Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Lecture – 53

Now, we define energy flux. So, idea is this equation I wrote sometime back know. So, this is T u k. So, I now integrate this over a sphere. So, I think I have the example of a sphere this is sphere. So, this is imagine a 3D sphere, yellow sphere.

So, I told you that energy is coming by a non-linear term. So, we focus energy by a non-linear term. So, this P mode which is a giver mode is inside the sphere P inside the sphere and k is outside sphere right. I mean the k x k y k c that is outside.

So, it turns out energy in 3D typically flows from P to k small wave number to large wave numbers. So, there will be energy going from P to k on average ok. So, that is in fact, is breaking the detail balance somebody is giving energy to somebody ok, k is not giving as much energy back to P. Sometimes it may give, but it is basically taking from P ok. So, now, we want a one number.

So, I would say all the P modes inside the sphere how much energy it gives to the modes outside the sphere it is an one number. So, the energy flux is from all the P modes. So, this is a call we call it energy going from P to k, P is a giver, k is a receiver and q is a called mediator ok, q is not really giving or taking, but it is like a lawyer ok. So, P is given to k, but P is inside the sphere, the sphere reduces k naught, the radius of the sphere is k naught. So, P is inside the sphere, but k is outside the sphere, k is outside the sphere, k is greater than k naught.

So, this is the flux this is the definition of energy flux which is energy going from mode inside the sphere to modes outside this sphere correct and in 3D this is greater than 0. If property of the Navier-Stokes equation ok it has been proven by Kolmogorov the energy will flow from large scale to small scale, large scale means small wave number ok. So, large scale is here and is going to a small scale here. In fact, we inject energy at a close to the origin small wave number. So, ok I am not proving it this is not objective of this course.

Well I will show by field theory that this will be indeed positive, but I will not prove by Kolmogorov theory ok. Kolmogorov showed this in his 1941 paper ok. So, that is by non-linear term, but this equation has other terms this force injection and the dissipation. So, any questions on what I did so far I am kind of going very fast I would say, but any questions on what I have done talked about so far? Everybody is clear? Well some of these guys know turbulence. So, if you do not know turbulence you should ask.

Any questions? Just to recap so, in hydrodynamics we are like in this room. So, this is the source of well fan we should think of AC is with both. So, fan is giving energy in this for the fluid in this room, but it is large scale fan is not of size of the room, but of course big fan and this energy which is injected by the fan is getting dissipated at small scales, but via cascade energy injected by that the fan scale is going down down down down down and finally, dissipates. So, that is a picture ok. I will show you some more results, but that is the idea you should think of and this is not a thermodynamic system right.

You can see that definitely not like a thermodynamic gas where all. So, thermodynamics is single scale phenomena. So, I think let us get it recorded I think this is a good example. So, I will so, let us let us record this as well. So, is that clear this is the idea is clear and non-equilibrium is basically multi scale ok that is what I would say ok.

So, basically we discuss this non-linear term how it is giving energy from large scale to small scale this is a which one this is a flux the blue arrow, but it requires that I have something injected here. So, in a room example would be a fan or in a coffee cup I stir by the spoon. So, that is large scale the energy faded that large scale is getting down and it goes down like that it goes to a small scale and dissipates at small scale. In fact, kinetic energy is lost by viscosity, viscosity basically eats up the kinetic energy ok. And so, we are injecting energy this is the injection by large scale ok.

And this naturally a multi scale phenomena right is going from a scale of the fan in a room to small scale at a mostly like well like micron scale that is what micron scale. So, this is a multi scale transfer and it is very different thermodynamics which is a single scale phenomena. So, thermodynamics what do we have these molecules are hitting each other and the length scale there is the mean free path length. We do not have phenomena across scales in a thermodynamic gas of course, the gas in this room is multi scale, but then we have this big it turns out there are big vortex where the molecules are of course, going locally, but also there is a net velocity ok. So, these are multi scale phenomena we got ok.

Now, this one example which I would like to emphasize and with that I can derive Kolmogorov theory ok. So, I can intuitively give a stuff. So, imagine so, this is a white sphere and there is a orange sphere. So, the white sphere the flux is pi uk this is a flux coming out there is a flux coming from the orange sphere i k plus d k are they equal in general scenario? They would not be equal if there is a energy injection by this green arrow of course, normally I said in Kolmogorov theory energy is injected at large scale small k, but energy can be injected at all scales like in a gravity. So, if there is energy injection then this will be bigger right orange sphere will give more energy out under steady state.

So, if I inject something then this is coming in, but I give some more money in the system. So, this will have more money going out. So, that is a positive source of flux, but this is a dissipation in this orange shell and that will decrease the flux. This is the competition between pi uk and d uk and that is d pi by d k ok. So, so this is one more example the coffee cup.

So, this is a multi scale transfer like that ok. Now, imagine for Kolmogorov theory we say energy is injected only here in large scale. So, in between the region this is 0 also in the intermediate range the d uk is small it is not 0, but small. So, we can ignore it then what happens this pi uk is constant ok. So, that is the Kolmogorov theory.

