Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

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Lecture – 69

Alright. So, in the last class we were started discussion on weak turbulence and I kind of took a toy problem where the fluctuations were moving with a certain speed. In fact, so some of weak turbulence involves a linear term which corresponds to Hamiltonian H naught of quantum mechanics. So, the linear term was a much stronger compared to fluctuations ok. So, the waves you know linear term is driving the wave, but because of non-linear term which is weak these waves will get modulated ok. They will generate more waves, but the effects are small.

So, we showed that the spectrum for wave turbulence with constant u naught is minus 3 half ok, under some limit, but there is another theory which is k perp minus 2. So, I showed it in the last class. So, I am going to do some more examples ok. And so we will not want to discuss the physics of this example which are quite complicated, but we are just going to treat it as a mathematical problem ok.

So, I am not expecting you to become an expert of rotating turbulence or even get a full gist of it, but I will show you how to derive the spectrum for rotating turbulence using weak turbulence theory. So, first thing is what is rotating turbulence? We have a fluid you can think of a fluid in a container which is like 3D box which is rotating very fast along z direction ok. So, this omega is very fast. So, there will be a coriolis force, it turns out the centrifugal force can be absorbed in pressure. So, we have coriolis force and there is equation in fact Navier- Stokes equation and we can write down the equation of course, but the turbulence is highly anisotropic, it tends to be two dimensional like earth you know.

So, because of rotation we are having a two dimensional. So, fluctuation parallel to the rotation is suppressed and there is a fluctuation perpendicular to the rotation and so we have ok. So, now, I am not going to tell you what this u plus and e minus are these are helical modes ok. They satisfy with a non-linear term this simple relation. So, two waves one plus wave which is i omega and minus wave which is minus i omega.

So, they are linear waves right and they have the speeds well not speed actually. In fact, you can look at here k. So, by the way omega is a rotation. In fact, I should really use capital omega. So, let us use capital omega.

So, this is a rotation frequency that omega ok. So, we can write down. So, I will not derive it actually in fact, is small omega only. So, this is the frequency. So, this was small omega which is the frequency which is connected with capital omega is 2 omega cos zeta.

What is zeta? So, this z axis and a wave number k this vector. So, this angle is zeta. So, omega is not linear in k. So, this is not a wave I mean this is not like a non-dispersal waves this dispersive waves omega is not a linear in k. So, it is not like a normal waves where omega is proportional to k.

So, this is property of this system. So, omega is proportional to 2 omega and this is this k dependence which comes by zeta. So, cos zeta will be k parallel. So, parallel along z direction k parallel by k. So, this is my omega.

Now, I play the same trick. So, I need to write down this energy transfer. We need to write down the flux formula. So, some of you are absent in the last class. So, we need to write down how much energy is going across scales.

So, I did in the class this energy transfer from p to k. This is the q is a mediator and this pretty complex formula which involves it is a for a given triad. So, this is a triad k no sorry k prime t q. So, we had seen this before and their angles. So, in front of k is alpha, in front of p is beta, in front of q is gamma.

So, these are these are formula it has been derived before. You can look at my book where I have all these derivations. So, this is the formula which is connecting this interacting modes. What is S p and S k? They take value plus or minus 1. So, this is the helical modes.

So, plus helical plus mode is circularly polarized clockwise and minus will be anticlockwise. So, these are circularly polarized waves in fact, they are helical modes. So, we have this non-linear interactions here. So, this we have studied in well briefly in our course. So, this is non-linear interaction and that leads to energy transfers across scales.

So, I can compute the flux and the flux formula is given this energy transfer I just integrate. So, flux is energy coming out of sphere of radius k. So, this given by that. Now, I will not I discussed this slightly in the last class, but this is that energy formula.

So, I integrate. So, giver is inside the sphere and receiver is outside the sphere, k prime is the receiver. So, this is a k or k prime, k and k prime are same well k prime equal to minus

k. So, now since I said this is two dimensional quasi 2D, flow tends to become quasi two dimensional. So, I am going to divide this integral dk which is three dimensional integral into k perp and k parallel. So, this is dx dk x dk y dk z which is k x k y are here and k parallel which is k z is that.

Similarly, you do it for dp d this one ok. Now, so there is one more term which I missed. So, there is a delta function delta omega minus omega k minus omega p minus omega q ok. So, this I derived in the in the last class. So, this is a coming from the time dependence part ok.

