Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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This part will be bit difficult, we will do renormalization group of turbulent flow, but I am going to take a simple example. So, let us so that was kind of overview of turbulence phenomenology, but now we can do some field theory. So, in fact the idea is that we had this viscosity you know which is l to the power four third diffusion coefficient, can we derive this from field theory. So, you see the viscosity is increasing with l. So, we had viscosity nu l which is l to the power four third epsilon one third, l is increasing you know it is similar to what we did for Wilson theory where my parameter the mass parameter, R was a parameter which was changing with scale. In fact, we did mass renormalization of phi-4 theory it was changing with scale.

So, that is what we would like to do here, but it is a non-equilibrium system. So, there will be some differences. So, that is let us let us get started. So, I will renormalize the viscosity of not the Navier-Stokes equation, but simpler model of Navier-Stokes equation called Shell model.

The field theory is simpler here, we will do Navier-Stokes later in the next class possibly and let us start with the shell model. So, shell model has fewer variables. One drastic assumption is that we divide this 3D sphere into shells, overlapping shell well not overlapping, but different the shells like this. So, this is shell with radius kn kn plus delta. So, this is un, un is the velocity field.

So, in real flow we will have lot of Fourier modes and each Fourier mode has a velocity, but in shell model we just give one number un. So, instead of k squared modes having different different values, but I have single wave number called un. Now that is one. Second is wave number of the shells. So, shell radius is basically specifies the of the k that goes as power law.

So, kn is k naught b to power n. So, the first radius is k naught. So, I can draw this stuff here. k naught next is k1 which is b times k naught b squared k naught b cube k naught like this. It is in log space is linear in log space, but if you look at the real in linear scale this will be this gap will be bigger b is greater than 1, b is greater than 1.

So, this gap will keep increasing. So, this is called shell model and we will also make another assumption that this shell in. So, we had interaction right u dot grad u that had interaction u p u q p p plus q is equal to k. So, any p can interact with k right. So, if you have wave number k then the p can be any size.

In shell model we do not allow that. We say that kn will act only with modes few modes in fact, 5 modes 2 in the left and 2 in the right. So, I will show you that model now. So, these are model. So, d by dt of u n.

So, my Navier Stokes equation has been greatly simplified this one my viscous term is nu bar kn squared this is a kinematic viscosity. Now, the look at the non-linear term it is. So, this kn we are interacting with 2 modes to the right u n plus 1 u n plus 2 this n plus 1 n plus 2. The next is 1 to the left and 1 to the right. So, 1 to the left and 1 to the right this is 1 to the left and 1 to the right third is 2 in the left n minus 1 and n minus 2.

So, as I said these are basically 5 modes that involved in interaction. I do not this k k un does not interact with shell here does not interact ok. This is called local which has been proven to be correct well proven to be valid for Navier Stokes especially in 3D 2D has some issues, but 3D it was. So, now what now I start doing if you recall Wilson theory please we got to go back to Wilson theory. So, what did you do in Wilson theory? I want to see what is the viscosity at different scale.

So, the nu bar will change. So, my nu bar basically is the very large n nu bar, but I keep going to the left means I keep coarse graining and coarse graining means I remove this variable and see the effect of it. So, my viscosity will increase then I do it here I increase ok. So, my viscosity keeps increasing with scale. I am going to show you this will happen here in this proof in this scheme that is what happens in atmosphere.

Atmosphere the diffusion coefficient is bigger and bigger as you go to higher scales that is why pollution is spreads at different bigger scale ok. So, the idea is to divide this wave number. So, I am going to make some in fact following Wilson we are not going to do it step by step for all of them we just focus on kn in the inertia range. Imagine kn is the inertia range and see the effect of kn minus n plus 1 n plus 2 and assume that when I came here I already have viscosity modified. So, viscosity so, I came from here all the way, but I am doing the calculation here in this part is that clear I am not going to do all the steps.

So, we are going to see at kn what is the effect of n plus 1 and n plus 2 and how does it how is un going to be modified. So, the idea is ok. So, I take one more step back. So, a 1 a

2 a 3 are these are constraint that a 1 plus a 2 plus a 3 is 1 and there is one more constraint which you will ignore, but the coefficient we choose for our our work is this a 1 a 2 a 3 is this what happened 0 0 0 0 0 0 0 sorry sorry 0 yes the sum is 0 sum is 0 ok. So, let us look at turbulent case in the inertia range ok.

So, this one intermediate slide. So, it turns out for shell model which is mimicking the Navier Stokes mod un square. So, this is not the spectrum right spectrum is mod un square by kn. So, remember E k is un square divided by k. So, un so, this is five third.

