Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 68

Alright, so time short and I well it was my ambitious objective to give you overview of various aspects of equilibrium and non equilibrium. So, non equilibrium we started bit late maybe one third of the class. So, this is important topic which is not difficult this is wave turbulence theory. In fact, it is much easier than what we did not so far, but it is fascinating, it is also interesting we can predict a lot of things with this weak turbulence theory ok. Weak turbulence I am going to tell you what is weak turbulence in the next slide and there are lot of material, but I would say that if you want to look at it well my lecture notes should be enough for this course because you do not have much time anyway to look at reference, but if you want to work on this topic then you can look at them ok. So, this book by Nazarenko is one of the famous books on wave turbulence ok.

So, I am just going to do formalism using several examples ok. So, this first PPT I am going to use a simple example which is not really a great example, but it is a simple example where I can do these calculations. Ok. So, the one main idea is that we have a linear term and a non-linear term ok.

Here V is a constant so, which is a linear term for this PDE partial differential equation like other equations we studied and phi is my field phi is a dependent variable ok phi is function of x and t. So, this is a linear term and this is a non-linear term. If you like you can put a factor lambda in front or we do not need because V is a parameter for me. So, I can say that this linear term is much bigger than non-linear term. Is that idea clear? So, I assume that V d by dx this phi is much bigger than dx phi squared by 2 and this kind of things you done in your earlier course where we did not deal with non-linear equations, but you done this is a perturbation theory in quantum mechanics right.

We assume that the H naught which is a Hamiltonian for the free thing is large compared to H1 and H1 is a perturbation right. In fact, this whole theory was developed by there few leading names the one person who is given lot of credit is Zakharov. So, these are from Russian school ok. So, USSR school that time earlier times and by lot of things happened in USSR ok. When the well Russia because lot of people left Zakharov went to US after breakup of USSR, but like Landau, Kolmogorov all these big names are from that Soviet school.

Zakharov is one such person. So, his idea was to use a quantum formalism and apply to things like waves. So waves you know so if there is no non-linear term this kind of term then wave will go unperturbed just keep going. It may damp, but it will keep going, but in a putting the non-linear term the wave gets in fact the wave can generate more waves and there is cascade of energy and then what happens to turbulence properties. In Navier-Stokes equation when I did turbulence where in fact there is no linear term for Navier-Stokes right.

The linear term is viscous term which is small. For Navier-Stokes our linear term is small compared to the non-linear term, but now we are changing the role. We say linear term is big and non-linear term is small and that makes life this wave turbulence life is simpler. In some sense. So, please keep in mind that it is a wave turbulence so their waves are essentially unperturbed well waves are primarily unperturbed there is a small perturbation or these waves in ocean.

They are essentially waves which is travelling, but the amplitude of the wave or their which keeps damping or it generates some new waves, but this generation process is weak. So, as I said the non-linear term is assumed to be weak compared to the linear term. So, non-linear term does not affect the linear term ok. So, as a result we do not denormalize V, the V will remain unchanged ok. So, RG is not required ok.

So, it is a great thing one major task of field theory is not required. So, if you recall for Navier-Stokes equation we did two things, one was renormalizing the viscosity, a second compute the Kolmogorov constant using energy flux equation. Here I need only energy flux equation, I do not need the renormalization ok. So, that is my way of looking at it ok. So, I am trying to make it simplified and also connect with what we have done in the course.

But if you look at waves or look at Schrodinger equation energy is conserved know Schrodinger equation is well it is linear equation in fact, and there is a conservation of energy conservation of mod psi squared ok. But so, there is no in fact, it turns out there will be back and forth of energy coming from in space ok. But I want to create a cascade so, energy going from large scale to small scale then I need something ok. In Schrodinger equation typically well now there are systems which I am going to cover in the last lecture non-linear Schrodinger equation which Sachin should come well he is also missing. So, there is a cascade in quantum turbulence, but in if you look at Schrodinger equation there are things will go to small scale, but they will come back it is called revival.

So, this is in Hamiltonian system we cannot decay or transfer of energy from one scale to next scale, digital balance and so on will come into play. But in statistical physics we have many many degrees of freedom then it is possible. So, I think these are open questions ok there are people who are working on them, but for our thing I would say that these are Hamiltonian systems this will conserve energy this equation which I am I have written on top this will conserve energy. In fact, these are so, Galilean invariant if you treat phi as a velocity field ok in fact, we can be eliminated that is why I said the toy problem I am giving I am doing it right now to illustrate some idea it is not a great example ok. We need more complicated example which I will mention how to make it complicated, but for illustration this is ok.