So, this is this part. Now, I am going to just count the dimension this integral is reasonably hard to do, but we will just count the dimensions. So, let us see. So, this omega ok there is a so basically we get a delta function because of the time dependence part and that delta function basically has dimension of frequency and frequency is coming from here 2 omega is here and k parallel by k perp. The frequency comes below and if you remember for hydrodynamic turbulence the frequency was 1 over nu k squared comes below in the denominator. So, same thing here, but here I do not have viscosity.

In fact, I have this frequency u i omega t and that frequency is coming here. Now, k parallel will cancel with this k parallel. Well, we get a log stuff, but that is non-dimensional that basically dimensional it cancels. So, how many what is the dimension I am going to get? I get here k squared ok. So, and it is a so this k parallel will cancel and there is a k squared here there is a k squared it is a 2D.

So, this k squared will be k perp squared and I get from here k perp and k perp ok. So, there is a k parallel is cancelling here and I get k perp squared and I get here k perp squared and d perp will also give k perp square and p parallel is this k parallel and this k going up. So, k 7 this something is odd. So, this k parallel is here and I am over counting this k perp.

So, this energy. So, what we got what we get here is this C k which is modal energy right. So, if you look at the derivation I did that one C we will get two of the correlation functions C p C k C p. Now, I need to convert that to 1D spectrum. So, I convert this to 1D spectrum which is going to be E k for 2D is going to be k perp right 2D will k perp and this is going to be E p divided by k perp. So, these two k perp will cancel with this two k perp ok and this is anisotropic.

So, I am going to write this k perp k parallel and this is p perp p parallel ok. So, now, we get basically. So, this k is a quasi 2D. So, k is essentially k perp know.

So, I can replace these by k perp. So, I get k perp to the power 5 and 1 k parallel here and this 2 energy squared and divided by omega ok. So, now, from this I can derive the formula for energy spectrum which is given here is k perp to the power minus phi half and k parallel to the power minus half this guy. So, that is the prediction. This is by Galtier 2003 and we did simulation and we find a similar spectrum in our simulations ok. So, this for very high rotation we will not get this for low rotation because high rotation has the corollis force will be too dominant compared to fluctuation and that is where we will get this formula ok.

So, this is a simple illustration. In fact, the derivation is not too long. If you just follow this procedure we can get this derivation I mean we can derive the spectrum. Of course, you will be careful with this k perp k parallel for for this quasi 2D ok. Now, let us do another one which is called inertial gravity waves. Now, in earth actually in fact, in earth there is a gravitation field G acting this way.

So, it turns out dense fluid is below and lighter fluid is above ok. So, if there is no motion then it is going to be just a stacked up flow right, density decreasing with height. Now, you put some perturbation like the wind is blowing and that will give some perturbation and this creates something called internal gravity waves. Internal gravity waves is very similar to water waves in ocean, but ocean water surface is there is air and water right there is a two different fluids and the interface is water waves, but in atmosphere we do not have a interface right it is all air. So, there these waves are within the bulk ok.

So, this basically the fluid is going up and down. This is along wave number is along that direction k, but of course, wave number can also be along some arbitrary direction k ok. This is allowed and then we will have to take the components ok. So, internal gravity wave is basically like water waves, but within within the bulk. In ocean also we find these waves within the ocean.

Now, it turns out we can derive all this. So, the dispersion relation for internal gravity wave is omega is N sin zeta. Now, the zeta is the same angle zeta and this is k and this is omega this is the no sorry gravity iteration z ok. So, what is sin zeta? Sin zeta will be k per pi k. So, for rotation we had k parallel by k and here I got k per by k and the derivation is very similar. We can just repeat that calculation I am just writing the same calculation.

So, here I replace by k per by k and you can do the algebra and you find this formula ok. Now, we can invert it and you can get the spectrum like this k per to the minus 2 k parallel minus 1 ok. So, this is the prediction from weak turbulence theory ok. So, this I have not dug in to look for some numerical verification somebody would have done it.

I do not have a reference right now. Now, slightly more complicated thing is surface gravity waves. Like in the ocean we have on the surface we have surface waves right ocean gravity waves. Now, this is definitely quite complicated. The reason is we have so by the way just to give you a gist of the thing. So, we have two fluid one is air and water and we have two different pressure above and below.

So, to derive this surface gravity wave is pretty complicated, but so it has been derived of course, in it is a textbook material we can find in a standard fluid mechanics book you will find the derivation which is reasonably complicated, but I am not going to get into the derivation of surface gravity wave but with all that algebra it turns out. So, this is a paper which I am trying to refer to Zakharov. I told in the last class that he is a person who started this whole field of wave turbulence. So, Zakharov gave this equation where the non-linearity in real space is phi cube.

