Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, passive scalar turbulence, this is one of the important topics. So, we covered hydrodynamic turbulence in the past, but there are always some secondary fields along with velocity field. So, could be pollution right which is advected by the velocity field or could be some chemicals which are mixed by the velocity field. So, they come under scalar. So, this is scalar thing.

So, velocity field plus a scalar could be temperature, could be particle density, or it could also be magnetic field. So, then we have velocity field and magnetic field or it could be velocity field and a polymer. So, the secondary field can be complicated. So, I will just do one example, simple one with a passive scalar.

So, this is a quite important word. So, what does passive scalar mean? So, we write down the equation. So, I will write down the equation in the next slide. So, first is my equation of the velocity field everybody is familiar with that know. So, pressure gradient, euro gradient, this forcing for the velocity field.

It is a large scale forcing, you can think of some stirring and this equation for the scalar, theta could be particle density, ink concentration in a water or. And this kappa is a diffusion coefficient for the scalar, which is same dimension as nu which is L squared by t velocity times line. And f theta is a forcing to theta which is only function of, it can be function of u as well. But so, theta is driven by u, theta is this scalar is advected by the velocity field at least here or it could also affect f theta velocity field. But does theta respond, does theta affect u? So, you can see I do not see any u, I mean of course F u may have some theta, but I say well F u is not allowed to have any theta component.

So, F u is not a function of theta. So, this is not affected by theta. So, velocity field is not affected by theta that is why it is called passive. So, my theta is a passive scalar field. Now you might have heard this active fields nowadays, there are lot of people are an active field.

So, what is active? If theta reacts on this, it is possible for example buoyancy. So, temperature affects the velocity field, then theta is a active scalar ok. So, we are not going to work with active scalar which is more complicated, I am just going to tell you about passive scalar. Now if this is not affected by the velocity temperature or theta, then the properties is exactly same as what we did before. If you assume the F u react active only at large scale, then you get Kolmogorov spectrum right.

So, I did not want to touch, well I can just use the results of the earlier lectures, no need to do any work done. Now I need to worry about this ok. So, the two objectives, one is does kappa get renormalized ok or and what is the flux what is the flux for theta? The two objectives or I will have some spectrum for theta, theta mod k squared. This is in fact, this is divided by k will give me E theta of k. This is a spectrum of k right, I mean the way we define for velocity field, what is velocity spectrum? It is mod this squared by k, it is E u of k right.

So, I have this one. Now there can be some proportionality constant like Kolmogorov constant was there, similarly there is a constant for E theta ok. That constant can be obtained by field theory. So, both all this can be done, this is called Ubhokov coarsin constant ok, that is a constant ok. So, let me just sketch it, I am not going to do in detail derivation, but I will tell you how to renormalize kappa.

And idea is exactly similar to what we did for hydro, not much difference ok. So, first let us do RG of kappa. So this is equal, I mean there is a force, but force is activated only large scale. We assume the force F theta is activated large scale and is not activated k in the intermediate range, well I mean intermediate range, where I want to compute the renormalized kappa. So, this is gone and what is that non-linear term, which was U dot grad theta right.

So, grad will give me k, the U is here and theta is here, convolution. So q and p and q plus p is equal to k, straight forward ok. So, let us draw the Feynman diagram ok. I hope you will learn, I mean at least that part you will have courage to write the diagrams and you should be able to calculate yourself. So vortex will be k, if you do real calculation there will be some tensile stuff and so on, but we just have this vortex is k, it is a right now is a vector which does not look good, but I assure you that if I do it properly then I can get a vortex ok.

And I have velocity field U and theta, theta p and this velocity q ok. So, I am going to work for theta given U. So, given velocity field, so U do not touch, I am not going to I assume that U is given to me and I am going to correct kappa by expanding theta ok. So here idea is to expand theta by Green's function ok, this is G of p of theta. Now what will it give me? So, this is U here.

So what will be the legs for coming out from this vortex? This vortex also will be p in fact right, I mean there will be i p, this i k. What would be come out of theta? So, if I expand Green's function what will that be? So, this G inverse, so theta p is Green's function of p and this non-linear term which will be integral ok, instead of p I write dr p dot U of r and theta of s and what is r and s? r plus s equal to p right. I mean I just rewrite it, but we change the variables. I should not write pq because that pq is already taken. These are dummy variables r and s.