So, the idea is if I want to create a cascade which we want for turbulence then I need some dissipation. So, we need small dissipation ok. Now, dissipation what form of dissipation is not stated explicitly it could be viscous term or it could be term I need to dissipate energy ok. So, this dissipation is not specified in the derivation I am going to show you in when I derive it that way dissipation is coming, but you need this is normally hidden, but we need to know that there is a cascade and there is a dissipation well cascade is required dissipation ok. Now so, it is a conservative system.

So, this in fact, you can prove that this is conserved ok that is conserved stuff and so, cascades in fact, I am repeating this one. So, this part I just said a small amount of dissipation to set up a cascade and most of the derivation which is done in weak turbulence theory assumes that cascade the energy flux is constant and using this constraint we can derive energy spectrum ok. So, that is a typical framework that is how we do it. So, and basically energy spectrum on scaling relations is obtained and I am going to show you how to do it is not very complicated ok. So, I am going to use this example which is not the greatest example I told you, but it can help me to illustrate the ideas ok.

So, let us calculate. So, first is this dispersion relation ok, this is called dispersion relation you heard about this know frequency and wave number relation. How do you derive it? So, you write phi as phi k well Fourier transform e to the power i k x minus omega t right in 3D is going to be k dot x. Substitute it here this may give you i minus omega and this will give you v i k x right. So, note the sign change there is a minus sign there this minus sign there ok.

So, frequency and k have different signs. So, this should be you know the non-linear term linear term this is 0. So, that will give you omega equal to k x times v, v has the unit of velocity ok. So, this is dispersion relation. The first thing is you should know that is dispersion relation, it is a linear part.

And then like we did before I am going to write down the expansion. Basically perturb the non-linear term I am going to do Taylor I will basically write this convolution. So, write down the equation Fourier space ok in k t space I am not going to do k omega lot of people do k omega, but k t is convenient for me. So, this is equation in k t space right. So, d by dt I will keep it there ok and this gradient d by dx will give minus i k x and phi square will give me convolution all right.

Now, we need to write down equation for the energy. Remember I am not interested in renormalizing this V ok. So, why do I this equation is not of interest to me. Well is interest to me, but I have to go to the energy equation. I will not carry this forward for RG.

For RG this is starting point you know renormalization right. I mean remember so we expanded this and then write Green's function and and that gives correction to v. But we do not do it here I write down equation for the energy. So, what do I do? I multiply this by phi star of k t correct and add complex conjugate to it plus complex conjugate. So, that will give me d by dt of mod phi square correct.

Is it correct? I mean d by d t of phi phi star will be give you phi star d by d t phi plus phi d by d t of phi star. But what happens to this term when I add complex conjugate 0 it cancels right because look at it here is minus i k x v mod phi square mod phi k phi square. Now, when I do the complex conjugate then it will plus i k x same thing, but this phi k square when add them it becomes 0. So, this linear term in fact drops out. And the non-linear term in the right hand side I will get triple phi p phi q and phi star k right.

So, that is what we get. So, this algebra which we can easily do. So, the linear term has gone and this part. So, it is clear know I have phi p phi q and phi star k and I do complex conjugate adding, but there is a minus i string there. So, real part of minus i times some number, but then because of this i I will get imaginary part, imaginary part of this triple correlation or triple term right now I am not doing any correlation, but triple term and the coefficient in front is k x by 2 this one. And this is the term called non-linear energy transfer T k.

So, this phi k square is changing with time and because of this term non-linear term. So, had it been a pure wave T k will be 0 right pure wave means non-linear term is 0. So, phi mod phi k square does not change with time, but now we have the non-linear term that is why phi k changes with time and this is the T k. And this T k which I have written here this T k can so I just rewrote that equation and this T k can lead to flux right. Remember for Navier-Stokes there was a flux because of this non-linear transfer.

You have to argue that energy is going from a wave number sphere of radius k. This is a wave number sphere of radius k then is energy going out and that is the flux right that is the definition of flux. So, I need to compute this T k. So, let us compute T k. Now this is a cubic term you know it looks pretty odd I mean what to do with it.

