Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

Week - 02

Lecture – 08

So, now we will go to Green's function, continuation of Green's function. So, in the last module I had worked with Poisson equation. So, we computed the Green's function for Poisson equation. Now we will do some more, so, Helmholtz equation, you will find this in this course. So, what is Helmholtz equation? This is wave equation, well, I am going to derive a Helmholtz equation, just give me one minute. So, this is a wave equation, partial, so, this is time derivative, when I write like this is a time derivative, this is equal to d by dt of some function.

If I write like this, this is partial with h. So, this means double derivative in time, c is the speed of the wave. So, this is a double derivative in time minus Laplacian, pi xt. Now the right hand side, I am going to assume that functional form for time.

Is function of x it is rho x, but the function of time is exponential i omega t. Imagine that there is a reduced station which is transmitting a wave with frequency, fixed frequency which is i omega t, with frequency, fixed frequency of frequency omega. So my source will be e to the power i omega t times rho of x. A reduced station is not a point source, but there is some kind of, there are lot of charges which are transmitting electromagnetic field. Now if I have this kind of source, the source, I said this is source, if this kind of source, then for the steady state solution, we are not going to deal with transients in the beginning part to satisfy the initial condition, all that I will ignore.

The steady state solution, if you wait for long enough time, then the steady state solution phi xt should also have e to the power i omega t response. The time dependence will be e to the power i omega t. This is something very similar to what we do for oscillators. If I force an oscillator with certain frequency, then the response under steady state will be e to the power i omega t except at the resonance. But we are not going to have resonance for, well we will ignore resonance, that special condition we will ignore in this stuff.

So my response also is e to the power i omega t. Now plug that in here, if you plug that in here, what do you get? So this one will give you minus omega squared, you know, if you put e to the i omega t for here, so what I am going to do here, I just plug in e to the power i omega t phi of x. So double derivative of that will give us minus omega squared and Laplacian will give us plus k squared and this c squared sitting here, fine, straightforward. And we call this omega squared by c squared as k naught squared. These are also minus sign, they are both the minus signs, these are, well this plus actually.

No, so I have right now put Laplacian, I am not substituting here, this is Laplacian. So I have not put k squared right now, this is Laplacian because phi of x, phi is function of x. So do not disturb phi of x or Laplacian, just keep it like that. So this becomes minus Laplacian and minus k squared, so minus sign I send it to the right hand side. So this is my equation, this is called Helmholtz equation.

We got rid of time dependence by assuming that my wave, my source is of the form e to i omega t. So now what is the Green's function for this? So rho of x I am going to assume a delta function, then I get the Green's function and I did not write it, but you can easily see what should be the Green's function in Fourier space. Yes, so let me just write down for completeness Laplacian g of x x prime delta x minus x prime. Now when I do the Fourier transform, then e to the power ikx or k dot x will give me minus k squared, this guy will give minus k squared and this is my Fourier transform of the Green's function g of k. This bit of algebra but I cannot do every step, you should be able to see that this will be indeed the Green's function.

Now this is in Fourier space in any dimension, this is the beauty of Fourier space in this formula that work in any dimension. Now I want to go to real space. So let us work it out for 1D, in field theory often we will not work with 1D, we will work with 3D typically, but let us do it for exercise, you know let us do it for 1D. So my Green's function is this, this is the Green's function. So I want to get to real space, so by definition g of x is dk by 2 pi, where d is 1, so this is dk 2 pi, this this stuff.

So I am integrating along this line. Now there are singularities sitting at the x axis, k naught is real. So the two singularities sitting there, k equal to minus k naught and k equal to plus k naught, this is k axis, real k. Now that is a problem, something pole sitting on the axis is a problem. Now that is where we invoke boundary condition.