So I get r here. So U will come combine with U, U of q here and U of r here, U of r and this will be theta of s ok. Now what should be r in terms of q? r should be minus q right, I have to get non-zero value. If r is minus q then what is s here? What is s? p plus q which is k. So this is theta of k. So this part of the diagram is going to give me correction to kappa right.

That is what we did before. So this loop diagram ok, this is whatever number I get let us say integral I 1 and this integral I write I would write this as delta kappa minus delta kappa k squared. Actually this should be plus I am sorry this should be plus right. So that is plus. So minus delta kappa k squared and this I can take it to the left hand side to kappa plus delta kappa k squared ok.

So this is a correction to kappa. I skipped the step of postgraining and so on, but that part you know already I mean I am sure you are very used to that part ok. So this is a correction from Feynman diagram and if you do the algebra all of it properly you can get a correction. And this for isotropic homogeneous turbulence kappa of k also goes as k to power minus four third like viscosity. There will be different coefficients, but that is the stuff ok.

So this is for renormalizing kappa. What about energy flux per theta? So let us see this flux computation ok. So I can write down equation for the energy. So d by d theta of k dt equal to well let us put here plus kappa k squared theta of k and right hand side was k dot U p theta of q right integral minus i dp like that right. So what do I do? I should multiply by theta star k standard cosides and add complex conjugate.

So I get d by dt of mod theta k squared ok. So here I get dissipation which is two kappa k squared theta k squared right. So this is positive positive so they get added up. The Schrodinger equation that went away because of i because you have to you have to be careful. And here minus i integral dp k dot U p theta q theta star of k ok and add complex conjugate.

So we will find that this will be plus i and I will get real part of that. So I will get imaginary part of this ok. So if you do the algebra right I will get this. So this is the term which is increasing energy. So this T of k theta non-linear transfer to k, and wave number k by theta theta interaction.

So what should be the flux non-linear flux for a sphere of radius k? I have T of k so by definition 0 to k T k prime T theta k prime right. This that is the formula which I used very often. So if I do the integral I get minus dk prime dp there is a dp sitting here I am just going to substitute it here k dot U p theta q theta star of k. So it is a three wave interaction. So U of let us put U of p theta of q I am going to put a dash theta is going to put a dash and then theta star of k.

U is given to me U is I do not do anything on U but I am going to expand to zeroth order this will be 0 ok. These are random fields so they will give 0. I need to go to the first order next order. So I will get Green's function for I am expanding theta k ok or let us say theta k ok. There is a star there but do not worry about that part.

There is a vortex here which I had ignored. So what is going to come out of from the right hand side? When I expand theta what will I get? One U and one theta right. So one U will combine with this U, U p will combine with from this side I will get U of minus p and theta of q will combine with theta minus q. And there is a Green's function of k ok. Of course k equal to there is a complex conjugate that is why k equal to p plus q it will all comes out.

So this integral involves integral c of p which is velocity correlation function and temperature or scalar correlation function q and Green's function of k and I got dp and dk ok. So you do this complex integration and you will get the flux. For five third regime which is Kolmogorov regime this will be a constant which is epsilon theta ok. So all this algebra will give you flux and given this we can also derive a spectrum for temperature.

I may have one free slide. So I can write down the expression for theta, e theta of k. So it turns out to be if I get it right epsilon theta is linear in epsilon theta epsilon U which is a dissipation rate of velocity field minus one third k minus five third and k Ubhokov coarsen relation. This is a constant which is non-dimensional constant ok. So this is a spectrum for e theta and this number can be computed by field theory ok. So and this so field theory tools can be applied to secondary fields.

This is for scalar but we can do it for MHD as well. MHD is more complicated. For convection it is not very well done I would say but because it is anisotropic. Once you bring in anisotropic field theory it gets more complicated. There are works on that but it does get more complicated but it can be done and that is the problem.

I mean I think if you guys are interested in challenge then apply to thermal convection ok. Good field theory is lacking in this. I did lot of work on turbulent MHD turbulence ok. Still there is lot to be done. So these are the areas which are not touched by their hard problems. Thank you.