So idea is do perturbatively. So the idea will be to just basically I am looking for average T k. These are random things so I am not interested in exactly detailed quantities function of time. So, I am looking at time average and look at this statistical property. So, you assume that phi p phi q phi k is a Gaussian variable that is a key part ok. If it is Gaussian variable then what happens to this cubic term this triple term? 0 no Gaussian odd order term is 0.

Please at least learn this part from this course. Gaussian variable all odd orders are 0 even orders can be written as product of second order ok. So, first order the zeroth order is 0. So, I go to the next order ok.

So, again we write the Feynman diagrams. So, the the vortex is this term this vortex, vortex term this is a this vortex is kx by 2 and there are three phi's phi q phi p and phi k ok right phi p phi q and the phi star k complex conjugate ok. So, I I have to to zeroth order it is 0. So, I go to the next order. So, I write phi k as Green's function this Green's function this one and so, what will phi k give me? I will just write it here phi k is the non-linear term. The linear part will give you will not give you will give only phi k when I combine with them they will give 0.

So, linear part is not going to create any non-zero term ok. So, the non-linear term is G of k t minus t prime integral dt prime exactly like what we did before dt prime dt d phi k t is let me write it properly phi k t I will do complex conjugate bit later. So, there was a minus ik by 2 k is by 2 vortex term G k t minus t prime Green's function t prime and the non-linear term. So, what is the non-linear term within? So, do not use p and q ok use phi of r t prime phi of s t prime and r plus s equal to k correct.

Now, you put a complex conjugate phi star. So, this becomes plus and this phi star phi star right and I need to integrate dt prime dt prime goes from 0 to t and there is a integral of what dr dr vector. Well s is not an independent variable s is k minus r right exactly same what we did before ok. So, I am expanding this G k in terms of Green's function. So, look what happens I will get Green's function and in the left there are two phi's right there are two phi's phi q and phi p from the right two phi's will come these two phi's this phi and this phi. How will you get non-zero value for this correlation and this correlation this one? So, this is q.

So, this must be minus q right otherwise I will get 0. If this is p then this should be minus p. So, that is C p, but this is t prime ok this t prime here t prime and here it is t. So, time here are not equal.

So, I write t minus t prime and t minus t prime. So, this because of time dependent system ok. This is not like equilibrium stat mech. This is non equilibrium stat mech, ok. So, t minus t prime and p t is Green's function t minus t prime. Ofcourse I need to do the t prime integral and the k this p integral and also the r integral. Well r integral is done here. So if r equal to minus p, then what happens to s. s once be minus q okay. So, the Feynman diagram also tell you what should be the way numbers ok. So, we got this one diagram. We will also have other diagrams. So, how many more diagrams? So, tell me how many more diagrams? Two more diagrams, but phi q also I can expand by Green's function and phi p also I can expand by Green's function.

So, this Green's function and like that and this Green's function like this ok. So, let us ok there are three diagrams, but let us focus on one of them. If you know how to compute one of them you can do the computation for other diagrams ok. So, this is a dt prime integral I forgot to put dt prime here dt prime integral and these two correlations and Green's function ok. So, remember what is the assumption I made for this time dependence.

So, it comes from the linear term. By the way my viscosity is not renormalized my V is not renormalized. So, the Green's function will be very much e to power well you can look at the previous slide is going to be. So, our linear part linear part will give me the Green's function phi d by dt equal to minus i k x phi right. So, Green's function is we put a delta function. So, the time dependent part is basically the minus i k x t minus t prime t greater than t prime.

So, this is the theta function ok. So, this is part. So, we have V will be there V will be there V k x no dimensionally that must be. So, this is velocity times k x everybody is happy with this and we assume that the correlation function two have similar time dependence. So, this will give us exponential minus V times p t minus t prime p x and this gives me V q x and I do the time integral. So, that dt prime will go away and this algebra is very similar to what we did before. So, this is that I wrote there and the dt prime I am always missing the dt prime and that will go below know the exponential and assume that t is bigger than this relaxation times or not relaxation time basically the wave times.

This is a wave know right now. So, wave has a time period. So, assume that this this t is much bigger than time period basically 1 by V k x V delta k ok. So, V k x basically there is a time period for the wave and we are looking at last time where there are many many oscillations of the wave have happened ok. Just put V of k that has the inverse time know V k is inverse time.

So, that will. So, if I make the assumption then well I am rewriting. So, this will come below ok. Now whether waves if I do the integral plus minus plus minus they do not cancel. So, I am going to put a damping that part is critical ok.