So, Cn will be k minus two third I am take this kn and multiply there. So, Cn is k to the power two third epsilon is the dissipation rate is two third and full moto constant and the energy flux is constant and this has been verified by many simulation once a simulation is here. You can look at this is a spectrum which is multiplied by k two third and there are two runs, but ignore the two runs the red line. So, the spectrum is k two third k minus two third is constant because I multiply this by k two third. So, this becomes one right and the flux is constant ok.

This is a property of shell model and it is also seen in Navier Stokes equation. So, this is a non equilibrium solution which I am trying to show you that ok this indeed happens initial model as well equilibrium case the flux is 0 and the every mode has equal energy. So, it is not power law minus two third, but it is constant and this is the inside is the thing. So, the C n is constant and the flux is 0 ok. So, this is the equilibrium scenario ok.

Now, let us get to R G I will start the renovation group is ok you can take a small break shell model is clear to everyone ok. So, let us now. So, it is not well I mean it will take 15 minutes to get there, but there are some details in between RG analysis. So, this is the equation I wrote right before this one thing I am not focusing on there some are complex conjugate some are not ok and this called Sabra shell model ok. This is a it has some advantages, but we will not get through that part.

Now, we are going to divide the shells into less shells and greater shells ok. So, this is my kn this is the my lamppost or this is my marker. So, there are two modes two shells to the right of kn I denote them by greater and the shells to the left which is denote by less. So, my idea is right now I have n plus 2 shells ok. Now, what I want to do is I want to see I want to average over these two shells this is called coarse graining right we did coarse graining for Wilson theory.

So, coarse grained means I am going to average over assume that u n plus 1 and u n plus 2 are random variables with 0 mean and look at the effect of this averaging of coarse graining into viscosity this object will it increase or decrease or would not change that is the idea ok. So, we are going to divide some shells to greater and some to less. So, which are the u n plus 1 is greater right this is the greater u n plus 2 also greater this is greater here one is greater and one is less I this is a typo this should be less and this is greater n minus 1 is less greater and n minus 1 n minus 2 both are less. So, we do this coarse graining. So, coarse graining if you recall Wilson theory that we average assume that these modes here are random and we average over them.

So, of course, random if a single variable u n greater or u n plus 1 greater this will give 0, but what about u n plus 1 greater u n plus 1 greater star what will that be not 0 x and x star complex conjugate is not 0. So, it is a in fact, it is a spectrum ok. So, we have this kind of quantities you got mixed up. So, I am going to do some restructuring. So, let us do the averaging now ok.

So, this is called coarse graining. So, we do coarse graining of this way this equation. So, this is a average symbol norm and this is normal in statics, but coarse graining is applicable only to the modes greater modes not to the less modes less modes thing should remain as is you do not coarse grained the less modes. So, they will remain as is. So, what happens to this this will remain as is to the remain as is what happens to this less less remain as is.

So, that is remain. So, I take it to the left hand side there is a minus here and then minus here there is a plus I take it to the left hand side and this is the term minus a 3 k n minus 2 this one. Now what happens to this term if I do average. So, 1 is less 1 is greater. So, what is happens to that? So, less guys are not to be coarse grained.

So, they will come out of this. So, we have u n minus 1 star u n plus 1 greater. So, this will come out u n minus 1 star u n plus 1 greater and what is this 0. So, this is 0 this term goes to 0 what about this term 2 greater both are greater. So, multiply 2 random there is no guarantee it will become 0 had it been Gaussian then it will be 0, but it Gaussian right I mean Gaussian variable x and y this is 0 unless x equal to y ok, but there is no guarantee this Gaussian ok. So, we will not assume any Gaussian approximation.

So, this one is we need to compute this. So, I am sorry this got mixed up here. So, it is minus i a 1 k n equal to. So, I retain it in the right hand side u n plus 1 greater complex conjugate and u n plus 2. So, my equation in the left is this equation in the right this object this of the right object I will compute and show you that this object is some delta delta nu minus k n squared u n less the right hand side is appears like that this will be of this form. So, what will it do? I can take it to the left hand side and what is going to happen is nu.

So, left hand side one term is nu k n squared u n less. So, bring it to the left. So, plus delta nu k n squared u n less. So, these two can get added up right and delta nu I know is positive.

So, it turns out my nu viscosity is bigger ok.

So, that is how well as viscosity is getting enhanced. So, is the effect of this coarse graining this precisely what we did in Wilson theory ok. There was no d by d t term there, but the idea remains the same ok. So, let us my our idea would be to compute this right hand side and that computation is what we will do now.