Kolmogorov theory assumes that energy is injected at large scale and dissipates at small scale. So, that is where you get pi u to be constant and this is the non-recursive scenario where we are getting cascade of energy. Now, given the constant flux we can derive what is energy at a different scale. So, this is the called energy spectrum ok. Remember I had said E u ek dk is total energy ok.

So, what is the unit of E u of k is E total energy by k total energy is u squared by u squared not total and k is 1 by length. So, this. So, the unit of uk is L cube by t squared ok. So, now, uk will there are some more arguments, but I am going to do it fast. So, E u k is function of pi or the dissipation energy induction rate which is same as cascade alpha and k to power beta.

So, we know the units of epsilon which is energy supply rate U squared by t and k is L to the power L to the power minus 1. So, you match the units both sides then we derive this formula which one. So, kinetic neutral spectrum is k minus five-third this flux to the power two-third and a non-dimensional constant called Kolmogorov constant ok. So, this is the Kolmogorov famous spectrum and this number is approximately 1. 6 or 1.7 or 1.8 depends on people, but normally we believe is between 1.6 and 1.7 ok. So, this is the derivation I will leave it for you to complete as homework and you do that ok.

Now, given the spectrum I can derive quite a few interesting things. First of all there is a verification in simulation, a spectrum we did a DNS, but this is the like done by thousands of people both experiment in simulation this is five-third line a spectrum and flux is constant. This is valid only in the inertia range, this range is called dissipation range and this is the forcing band ok. Now, let us derive few interesting observations from Kolmogorov theory. So, I can look at the energy in a band of the wave number.

So, k to k plus dk ok. So, there is a band k to k plus dk. Now, again I am just going to say that it is a power law physics. So, this k plus dk is going in logarithmic scale. So, my k plus dk is 2k ok. Now, you just believe me that this is how we do it for power law physics.

So, when I integrate this k minus five-third will give you my k minus two-third ok and this is important step when dk is k. So, my ul squared is epsilon two-third, k is 1 by length so l two-third. So, what is ul? ul is I take the square root I will get epsilon u one-third l one-third this is the cascade rate. Given this I can look at the diffusion coefficient. What is the unit of diffusion coefficient? It is centimeter square per second or it is length time velocity scale.

So, what is diffusion? So, our molecules are colliding with each other with the mean free path length l, but it moves with sound speed. So, the diffusion at micro scale is sound speed times mean free path length, but in hydrodynamics it is not in hydrodynamics is packets these big packets of molecules are colliding with each other or transferring momentum and flux and stuff like that. And so instead of mean free path length I put the length of this eddies or length of this vortex and velocity of the vortex multiply the two and that will give me I just multiply l here is l four-third. So, my diffusion scale a diffusion coefficient is function of l is l to the four-third. So, my diffusion coefficient for heat diffusion was constant, but now it is l four-third.

So, these are multi scale phenomena and is a non-equilibrium phenomena and from this I can derive another very important result. So, let us do it what about the time scale is l by ul. So, just substitute ul from here ul from here I get this u divided by u one-third is l two-third. Now, invert it l squared is t cube by epsilon u I just cube it and you get that. So, this is the diffusion length or rather as I said know the heat diffusion one meter how long will it take the example we did.

So, the l squared is goes as t cube. In what happens in random walker l squared goes as t that is a diffusion rate by random walker the distance travelled in time t will be square root of t because it is going forward backward, but here it is not t l squared is t cube. I think I did mention in the class that it is faster than ballistic if this thing if some particle moves with constant speed what is the distance travelled l is proportional to t. So, l squared will be t squared, but I am doing faster than t squared how is it possible? It is possible because we hop from a I gave this example in the class I remember this one. So, we my eddies are at small scale I am going around, but then I jump on to bigger eddy and bigger eddy has higher speed.

So, example I go from a bus to train to flight. So, I go to flight then the distance travelled is or the distance travelled is more on the flight and that is what happens here l squared is t cube and this is called Taylor diffusion. So, in turbulence we have diffusion which is t cube l squared is t cube and this is fully non-equilibrium this is not an equilibrium phenomena and so, this is called Taylor diffusion l squared is t cube is a super diffusion is faster than ballistic atmospheric flows exhibit stellar diffusion. So, we are all living in non-equilibrium world now I mean this is like thermodynamics does not work that atmosphere and so, this is a picture thermodynamic is equilibrium, but I am going to give an example here. So, this is a what you should see this is a hydro part fluid part.

So, I supply energy at large scale and this is going into a smaller scale like that, but then finally, at the smaller scale it goes into thermal energy of the particles this particles come into play of course, particles get heated by this process. So, this is called quasi equilibrium should have constant temperature, but it is getting heated up slowly this quasi equilibrium, but this part is non-equilibrium this cascade in fact, very strong cascade is going through this. So, this non-equilibrium plus quasi equilibrium. So, this is what is happening in this system and the corresponding picture is there is a flux in this part and there is a dissipation in this part dissipation and then this is the hydro part where this flux is 0, but there is some heating because of the energy coming from the smallest vortex. So, I think let us summarize it now turbulent phenomena is a non-equilibrium phenomena is a multi scale unlike equilibrium which is typically a single scale energy transfers break the detailed balance.

So, we will perform no field theory of turbulence next that is my next task. Thank you.