So, I said I need dissipation. So, to damp this wave I put minus epsilon t minus t prime ok. So, that is the wave part. So, this exponential if it is oscillating then these are these are oscillating waves right, but I put this damping then this oscillations will damp with time like that right. I mean of course, you do not make epsilon small. So, damping is weak, but it will finally damp otherwise this integral is not well defined.

So, put this epsilon. So, if I do the algebra then I get it below like that. Is that clear to everyone? So, that is origin of dissipation ok that dissipation is critical otherwise this integral will basically 0. Look in a conservative system there cannot be net energy flux. So, flux will be 0, but by this damping I put damping at small scales and that will set up a cascade and that epsilon is coming there ok epsilon is basically damping at small scales ok. Now what is this you may recall we did in the complex variable course complex variable not course complex variable lectures.

So, this part I have in the next slide. So, I rewrite it I multiply by i in top ok. So, minus i times i will become 1 and this i comes here ok. So, 1 by x plus i epsilon is is a principal value 1 by x minus i pi delta x I derived this ok and a principal value is 0 for this part you believe me the principal value is 0 plus minus cancel for 1 by x principal value is 0 anyway right. You will get things like 1 by x and 1 by x.

So, if I integrate sum it up they will get it cancel. So, this is a non-zero part this i here and this i here. So, that becomes minus 1 there is another i. So, how many i's are there in top 2 i's here which is real minus 1 there is another i here which is a minus sign. So, I get imaginary part remember I have imaginary part sitting in front.

So, I will get a non-zero imaginary part by this strike ok. So, if I do all this algebra correctly then I get this is a delta x. So, delta k x minus p x minus q x C p C q which is coming from here and this k x squared and so this is my answer ok. So, this is for one of the integrals i 1. Now it has two constraints. What is the constraint? k equal to p plus q because that is the momentum conservation and this frequency also are matching omega k equal to omega p plus omega q.

These are the two constraints you will find when in wave turbulent ok. It turns out for this case this can be derived from here, but in general it would not be the case right because if I take the x component I will get this k x equal to p x plus q x and that is why I said what I am doing is very trivial example, but I want to time is short. So, I just want to give you a flavor of wave turbulence, but I will make it bit complicated ok and the last lecture will be some more complex examples of wave turbulence, but not too complex ok. So, I will just try to simplify. So, is it clear? So, this is how I compute i 1 and this integral will be non-zero. I mean I did check I can argue that this integral is non-zero, but this is only one integral.

In Feynman diagram there are two more. So, I put all of them together then I get this three terms ok. So, this is Cp Cq, but if I do the Feynman diagram for p part then I get C well I am sorry this is a mistake here p then Ck this Ck Cq Ck and Cp Ck. So, this is stuff I integrate it I get Tk is function of k, but remember we were looking at flux. So, what is the flux? Flux is given Tk I can define flux which is energy going out of the sphere.

So, it is defined like this. This is a energy going out of the sphere ok. So, Tk is energy coming into way number k. If I sum over all the modes inside this stuff is energy coming onto if I sum over all this without a minus sign what will that be? Net energy coming out in the sphere right this one by definition know like everybody gets some money from the institute the net money coming into the room, but what is the energy coming out of the room or money going out of the room? So, you put a minus sign right I mean if you are all spending then there is money going out. So, I put a minus sign that is the flux. So, there is another integral dk prime which will come there is a dp integral there is a dk integral and this is going 0 to k, but dp is from full space ok.

So, this is the flux. Now this integral is complicated well I have not done it myself, but it people have worked out similar integrals for different examples, but we do not want to compute the integral I just want to get the form of Cp and Ck that is spectrum know right. So, I am looking for what is the form of spectrum and that is when we will end the lecture in 10 minutes. So I want to by dimensional analysis I want to get the form of Ck ok assuming that this integral is finite and is constant is independent of k which is an assumption may not work. Let us assume that this flux is independent of k it is in some initial range ok.

So, left hand side is epsilon constant flux. Now this is Ck the modal energy, energy per mode and I defined 1 d dimensional one dimensional energy Ek which is Ck is Ek by k to power d minus 1 ok. I hope you remember this part. So, energy so Ek is the energy of a shell right. So, to get energy of a shell I multiply the modal energy by the area of the shell which is k to power d minus 1 ok.