So, I am sorry well this is the same equation. So, we write this in terms of Feynman diagram. So, this minus i a 1 k n this this guy ok. So, the equal this bit of transport from keynote to PPT that what happened. So, minus i k 1 this is the vortex is vortex ok u n plus 1 star this is u n plus 1 star u n plus 2 greater.

So, these two are greater. Now, I am going to well right now I do not know what that thing is, but I will do expand it in terms of Green's function ok. So, u n plus 2 I write as so, this we done it before on the Green's function is again coming into play. So, you can write down this equation well actually you should write down the original shell model. So, let me just I think I have it in my next slide. So, basically this is the Green's function inverse.

So, the viscosity for nu greater n plus 2 will be nu n plus 2 ok. So, I am going to say that viscosity at scale k n is nu n at scale k n plus 1 is nu n plus 1 scale nu n plus 2 is nu n plus 2 viscosity is not same at all the scale they are different and if this scale is nu n plus 2 k n plus 2 whole squared u n plus 2 greater ok and the right hand side will have those three terms 2 in the right 2 in the left and 1 in the left 1 in the right. So, how will I write down equation for u n plus 2 greater? So, these are Green's function inverse this is a time t. So, you write this integral g n plus 2 t minus t prime RHS at t prime d t prime it is going from 0 to t.

So, these are definition this is the definition of Green's function and using Green's function I derive the solution right I mean this is we did in the one of the first few classes. So, this is what you get here nu n plus 2 this one Green's function and the vortex for the Green's function. So, there is a vortex RHS will have vortex in front, but there are three terms, but it turns out the one term which is nonzero will be the three terms here. So, here what is a. So, in fact, is good to write in diagram this is the beauty of Feynman diagram we do not need to write down all the terms.

So, this is remember this was u n plus 1 star this is u n plus 1 star right. So, what should come from the left so that I get a nonzero value if it is Gaussian right now we will assume that if it is un plus 2 then it will be 0. So, un plus 1 un plus 2 average is 0 it is similar to what we have uk uk prime average is 0 unless k k prime equal to minus k. You have same mode then you get nonzero value or otherwise you have two random variable x y average is 0, 0 mean they are random they will be 0 ok.

So, this has to be un plus 1 ok. Now, you can see that u un plus 2 one term is un plus 1. So, what should be other term RHS has three terms the one which will nonzero will be un plus 1 and un and the vertex is i 3 kn ok. So, this is a term vertex here i 3 kn one term un plus 1 and it goes with complex conjugate. So, that will give me a correlation function t n plus 1 t. So, this is a time t and this is a time t prime remember this is a time t prime.

So, these are these are bookkeeping you have to be just do it carefully and then you get it right. So, this is the time t t prime that is why I have t minus t prime time is stationary if I shift the origin it does not matter, but it is the depends only on the difference in time. So, that is the correlation function n plus 1 t minus t prime I put a bar here for reasons which I am going to show you in a minute and this is un plus t which is same as this term. So, this part of the diagram this one is a number is in fact is integral well not integral here is just a number this is going to correct this new bar this new bar this one term, but this is another term which is I can expand this using Green's function right I mean why only expand the bottom one you can expand right one as well. So, these are Green's function which is n plus 1 and due to the same kind of stuff then you get correlation function below and is again u n less than t.

So, these two diagrams this two this call this is self energy this called self energy this is the name from particle physics and field theory self energy diagram and these are going to make corrections. Now, our objective is to compute this diagram that is it and we got the correction ok. So, let us do it now I will next slide I am going to do this algebra I am sorry this is 0 not 0 to t. So, we got so there are three terms, but the nonzero the term which will give nonzero value are only this. See there are some more I will send you my paper where I wrote all this.

So, u n plus 2 the term in the right hand side will be u n plus 3 u n plus 4 the ending is 0 right because I am keeping only u n plus 1 and u n plus 2. So, there are simple logic to rule out the other two terms and only term survives is this ok. Now, RHS which is minus i a 1 k. So, this is the Feynman diagram I wrote n plus 1 n plus 2 I am going to substitute this u n plus 2 here is that clear these RHS of the equation which I wrote in my previous slide and this is going to make some correction to nu bar ok that is what I am looking at.

So, un plus 2 I just substitute. So, this part is you can see u n plus 2 is here i a 3 k n is here a 3 the a 1 is here, but I and minus I will give you 1 right. So, this is become 1. So, a 3 a 1 is here kn and kn will be kn squared this un less which is here and un plus 2 and un plus 1 right. So, this un plus 1 which combines with this un plus 1 star. So, I got this, but this is t and t prime please note this t and t prime two different times and this guy also t prime.