So I will say that I am looking for waves which are going right, well my source is transmitting waves, outward going waves. So my source is here and it is going to transmit waves both left and right. So I am going to choose my waves which are going to the right in the right hand side, x greater than 0, x greater than 0 and left for x less than 0. These two illustrate the power of shifting the poles, that is in my hand, because my boundary condition is not specified, so I am going to specify boundary condition by shifting the poles. So I am going to shift the pole, the right pole upward and the left pole downward.

Now I am going to do the closed loop integral. Closed loop integral means I need to close the contour. So which side of the contour, upper plane or lower plane I should choose so that my semicircle should give you 0. I want this integral over a semicircle to 0, to go to 0 by Jordan Lemma since k is positive and x is also positive, well sorry x is positive not k, k x is positive, right x is positive, k can be negative or positive, k is a complex number. I am extending my k to be complex because it is going the full complex space.

So x is positive, so if x is positive I will get 0 if I choose my semicircle in the upper half, right Jordan Lemma we did that. So I close from the top, okay. So what is the contribution from, so the closed loop will be not 0 because there is a pole inside, okay 2 pi I integrate that at the function, okay. So this algebra I am going to leave to you, you should work it out. So do the closed loop here, this part will give 0, let me just do it, the sketch this part.

Integral has from normally people use this gamma, integral gamma plus g of x, right. This is g of x and the integral will be gamma, this is 0, okay. This must be equal to the Cauchy theorem, 2 pi I value of the function at the pole, okay. Now look at here, this part. So we write bottom as k naught minus k, k naught plus k.

So pole is at k equal to k naught, this is k naught plus epsilon, okay. So pole has been shifted but epsilon is small, tiny. So this is pole is coming here but it is k naught minus k, so I need to flip it. Remember it is z minus z naught. So I need to put a minus sign and put k minus k naught, k plus k naught, okay.

So pole contribution will come from here, okay. So there is a minus sign, you pull it out, okay, this is put a minus sign here and then value of the function at k equal to k naught. So what will I get? e to the power i k naught x. So this is k is replaced by k naught and here also k is replaced by k naught. So divided by 2 k naught, okay and this 1 by 2 pi sitting there, 1 by 2 pi, this 2 pi is also sitting there.

So this 2 pi, 2 pi cancels and I get minus i exponential i k naught x by 2 k naught, okay. So that is my Green's function, is that clear? Now this is the space part but now recall what was the source? Source is e to the power i omega t then delta function, okay, that was source. So my response is this but my source e to the i omega t will be the same. Oops, well, I am using here minus i omega t, okay, this is all hidden there. So my source I am using is e to the power i omega t e to the power i k naught x, okay, this is red, red is cancelled.

So I am using my source to be e to the power i minus i omega t, okay, that is to make sense and that will also give you minus omega squared. So this k naught squared will come for both e to the power minus i omega t as well as e to the power plus omega t, okay. So this is a wave going to the right, is that correct, this one? Now for x less than 0 what do I do? For g x less than 0, now here even this x is less than 0, negative. Now which side should I close the semicircle? Bottom right because x is negative, Jordan Lemma again. So I close it from bottom and do the algebra then you find that my answer is this, okay.

I recommend that you do fill this algebra, it is straight forward but you should do that part and my wave will be looking like this, e to the power i k naught mod x minus omega t, okay. Now which way the wave is going for x less than 0? Is it going like that or is going like that? Is going backward because mod x is positive number, so imagine I go from here to there, so mod x is increasing when I go from there to here and time will also increase that way. So my wave is, this is the wave which is going that way. So my Green's function by choosing these poles like this, the right pole above the axis and left pole below the axis I am able to achieve my waves which are going away from the source, right. So radio station is a 1D radio station, okay, which is transmitting waves this way and transmitting waves this way, all the time, okay.