So, the Hamiltonian is phi 4. So, in fact it is phi 4 theory time dependent phi 4 theory. So, our interaction is phi cube. So, d by dt plus I so we have some motion so I need to write d by dx of phi times the velocity of the wave and right hand side is phi cube. But it turns out the frequency the omega k is not linear in k or surface gravity waves.

The omega k is square root g k. This is a property of this is a dispersion relation square root k and the non-linearity phi cube is going to be convolution with three phi's phi k 1 phi k 2 phi k 3. Now, so far for hydrodynamics we have u dot grad u. So, in Feynman diagrams we write this u and u bit easier to handle but now I have three of them. So, we can but it is possible to do this algebra is pretty complicated algebra but I try to simplify it. So, now here this is something which you should keep in mind in wave turbulence that we have something called particle density.

In quantum mechanics we have wave function psi. So, mod psi square is a density of the particles right. You understand? So, it is like a density of the wave function. So, it is proportional to particle density. Yes. Is k of course that that has to be the case k 1 plus k 2 plus k 3 is equal to k always the solution has this property.

So, remember this is a density and what is energy? If the frequency is omega for this for the mode then h bar omega mod psi square right. For photons each photon has energy h bar omega and density of the photon particle density. So, this is N k. So, this is what I write is particle density N k and this is proportional to mod phi square. But what is the energy? So, energy I need to multiply by this well in quantum mechanics I write h bar omega but here I write just omega.

So, we can dimensionally we can get that energy it is tough but you just write omega without getting into complications. So, omega k the but origin is that quantum mechanics formula omega k times particle density. Now, we have for linear problem we have two conservation laws N k as well as energy both are conserved for linear problem. You put non-linearity then of course there are variations, but we have this both these conserved quantities. So, now I need to have two spectrum one for particle density one for energy.

So, now let us look at. So, it is easier to derive formula for N k evolution for N k. So, if I derive it then I get d by dt of N k which is d by dt of phi k multiplied by phi star k and at complex conjugate. So, what is going to happen in this process the linear part will go away but the non-linear part will remain and the non-linear part will give you three phi's which was just before I am multiplying by the phi star k and real part of this right. I mean this is we have done this many many times in this course. So, this is the energy transfer well actually is not energy transfer is a particle density transfer 2 k.

So, particle density can change because of this term. So, this is basically d n k by dt fine, but we have quartic interaction 4 waves are interacting right. So, we write this 4 waves like that like this phi k 1 phi k 2 phi k 3 phi k 4. Now, what is this value? So, we can work it out under linear approximation. So, it turns out so if I assume Gaussian approximation so which is assumed all the time we did this in the course for a statistical field theory well not course in this course for statistical fields we assume the phi to be Gaussian and for Gaussian fields what do you assume? So, I am just basically doing the average. So, what will happen to this? Product of 4 phi's will be sum of products of 2 phi's correct.

So, I have it here ok. So, well I can do the flux by integrating this, but the idea is I think is the next line. So, I will get this. So, I get this delta's ok. So, I got this ok.

So, here so how do I do this? Now to zeroth order. So, we need to keep this mind to zeroth order. So, I skip that step to zeroth order I will get 2 energies right. In fact, I have done in my other example of a Schrodinger equation, but this will involve 2 correlation functions right product. So, phi k 1 phi k 2 average phi k 3 phi k average star ok. So, this will give you that k 2 should be minus k 1 and this is real number this is real number ok.

So, this is going to give us omega k coming from there phi k. So, this is using Green's function.

In fact, so the real part ok. The idea is the following. So, to get this formula we need the following ok. So, I need to expand it. So, I write this 4 functions phi k 1 phi k 2 phi k 3 and phi star k ok. Now, we expand using Green's function. So, I do Green's function and this is going to give me another 3 phi's right and 3 phi's will come from this side and they will merge together ok.

So, these 3 correlation functions are here c k 1 c k 2 and c k ok and this delta omega is the frequency which is coming from omega this will go to be omega k 1 plus omega k 2 plus omega k 3 minus omega k. So, this will be minus omega k ok. So, that is delta omega. So, these 3 correlation functions are coming from this first order perturbation ok. So, what I was trying to get to which is not clear to me if I just compute to first this order then I should get some value which is unfortunately is not what is here ok.

