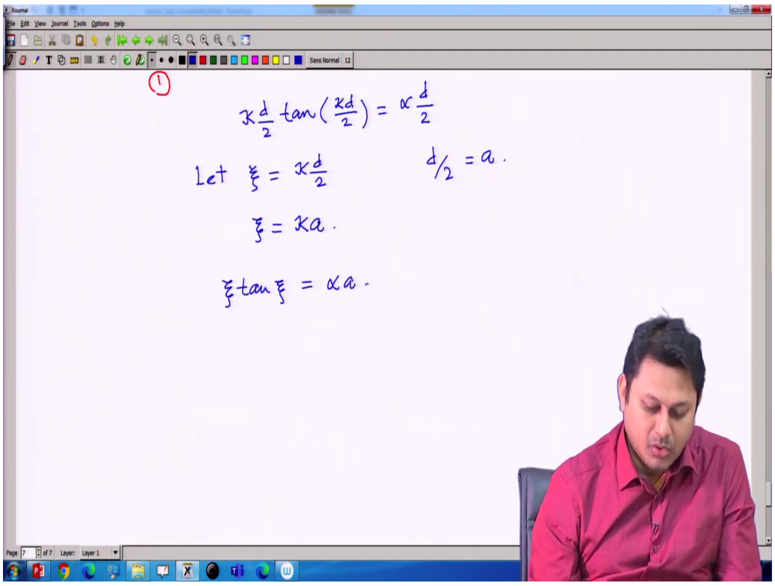

So, let xi which is a new variable is defined as kappa d by 2. And this d by 2 I also every time I am writing d by 2, I just replace this at a. So, kappa is essentially epsilon and the xi is essentially kappa multiplied by a. So, my equation becomes xi tan xi is equal to alpha a. Well, there is a relationship between alpha and kappa.

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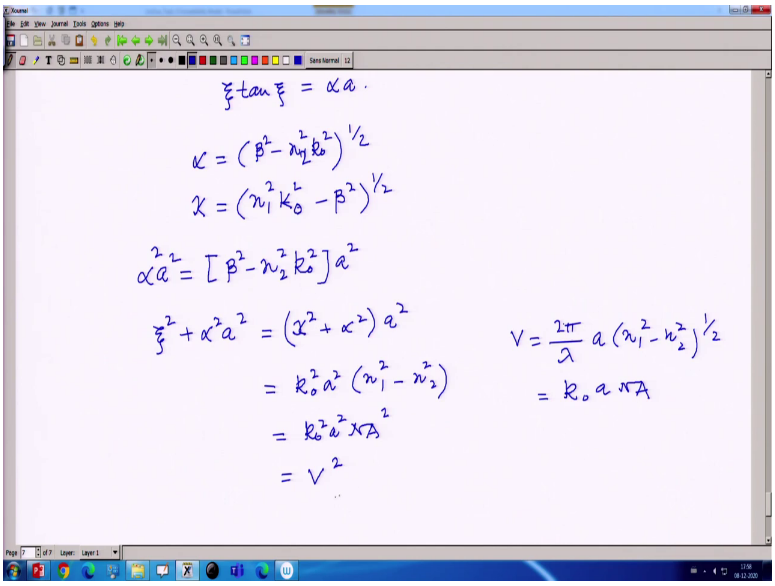

So, let me try to find out what is this relation. So, alpha was beta square divided by n 1 square sorry it is n 2 square divided by k 0 square whole to the power half. And kappa was n 1 square minus n 1 square k 0 square minus beta square whole to the power half. So, alpha and k we know. Now, alpha a will be simply. So, let us make. So, alpha square a square is simply beta square minus n 2 square k 0 square and then, a square.

Now, xi square plus alpha square a square is simply kappa square plus alpha square into a square and kappa square plus alpha square; if I calculate, then this beta square will going to cancel out I have something like. So, here I have k 0. So, k 0 square a square n 1 square minus n 2 square. This quantity is well known to us.

It is k 0 square a square N A square, because N A is a numerical aperture we know. So, this quantity is nothing, but V square, because by definition V is 2 pi divided by lambda, then a n

1 square minus n 2 square whole to the power half. So, this quantity is simply k 0 a N A. Now, we have k 0 square a square N A square so, it should be V square.

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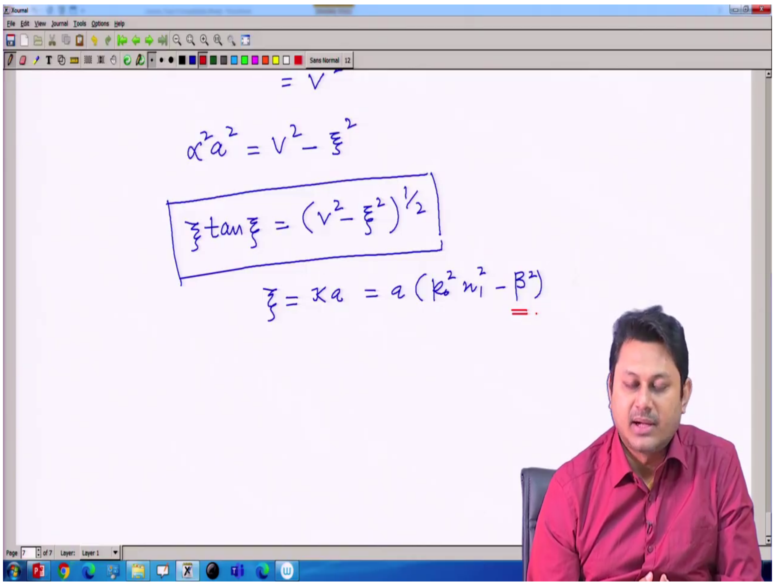

So, alpha square a square is V square which is a constant, because it depends only on n 1 and n 2; nothing to do with the variables minus xi square. So, eventually I have xi tan xi is equal to an equation V square xi square whole to the power half. So, this is a transcendental kind of equation and in order to find out the xi the solution xi of this equation through which I can calculate the propagation constant beta mind it xi is kappa a and kappa is k 0 square n 1 square minus beta square.

So, my goal is to find out here. My goal is to find out this beta which is a propagation constant of a mode. So, if I have a solution of this transcendental equation whatever the transcendental equation is written here, then I can have the value of xi. Once I have the value of xi, I can have the value of beta.

So, that is why this equation is very very important. So, in the next class we will try to solve this transcendental equation. And you all know that when we have a transcendental equation.

So, only one way we can solve that, and this way is the graphical solution. So, we plot the left-hand side and right-hand side as a function of xi and then, we find the cutting point. So, the cutting points are essentially the solution of xi, once we have the solution of xi, we will find the value of beta which is the propagation constant of the modes different modes.

And interestingly we find that there are certain discrete values. From the Ray theory also, we know that there are certain discrete ray's that are allowed to propagate in the waveguide. Here, we have some discrete modes through the value of discrete beta's that is allowed to propagate inside this waveguide. So, we will learn all these things in the next class. So, see you in the next class.

And, thank you for your attention.