So that is the difference, it is not delta function in time, you see e to i omega t or e to the power minus i omega t in time, okay. So this is 1D, straightforward, right. What happens in 3D? Okay, so 3D, of course I need to integrate in 3D, so this is a really a source which is e to the power minus i omega t in time but point source in space like a charge, okay, charge is oscillating continuously, exponentially in time, well oscillating in time, oscillating in time, okay. So imagine this is a charge which is oscillating like that, so this is a relay antenna, 3D antenna but it is a point source antenna, okay. Now the Green's function is exactly the same as before but now I need to do integrate it in 3D, okay.

Now we need to follow exactly same steps as what I did for Poisson equation, you can choose your R to be this along the Z axis, R vector and I need to integrate over k, okay. So by the way this, okay, so you just follow the same steps, okay. Now I have some steps here, I will not do it in detail but you can just do it yourself. So this k dk will in fact there is a k square dk but there is a one k will come from here, k square dk, this is cancelling here, so that k dk, there is a 2 pi coming from d phi integral that will cancel cube to 2 pi square and now there is a d theta integral which I did in the last class for Poisson equation, d theta will be given as e to i kr minus e to power minus kr by 2 i kr. Remember this was sum of e to power i kr cos theta, d cos theta, okay, so it comes like that.

Now, let us try to do a nice trick, this function, okay, the negative part, e to power exponential minus i kr, you make a change of variable minus k to k, okay. So this is a k sitting here and this k square does not change sign, so this minus will get absorbed by this minus k and you can go from 0 to infinity, rather transform this integral from 0 to infinity to minus infinity to plus infinity with only one term e to power i kr, is a nice trick, just change a variable for the this part, okay. Now, this is nice, right, I mean, you always convert minus infinity to plus infinity, okay, try to do it, sometimes well, we will not do that kind of integral, but branch cuts when you got, then there is a problem, okay, branch cuts we will not do in this course, okay, that is also very nice and cute integrals, but we will not get into that, but otherwise we can typically can manage to get minus infinity to plus infinity. Now, this integral is pretty much similar like what we had in the last slide, okay, this k sitting here, that is the only difference, right, otherwise is same as before, okay, then apply the contour, r is always positive, right, r is a magnitude of r, so it is positive, so I close from the top, I close from the top and you do the algebra and the answer is this, exponential i kr by 4 pi r. So, this is the in fact amplitude of the electric field, well, I mean, okay, electric field is a vector, but you will find something like this, when you potential will look like this, you do i kr by 4 pi r, anyway this is a Green's function, a real space for Helmholtz equation, okay, this is a pretty simple derivation in 3D, we need some of this stuff when we get into the real field theory.

Now, a related problem is this, instead of k squared minus k naught squared, I have k squared plus m squared, okay, now we can follow the same scheme, where are the poles by the way for this, it is an imaginary line, here I do not need to shift the poles, it is already an imaginary line, plus i m and minus i m, we follow exactly the same procedure in 3D, G of r is this, exponential minus m r by 4 pi r, okay, this is the shielded electric potential or in field theory, m is like mass, okay. So when I do the field theory with massive particles, we will get this Green's function, and this will be the real space Green's function. Now, one more point I like to mention that this G of r is not dependent on the direction of r, right, it is a scalar function, it does not depend on which direction is r, this way, that way, that way, it is a same formula, so it is basically scalar G of r, okay, so we should also notice that, okay. Now let us go to slightly more complicated thing, wave equation, I can take a break here, any questions on what I did so far, it is all clear, no, I mean, this bit of mathematics, who does not follow the steps, anyone is not following it, or who is bored with this algebra, just raise your hand, we will edit this part of course, so your phase is anyway not being shown, okay, so you do not hesitate, are you bored with this or if not, I mean, it is good mathematics, no, I mean, this is something which should learn, but we will need it for this course, I assure you, and ENM, somebody pointed out, in aerodynamics, you have all this stuff in your course, how many of you are doing ENM, ENM right now, anyone know, okay, so let us go to the next step, wave equation, so this is same equation which I wrote in the last stuff, but we have a source, a source is that antenna, you know, that is the source, but now, this rho will be a delta function in time, so it is a pulse, imagine the electron just blipped, okay, it is not oscillating continuously now, it is just a blip, electron just jumped like that, okay, so at time t, so I am going to say that my source is delta x minus x prime, this is source at x equal to x prime, and in delta in time, at time t prime, it did this blip or you can think of, you would leave a stone in a pond, that will create waves, but there is a point source in the pond, at time of the impact, okay, so that is the source, which is going to generate waves which are travelling outward, it is not a continuous wave which is going on forever, this front will just move forward, okay. So, so what is the Green's function for this? So, now I need to use a four dimensional Green's function, right, e to the power i, k dot x minus i omega t, because this is a time variable sitting here as well, so this is what I need to use, my function, well, I mean, this is a general formula which I had written in the, in previous slides, so my Green's function will be function of k and omega, okay, so I plug G k omega, in fact, instead of phi, you just put G, G here, okay, and I assume t prime to be 0 and x prime to be 0, okay, that is for convenience, otherwise I have to carry this x prime t prime.

