**Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives**

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**Week - 12**

**Lecture – 70**

So, these are non-linear Schrodinger equation. So, you have seen this before I am setting h bar as 1. So, i h bar d by dt psi is h bar square Laplace n square by 2 m, but I am set m is 1 and h bar is 1. So, this is the linear part, but we have non-linear term here non-linear mod psi square psi, a psi is a complex function. So, that is why we need to I mean in the real space itself my function is complex. So, what is the dispersion relation for the linear wave or linear system omega k is k square no.

So, this i d by dt will give us minus omega right and that will give me the omega k is k square right as a straightforward yes or no. So, you should put e to the power i k dot r minus omega t. If I substitute it here then I get minus i omega and there is i already sitting here. So, i square will give you minus 1.

So, this becomes omega and minus Laplacian will give us k square right. So, the omega equal to k square is a dispersion relation ok. It is a dispersive wave because omega is not linear in k. So, I think I hope everybody understands that omega is the linear function of k, then all the waves move with constant speed. So, if I take a packet, we take a packet, Gaussian packet then this will not spread.

So, you should think that each component moves with constant speed, same speed not constant, same speed. So, this packet is non dispersive that means the packet will the width of the packet will not change with time ok. But we know in quantum mechanics one the width spreads because this omega is k square ok. So, this is called dispersive waves, it disperses. The conservation laws one is mod psi square dx right.

This is everybody knows right I mean one of this is a probability conservation, but we can derive this by Noether's theorem. I did this in the class by gauge, gauge invariance local gauge invariance. Second is conservation of total energy which is not this energy is different than h bar omega times mod psi square mod psi k square ok. So, this is for the linear part well motivated by linear part ok I will say that. So, this is the total energy which is mod psi square plus lambda psi 4 which comes from which symmetry property? What time translation and space translation gives you linear momentum which is the particle current ok.

Now in Fourier space we can rewrite that equation. So, this part psi cube well it is not psi cube is mod psi square psi. So, that becomes a convolution. So, we have psi k 1 psi k 2 and psi star k. So, psi star k is coming from one of the psi's here psi star k 3 all right.

So, that this one complex conjugate in the right itself right side. Now I can write down equation of the energy. So, what do I do? I multiply by psi star k and and then add complex conjugate whether I have taken the i from left to the 2 here ok. So, I just shifted i is convenient for me. So, this i is coming from i h bar dt.

So, i has been put there and add complex conjugate you will get. So, this linear part will cancel know complex conjugate if I add I will get cancellations. So, we get equation for the energy if I write that and please note that k 1 plus k 2 minus k 3 is k because of this star is it clear to everyone this should be minus k 3 or k plus k 3 is k 1 plus k 2. So, 2 in the left 2 in the right and the one with the star is going to be. So, if I multiply by psi star I will get from both sides I will get psi star k.

So, these two wave numbers are added and these two wave numbers they are equal to the wave numbers for unstarred wave function. Now, particle density is mod psi k square and energy is omega k mod psi k square and this is the equation which I can derive easily by multiplying by psi k star and adding complex conjugate. So, the right hand side has this quartic term four terms like water waves and this is the energy transfer well particle transfer function I will should not call it energy transfer because it is a it changes the particle density in a different wave numbers. Now, we can compute this T n and that is of importance. So, if I compute T n I just want to do this average first ok.

So, I will get like this now I had forgot one part this is imaginary part know this because of this I this I sitting here and you might add a complex conjugate I get a plus psi. So, when I add I get a imaginary part here ok. So, that imaginary part gives you something very interesting. So, if I do this that is what I was water wave I was bit confused if I do the to zeroth order. So, this is zeroth order.

So, quartic function of function of four psi's will be product of a sum of products of two psi's. So, I get this plus different combination how many combinations two more a b c d a c b d a d b c and these are real right here k 2 is minus k 1 when I do the integral this is going to be energy and this also energy. So, they are real functions imaginary part of real function is 0. So, zeroth order I get 0 ok. So, T n is 0 to zeroth order of course, I am assuming the psi is Gaussian which is an assumption, but that is the starting point for most of the turbulence calculations ok.

Without that I mean Wilson theory also we assumed psi to be Gaussian for KPZ for our hydrodynamic turbulence all the time we assume psi to be Gaussian ok. So, we need to go to the next order. So, what do I do for the next order? I need to expand one of them using Green's function. So, I have this four psi's and I can take one of them and expand using Green's function. So, let us write down this stuff.

So, psi k 1 psi k 2 psi k 3 and psi star k ok. Let me just check which one I am doing is a Green's function ok. I am doing psi star no psi 1 psi k 1 psi k 1. So, I expand this is a Green's function psi k 1. So, standard stuff know we have the linear problem what was linear thing? Theta is psi equal to minus i k psi 1 omega k sorry not k omega k psi k plus non-linear term which is minus i lambda integral and three psi's.

So, I am going to write psi k 1 here my wave number of interest is k 1 right this one I am expanding. So, this will be omega k 1 psi k 1 and here I will get d S 1 d S 2 d S 3 I write that three wave numbers and I am going to write as psi S 1 psi S 2 psi star S 3 and what is the condition for wave numbers k 1 equal to S 1 plus S 2 minus S 3 correct. So, these are condition for the wave numbers. So, I am expanding this it does a Green's function. So, this is what I have done Green's function and I got 3 s 3 psi's here and 3 psi's in the left psi k 2 psi k 3 and psi star k ok.

So, now, three guys will come from the right three guys will come from the left there are many possibilities of putting them together ok. So, one possibility is that I put this is acting a psi star S 3 or you can put psi star S psi S 2 this complex conjugate well I mean I can manage it, but psi S 2 and this one is going as psi star S 3 and this is psi S 1. So, basically I am going to put them together and there will be a wave number must match ok. So, that. So, we will get three correlation functions or three particle function particle density functions right correct.

I mean this is particle density N k 2 here I get N k 3 and here I get N k correct. So, these three N k's will come and that is what I got it here I mean this is just algebra I get N 2 N 3 N k and by the way this is a time integral d t prime and that gives you that this function should be delta function this I did in the last class. So, you should see the video of the last class and the delta function comes because of this d t prime integral t going to infinity t going to large value ok. So, this is for one of the terms, but we have more Feynman diagrams. What are the other Feynman diagrams? How many Feynman diagrams will I have? So, these I am doing only the psi k 1 I should have a Feynman diagram from psi k 2 like this like this and like this for each leg I can have Green's function expansion.

So, there will be four terms correct other term is Feynman diagram here two three fourth one is Feynman diagram below like that. So, these are the four diagrams and so we got four of these terms and the sign I checked this is what we get and ok. So, this is a lambda square why lambda square because one vertex is in the left and one vertex in the right lambda lambda ok. So, these T N k this integral can be done mostly numerically I would think people have done analytically as well and we can well this is rewritten as N 1 N 2 N 3 N 4 n k and divide by this. So, your choice you can write like this or like write like that.

So, you see N k and N 3 are competing with complex conjugates remember in the first slide and these are coming with without complex conjugate and this is a form ok. So, this will give you the non-linear T N non-linear particle transfer function. And using this I can compute the flux ok. So, Sachin you have to do all this work. So, summary is that we can evolve particle density and energy density function.

So, this d by dt it may not be in equilibrium. So, flux may not be constant, but well that is a starting point for lot of calculations analytical calculations and derive energy spectrum etc ok. So, I will end this part here it is ok everything is clear. So, basically we have quartic interaction this is an example where we have 4 waves interactions it is like Phi-4 theory definitely. So, but we can I mean this derivation is not complete ok. So, I will do the last topic.