So, this is one thing I need to use. So, I need to replace them by this and what is dk? I have it in my next line dk is k to power d minus 1 dk multiplied by the area which is a non-dimensional number kd right. So, here I get k to power d minus 1 and here I get below k to power d minus 1. So, this guy will cancel with that. one Is that clear? When I will get dk here and dk dp here.

So, please look at this carefully ok, you will get the right answer no problem. This 1 kx here and 1 kx here and there is a delta function here ok. So, we just need to count the dimensionally argument ok. So, let us just do that here. So, this delta so I am going to do it in the rough work here ok. So, this 2 kx square no this kx and kx remember this kx is here kx square and 2 kd minus 1 cancel, but here I have so this e square and what about here dk.

So, this dk has dimension of k right. So, I will get dk and dp, but there is a delta function here and what is the dimension of delta function? 1 by k. So, 1 by k will come and this will cancel with that ok and this 1 by V will come right this 1 by v sitting there V pi has no dimension. So, you can ignore pi. So, this 1 kx square and this 1 k.

So, this k cube e square which is e square 1 by V is epsilon. So, what do you get for ek? ek is V epsilon one third one square root k minus 3 half ok. So, this one a famous energy spectrum formula this was derived by Kraichnan for hydrodynamic turbulence and not by this procedure, but some other procedure, but this is the mean flow comes into play this mean velocity fields V this V is the mean flow no, but more importantly it comes in magnetohydrodynamics Kraichnan-Iroshnikov theory which some of you have learnt about it. Kraichnan-Iroshnikov theory the derivation is bit more complicated because we have alpha in waves going in forward direction backward direction, but what plays is important role is this V alpha in wave speed ok. Interactions are more complicated and that is more realistic of alpha in waves, but this is the formula in fact we derived by Kraichnan and I did it well by pretty rigorous argument except my equation begin beginning equation is not very good ok. So, in fact for MHD we should work with z plus dot the wave this is a like a wave and that should go as minus B naught dot k z minus z plus and z minus dot equal to plus B naught dot k z minus ok.

So, the minus here and the plus here. So, the two waves they interact in opposite direction and that is more realistic, but this will involve more work. So, I cannot finish it in a short time, but if I do it and the non-linear coupling is z minus dot grad z plus these are non-linear coupling not phi squared. So, I have my coupling right now is phi grad phi ok, but here is a cross term ok that is if I do this stuff with this then you will get the same result ok. Now, this one result, but there is another variation ok. If V is large by the way for weak turbulence we assume that V is large right V is large and the magnetic field is large then the flow becomes two dimensional or quasi 2D there is not much activity in the perpendicular plane.

Then I assume this to be 3D right now I assume that well D is 3 ok for this flow, but then flow is 2D then the spectrum changes ok. So, these are theory by Galtier and Nazarenko and the argument there is the following ok. So, here we my spectrum is 2D two dimensional and then so this is not 3D.

So, I this dk per d. So, these are 2D. So, this 2D. So, my velocity field my phi field is in the plane ok. So, the 2D. So, D is 2, but interesting point is that here this integral is not for 3D ok.

So, you can count dimensionally. So, in fact, I have done it here. So, 1 k from here 1 k from here ok and this will cancel and the k squared here. So, you write is that. So, I wrote dk is dk k k squared know or k to the d minus 1. Do not write that right you write k perp and k parallel this one and k parallel is cancelling with this delta, but k perp remains as is or p perp remains as is and that gives you dimension which was in the previous slide I cancel this with that, but according to this authors they said well do not cancel p perp you cancel only px with that ok.

If you cancel that, but keep this. So, this k here k here and the k squared coming from here ok. So, that is k 4 and that e k squared according to this theory is k minus 2 ok. This is because of quasi 2D the spectrum has become 2D and k perp and k parallel both had to be taken into account and that gives you this spectrum ok. So, let us summarize my introductory thing on weak turbulence. Linear term is much bigger than non-linear term. So, for MHD I need to assume that d naught is much bigger than fluctuation and write down equation for the energy flux assuming it to be constant because all that derivation will not work if epsilon is also function of k right.

And, then non--renormalization of linear parameter ok. So, that is a good simplification and that is a basic idea of wave turbulence. I think so, what I will do next is I will take some more examples like water waves we can do rotating turbulence. So, I can just apply this idea without getting too much into detail. Some of it if you are not aware of rotating well you are aware of Coriolis force.

So, I am just motivate the equation and I derive the spectrum. It will look magical, but I hope you will get the idea.