So, I plug this here and the right hand side is, we write as a delta function, this divided by dk d omega by 2 pi to the power d plus 1, okay, and assuming the completeness of basis and orthogonal basis, we get this formula, gk omega is k squared 1 divided by k squared minus omega squared, here k is a variable and omega is a variable, both are varying, in the Helmholtz equation, for me, k was variable, but omega was not there, it was k naught squared, but now my omega also is varying with, omega is also variable, okay, so that is why I am going to get g is a function of x and time, okay. So, first we will do the time integral or d omega integral, which will give me a function of time, so I am wanting to get gk of t, only thing is of course, my, for my slide I am not writing it, so things tend to move fast, but I am going slow and please remember this, it is coming below k squared minus omega squared and the signs are different, when it is positive, k squared omega is negative, because of this one, this minus omega squared and Laplacian gives you plus k, minus Laplacian gives you k squared, okay. So, this from the structure and you can guess that pole will get k equal to omega, you know, that kind of, so is going to, so that is what we get here. Now, let us do the, okay, this part is, okay, now, so I do this time integral, okay, G k tau, from here I am going to do d, so keep k as is, but we do integrate what omega, right. So, d omega by 2 pi, this comes here, e to power minus i omega t, it is not plus, it is minus and this integral needs to be done.

Now we have two poles, right, here omega is a variable, but k is not, so where are the, so this omega real, okay. So, the poles are at plus k and minus k. Now, minus omega t, so I need to close from the tau, well, depends on tau positive, tau negative, tau can positive and negative, both can occur. So, this algebra will leave it for you, okay. So, for tau positive, so I am going to shift the pole, okay, I will shift the pole downward actually in fact and do like this for tau positive.

So, tau positive I should close from the bottom, okay. So, similar contour integral, so that the semicircle is 0 and we get the contribution from both the poles and it is not e to the power i kt, sine kt means I am getting contribution from both the poles, okay. Now this is by choice, I am, in fact, I am looking for a solution which is of, these are, if there is a pulse here, then it should propagate both sides, okay. This by, in fact, I know the answer, what should it look like and that is where I am tweaking my poles is, I am doing the trick. If you do not do that way, then you will get probably a wave only going to the right front which is not sensible, I want wave to go both sides.

So, that is what I think this boundary condition is not imposed, well, we just choose boundary condition by poles by, sorry, impose the boundary condition by shifting the poles, okay. So, I get this by shifting the poles and this by g kt, okay, sine k tau by kau. Now I am going to do the k integral, right, I just integrate k part. So, this was omega part is done already. Now I am going to do the k part, do in two steps.

Of course, if you want you can do in one step all of it, but it is good idea to do separately that I think I find it easier to manage. Now, so this is g k tau e to power i k dot r. So, this is what I have written here, you know, so I am going from k space to r space, okay. Now g k tau is sine k tau by k. Now this formula is valid for any dimensions, 1D, 2D, 3D is all valid, okay.

