Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Week - 01

Lecture – 06

This I am going to give as an exercise. I will give some homework. Exercise is I computed G of x, right, for point source at 0. I showed that to be this function. Now do the Fourier transform of this, then you should get this. G of k should be minus 1 over k squared.

This is straightforward. For other way round, given this. Derive this for 1D. That is not so straightforward, but I recommend that you do it.

So I did G of x in 1D. Now given that you can compute G of k, you should verify that is 1 minus 1 over k squared and vice versa. For minus 1 over k squared in 1D, you get this function. So this I will give as a homework and you do it yourself. Now let us compute this in 3D.

It is an interesting exercise to do it in 3D. I will do some of this integral because we need to know how to do this integral. Now in 3D, well, I forgot the minus sign. You should put a minus sign, minus 1 over k squared? G of k is 1 over k squared minus sign and my response is G of r. I am assuming source is at 0.

Then I, well, I should say that if your source is not 0, then this part is not r, but r minus r prime. But I do not want to carry this r prime all the time. So I assume source to be at 0 and if you really need it, then you need to put that. Otherwise, it is okay to ignore it. So let us do this integral.

I mean it is not difficult, but it is nice to see how to do this. So my integral is over k, but my r is fixed, no? In the integral, r is a fixed vector, yes or no? So choose r along z axis. This is my vector r. That is number 1. It is my choice.

My integral, okay, I am choosing my r vector along z axis. Okay. So I mean if my source is sitting here and if I am measuring my potential in some direction, I choose that with my z axis. Okay. Now k is of course varying, right? So my k vector can be anywhere and this angle is theta.

So we will use spherical geometry, spherical coordinate system, r, well, the magnitude is k and d theta d phi. So k is varying on a sphere. In fact, not on a sphere. k goes from 0 to infinity, the magnitude of k. So 1 by 2 pi cube, minus sign I keep it outside and integral.

Now dk is k squared dk d cos theta d phi. Now this is azimuthal symmetry. If I let go in that direction, my integral is unchanged. It is not function of phi. So I do the d phi integral in one shot and 2 pi will come out.

D phi is 2 pi, no? This is azimuthal angle 2 pi. So d phi I don't need to worry and k squared is here and I get e to the power i k dot r. What is that? K dot r. Cos theta, no? So e to the power i k r cos theta.

That's it. Now my k goes from 0 to infinity and cos theta goes from minus 1 to plus 1 or theta goes from 0 to pi. So this cancels, 1 over 2 pi squared minus sign, k squared k squared cancels. Now what is this integral? d cos theta e to the power i k r cos theta. So this we can call a variable s. So this will be e to the power i k s goes from minus 1 to 1.

So e to the power i k minus e to the power minus i k by k r, i k r. So that is the d cos theta integral. It has been done in your quantum course, I am sure. So the cos theta integral also is done, d k and k goes from 0 to infinity. Now what is this? i k r e to the power minus i k r is 2 i sine k r.

So minus 2 pi squared, this minus sign, I need to worry about it bit later, sine k r 2 i divided by i k r. So i i cancels and that is what we get, k r, and great. Now what should I do after this? So this function, this r here, r here and this is a d k. So one thing is to make this integral non-dimensional. This integral has, what is the dimension of this integral? One by length.

So I want to make it non-dimensional. So multiply this by r, r is a number, for this integral k is variable but r is a number. So multiply by r and divide by r. Now this integral, in fact we already know what that is. Pi by 2, no this we already did.

Sine x by x d x, this is pi by 2. So this is 2 pi squared 1 by r and pi by 2. So one pi cancels and so I get 4 pi r. Now there is a minus sign which is 8 pi r. So there is a 2 somewhere, 2 is, I thought this 2 is there, 2 cancels, 2 cancels with that.

So this is 1 by 4 pi r. This is a, in fact we know this potential, right? For a point charge in 3D the potential is 1 over r other than this pre-factor. In fact pre-factor is also correct.

Only thing I need to worry about is the sign. This is a negative but I should get positive. Well, let us put a negative here.

I think it is because Laplacian I have put a negative. So for 1 over k squared integral is 1 over r. You put a minus then I will get minus 4 pi r. So this is how we derive it. In fact we know it, this is correct.

I am not 100 percent sure about the sign but you will find that part. So this is how we do the integral, 3D integral. Most of the time you will be basically doing this. You will use your r along z axis and do this integral. Sometimes of course we do not use sine k r.

I am going to just show you some more integrals but it is straightforward, no? But you should know how to do this. So any questions on what I did so far? These are straightforward definitions but you need to remember these answers. The Green's functions in for Laplacian operator is 1 over k squared and in real space it is, if you know the potential electrostatics you can easily do this. What about in 2D? What is the answer in 2D? 2D is log r.

2D potential is log r. For line charge potential is log r and you should be able to describe it. The derivation is not that straightforward. It involves the special function but you should be able to, well I mean that is an exercise for interested students. G of r is log of r. G of r is potential for a point source and if you know electrostatic answer you should just be here.

So any questions on this? In well basically in general yes. So Fourier it is 1 over k squared irrespective of the boundary condition. That state came from the formula G of r. Now it is a very good question. Finally you need to compute this potential in the real space.

So given 1 over k squared I need to compute G of r in real space and that is where I need to put the boundary condition. So inversion from Fourier to real will involve boundary condition but in field theory we will assume infinity. So that is where we are avoiding this boundary condition details. So like in quantum interlude dynamics you are not considering walls. These photons are just going to infinity and so we don't, we are too far away from the wall.

So that works reasonably well for most of the time but of course fluid in a cavity we need the boundary condition. So in fact there is another advantage of Green's function in Fourier space. You don't need the boundary condition. Most of the time you don't need the boundary condition but when you get the real space then you need the boundary condition. But here we will assume infinity, infinite extent.

So this is the one place where we can stop. Thank you.