So, we can get this T will have 3 correlations and 1 delta omega this following the standard procedure ok. So, I can compute the flux, flux is integral of T k from 0 to k this this stuff. So, how many k primes I got? I got 1 d k prime and 3 d k's and I have got this correlation functions. Now, I did in the last class. So, d k is d k times k to the power d minus 1 for any dimension right this by definition isotropic I do k d minus 1 d k a c k is e k or so whether this correlation is particle density phi square is particle density is not energy density.

So, N k divided by k to power d minus 1 ok. So, this k to power d minus 1 cancel with this k d power minus 1 ok. So, that is what happens. So, this 3 will be N k cube this one. Now, I got 1 k cube here and 1 k p cube here right this is sitting there and this delta omega will cancel with this omega I need to go back. So, this is for the energy this T n is for particle density and this T is for the energy.

So, I need to multiply omega and this omega k is cancelling with this delta omega delta has 1 by omega. So, this is cancelling now here I am going to get 3 k s ok. So, one delta function will disappear because of this condition delta k 1 plus k 2 plus k 3 minus k. So, one of them will go away.

So, I get k cube here because of this d k 1 d k 2 d k 3. So, I will get k cube from from these guys. So, I get k cube k cube k cube. So, k to power 9 ok. So, N k is k minus 3 epsilon one-third ok. So, that is what N k this epsilon is a energy transfer rate or energy cascade for the energy ok.

Now, we can do some for energy I need to multiply by omega k right. So, E k is N k multiplied by omega k. So, that will be this. So, when I multiply this. So, there is a k minus 3 n square root of k that gives you k minus phi half square root g and epsilon one-third ok. So, these energy spectrum for water waves ok and here we assume that there is a constant cascade of energy that is an assumption and that will work if you have enough range if your dissipation is weak and you have enough energy to push the waves.

Now, the story does not end here this is for the energy spectrum which is in the forward direction it goes from large scale to small scale like Kolmogorov 3D energy goes from large scale to small scale and the spectrum is k minus phi half, but it turns out in weak turbulence if you have two conservation laws for non-linear term if it is turned off then two conservation we have one energy conservation and particle density conservation both are conserved ok. So, that is a additional thing in weak turbulence. The one with higher k. So, E k has E k is N k times omega k which is square root g k.

So, I am just giving you a rule of thumb ok. I mean the proof you can not it is not a proof, but the more detailed argument is given in Nazarenko's book, but you typically find that one with higher k which will have higher dissipation at large wave numbers. So, higher k guys will go forward direction. So, E k is forward direction, but what happens to N k which is conserved quantity that will have inverse cascade which has lower number of k. So, look this is some k to power minus alpha.

So, it has lower wave k dependence compared to E k. So, particle density N k has a inverse cascade ok. It happens in 2D turbulence. 2D turbulence enstrophy goes forward which is E k times instead of E k times k squared k vorticity squared that is forward, but energy E k is backward in 2D. So, similar motivation leads you to a conjecture that N k has inverse cascade. So, that is inverse cascade of N k and we can derive in the following the same set of logic.

So, particle flux I am going to assume particle flux to be constant and then similar arguments, but here I am not multiplying omega k. So, earlier slide was energy spectrum energy flux. Here I have particle flux and do the dimensional counting and you find the zeta it is that I mean ok. We can just do it carefully we will get this stuff and the particle number is this pretty complicated looking formula 17 by 6 is around minus 3, 18 is 3.

So, that is a formula ok. So, this pretty complicated set of arguments, but you can get this spectrum by applying this logic. So, we assume that there is a constant flux which is an assumption which will work under the assumption that there is enough non-linearity, weak, but enough non-linearity and the dissipation is at very distance scale. So, you should get dissipation should be far away ok.

So, basically for this classical problem so, we have waves phi k. So, phi k square ok. So, this is the spectrum. For quantum problems we will have particle density for example, for photons we have photons of different frequencies and they are different distribution know. So, in fact, the Wien's distribution right I mean what is Wien's distribution it is like that or blackbody radiation these are particle densities ok. So, photons of different frequency they have different number density. So, for water waves of course, look this is the part this tells you what the part this amplitude square is dropping as k minus 17 by 6.

So, high frequency waves are less common. So, the frequency low frequency waves are more common that is what it means. They have more energy that means their density is more, but indeed I mean this is not quantized. So, we cannot really say particle density.

Particle density makes sense basically in quantum mechanics is true ok. Any other question? Ok. So, this is end of this topic ok. Once you have particle density you can get E k and so let us summarize this. So, you can apply weak turbulence theory to many applications. I am going to do one more example nonlinear Schrodinger equation and then a possible inverse cascade right.

So, if you have two conserved quantities then there will be a possible inverse cascade and end. Thank you.