So, let us look at 1D first, this same formula like that. In 1D this exponential will be e to power a cos kr plus i sine kr, correct. Now what happens to sine kr? So, sine kr by k, sine kr by k sine k tau, no, no odd, this k there sine sine kr by k is e 1, but sine k tau is odd. When I change sine from positive k to negative k, I will get exactly opposite. So, this part integral will go to 0, imaginary part, what survives only the real part, okay.

So, now the real part, in fact, if you see it, we can, anyway, let us look at doing two steps. I use the product formula. If I use the product formula, I will get sine k tau plus r plus sine k tau minus r. This, you know, right, sine cos is sine A plus B, sine A cos B, twice is sine A plus B plus sine A minus B, okay.

I just use this formula, trigonometry. I do not know how many of you remember this, but this is a trigonometry formula. So, it gives me that. Now, this integral has a, this is a homo problem, sine ky by k dk is a sine function, pi times sine function. That means sine function is, now this is doable by the tricks we have done in the past by shifting poles and well, you can do this part, struggle a bit with this, but we can show it. So, sine function is for y positive is 1 and y negative is minus 1.

This is sine function of y with y. So, y positive is 1 and y negative is minus 1. Theta function is y positive is 1 and y negative is 0. Sine is combination of the two. So, this is a nice function and we, we need this here.

So, I get sine, the k here and dk there. So, you will use combination of sines. So, I get sine tau plus r from here and some tau minus r here. Some of the two sine function. Now, you can just play around with this and you will find this to be half or minus, this is a 2 will come here.

So, minus t to t will get half and 0 otherwise. So, what we get here is like that, is a function of time. This is 0, this is t and minus t and these values half. This is my g of r. So, it makes sense.

You may not be able to guess half, but this is a front. So, I did some shaking up in the at the origin in 1D and this front is just going both sides. In 3D is like a, is a spherical front, but in 1D this is what we get. It is like a, it covers that region. It is not that we get delta function in both left and right. In 1D we get basically this 1D space is all covered, covered, covered.

It keeps moving like that in both sides. 1D is a, 1D, 2D, 3D they have very different properties. Like potential for 1D we saw that it goes linear in x for Poisson equation. That is like capacitor plates, electric field is constant and the potential is linear. In 2D it is log and 3D is 1 over r. And we can also argue why it should be so by various physical arguments, but this is the front which is covering in the region.

It just follows from mathematics as well, but I will not delve into physical arguments. So, this is 1D. The wave front propagating in both minus and positive direction and this happened because I chose the poles appropriately. I had shifted poles for both the poles in my omega integral.

Now, it is just algebra actually. I think I will not, you can fill up the steps, but this what we get. In three dimensions pretty similar algebra and this d cos theta, so here again I use the product rule and I get, in fact there is no 1 by k here. So, it is not a sine function anymore. What is this stuff? Integral dk, this one. So, what will this guy cos k times of positive number, this is a positive number, integral of cosine 0 to infinity cos k times a dk.

Well, I am going to infinity. You can make it minus infinity and half if you like. It is even function, so I can divide by 2. So, what is cosine integral, cosine k of a, k times a dk, a positive, it is 0 unless k equal to 0. If k is 0, then it is infinite, but if k is not 0, cosine will cancel now and this is just straight forward.

Positive and negative will just cancel. So, it is a delta k equal to 0. So, basically this integral gives you 0 and this will give us non-zero value when t equal to r, delta function, that is delta function. So, that is the next step, delta t minus r divided by 4 pi r. This is a pi coming from this dk integral, that pi cancels with that. So, this is the front, now it is a delta function front.

The front is not, so we get a delta function which is moving outward. So, there is a blast at origin, then that will create a front. Of course, inside there is no G if G is 0 inside and G is 0 outside, only at the front it is non-zero. This solution is different, but here is a proof.