So, there is a bit of problem I want u n of t right my equation in the previous slide is u n of t, but this guy is u n t prime problem know because I want at time t and I have this integral inside and I need to integrate this stuff. So, what should I do? So, I make a approximation this is called Markovian approximation. So, Markovian approximation says that this integral is integral gets maximum contribution near t equal to t prime. So, this t prime is from 0 to t, but it gets maximum contribution when t prime is close to the upper limit.

So, I have integral 0 to t e to power minus. So, there is let us imagine that I have e to power minus t alpha dt. So, under what condition of alpha I will get maximum contribution for t in the upper limit. You can easily argue when alpha is large then you can get maximum contribution when t prime is close to t alpha should be large and this is called Markovian approximation ok. So, this approximation is made then this guy comes out as a t ok.

So, basically we can integrate this minus alpha t ok. So, you need to do plotting and so on you can convince yourself that this is happens when alpha is large. If alpha is small then all of them will contribute. If alpha is large then it turns out till it is. So, it drops suddenly near t and that is where big contribution comes from here.

You reverse bit of plotting and it you get that. So, this un t prime becomes un t. These are critical approximation is called Markovian approximation which is made and then we get ok. So, this guy will come out. Now, I have I am left with these two Green's functions sorry Green's function and correlation function. Now, I need to do the integral otherwise I cannot get a number and these are some funny functions right.

I mean I do not know what they are well hopefully we can simplify it, but it is a I do not know what their function is. So, we became in other approximation that this Green's function is exponential minus nu kn square t minus t prime. So, this nu n is the renormalized viscosity or viscosity at kn. So, nu n kn square t minus t prime this is the Green's function and correlation function t bar the time dependence is exactly same as this nu kn square t minus t prime. You can easily check that nu n kn square is dimension of time inverse ok and this c n is equal time correlation.

This is five-third or this minus two-third. This is the Kolmogorov spectrum this guy ok I plug it in. Now, it is easy you know when this integral are basically exponential I can easily integrate them right. The two exponential in fact, they get added up ok. So, these are critical assumptions of RG and once I put that then I get this. So, the integral you see there are two exponentials I can easily integrate them and that comes in the bottom.

So, my integral of the first diagram the two Feynman diagrams the first diagram is this and this is nicely u n t and this is well I do not know what is cn plus 2 right now, but I will assume well I am going to do some the assumption again and I do not know what is nu n plus 1, but I will make some assumptions and I get a number ok. Right now I 1 is some function of cn plus 1, cn plus 2 and nu n ok. I need to do some more work similar procedure can give you I 2 the second Feynman diagram this second Feynman diagram ok. This is cn plus 2 you can look at the Feynman diagram and this is the correlation of the u n plus 2 ok.

I can add them up and I will get this. So, nu n plus 1 squared is nu bar k n squared. So, this I am sending it to the left I did not know I mean. So, this there was plus sign in the left in the previous slide it became minus sign and I get this. Now, believe me this quantity is small nu bar is much small compared to nu n ok. So, this guy goes as k minus four-third for small k this will be large number right and nu bar is a constant.

So, we can ignore this. So, that is where the microscopic kinematic viscosity is not playing much role in atmospheric diffusion you can ignore it. So, now, I need to compute this. So, basically I know what is nu n, but nu n is function of nu n plus 1 and nu n plus 2 and correlation functions. So, this is where a new scheme well rather one new idea comes for self consistent closed.

So, assume that nu n plus 1 follows something following Kolmogorov theory. So, correlation function I just replace Kolmogorov spectrum and for nu n I again use dimensional analysis and I write nu n k n squared is nu star Kolmogorov constant square root epsilon one-third k n two-third ok nu n is k n minus four-third multiplied by k n squared is k n two-third ok. Substitute it here. In fact, this pretty funny you know why I am using a k n square root. It become clear that k n Kolmogorov constant sorry not k n k Kolmogorov square root k Kolmogorov will get cancelled is easy for to see. There is nu here and there is nu here nu here and this nu will become square of square root and that will cancel from Kolmogorov here.

So, Kolmogorov constant gets cancelled and you get only one variable nu star ok. So, there is a nu star coming from here and there is nu star coming from here there is nu star squared Kolmogorov constant will be 1. So, this together will give you k Kolmogorov and this also gives you k Kolmogorov they cancels epsilon will also cancel right. This epsilon one-third epsilon one-third epsilon two-third and epsilon two-third here cancels. So, everything is gone now you get nu and nu star squared is just this. This function of a1 a2 a3 which we know and b, b is that expansion parameter for the wave number.

So, I know the nu nu star ok. So, I have computed this viscosity, renominance viscosity ok great know this is what is done. So, it will function of b small b and b can range from as I said it should be greater than 1. So, we took from number from 1.2 to 2 ok and a1 a2 a3 already have told you.