**Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives**

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**Week - 01**

**Lecture – 05**

So, let us use the definition, first linear operator, okay. So, derivative is a linear operator d by dx Laplacian. So, what is a linear operator? If an operator L, then I put f phi and they will get a some zeta, right. f of x acts on it and I get zeta x. So, I put factor phi what this, multiply this by a factor, then my output also will be multiplied by the same factor, right. Also superposition principle, L acting on phi 1 plus phi 2 is same as L phi 1 plus L phi 2.

So, these are linear operators which won't work for non-linear function. For example, quadratic function. So, my f x equal to x squared, okay. Now, you see x plus y square is not x square plus y square, okay.

So, this non-linear and f x equal to x is a linear function, I mean linear straight line. So, okay and this is of course standard easy function but Laplacian is also a linear function, linear operator, right I mean, okay. Anyway, so this is standard but so we are dealing with linear operator for definition of Green's function. We are going to use it for non-linear as well but that is for later. Right now, right now in this course, in this class we will use only linear operator.

Now, we have source term. To give an example, to compute a potential for a charge, so Laplacian V equal to minus 4 pi rho. So, rho is a charge, it is a source. I put some charges and compute the potential. So, potential is a function I want to compute V, okay.

So, this is called source term and this is my solution or it is also called response, okay. So, response, so we do burn some source and look at response. Like in spring, I apply some I do some excitation on spring and I look at the response, okay. So, that is what these are the two things. V is solution is response and rho is source, okay.

So, this is what I have written. L phi, phi is my solution and rho is a source. So, solution is phi x and consider source, okay. Now, we can consider source at x prime. Now, it is a point source.

So, for definition of Green's function, we think of point source, okay. Or it could be point in space and point in time. It is a delta function, no? So, this is the critical part. Green's function is for a point source. So, my rho of x is not a general charge distribution.

You make a point charge at x equal to x prime. It do not be charged. It could be some source for a wave I do some disturbance at a point at a given point, okay. So, the response is the Green's function. That is the definition.

For this one, for any continuous rho, I get a response phi x. There is a solution. But for definition of Green's function, I put a delta function source, okay. And then look at the response and that is the definition of Green's function. So, that is what is the definition of Green's function.

Linear of L acting on G x x prime. So, now my source is at x prime. My response I computed x, right. So, there are two arguments x and x prime. So, because you put the source at x prime, so imagine that I have heater here and I am point heater and I am looking at temperature at different different points, okay.

So, x is at arbitrary point, but x prime is my source, okay. For convenience, sometimes you put x prime is 0, okay. This is hard to carry out x prime for calculation. So, sometimes for convenience, you put x prime at 0, but Green's function is function of two variables, two two independent variables x and x prime. But my solution is function of one variable x, right, this one.

Now, given Green's function, you can easily compute phi, okay. That is the beauty of Green's function that once you know G, then you can compute phi. Now, this I will not derive it. This follows from superposition principle. Pi x is G x x prime rho x prime.

I am sure you you can derive this easily, but this is the advantage of computing Green's function. Once you have Green's function, then you can compute solution straight forwardly, okay. So, this is the definition part and please keep this in your head, okay. Green's function contains lot of information. In fact, as I said, once you compute Green's function, you can compute solution for any charge, any source.

And and we will of course, important point is boundary condition, no? So, Green's function by this differential equation itself, remember I had L operator G x x prime equal to delta x minus x prime. Often this does not give you unique Green's function, okay. This I want to make it very clear. Green's functions requires boundary condition or your solution requires boundary condition, right. If you do not express a boundary condition, then your solution is not unique.

Now, in contour integration or what we are going to solve, this boundary condition I am going to change by changing the poles. There will be similarities in contour integration and I am just going to shift the pole above the axis or below the axis. I will illustrate it, okay. You will see that that fixes the solution by changing my my my similarities, okay. It will become clear when I show some example.

So, one point you should remember is that the equation itself does not uniquely give you G. You need boundary condition and that is in complex integration it is done by shifting the similarity, okay. But this is of course not a course on mathematical physics or Green's function. So, I will do it quickly, okay. So, I will just show some examples and we will move on, okay.

So, this is called Poisson equation, right. You have seen this for electrostatics, you compute potential given charge, but of course I am not putting the minus sign 4 pi, because this is a general equation, okay. You find it in fluid dynamics for pressure computation, you find it everywhere, okay. Poisson solver is really there everywhere.

Now, we will do it for 1D. So, I am just going to do it some examples bit carefully, but after that I will just state the results, okay. So, the Green's function for 1D, so Laplacian in 1D is d squared by dx squared, this one, G x prime, my source is at x prime, okay. So, you should think of is 1D problem, so my source is a plate, charge plate, remember. So, what is meant by one dimension? So, things do not, so this is x axis, things do not change with y and z. So, is assume that infinite plate is kept at x prime and compute the potential for x greater than 0, x greater than x prime and x less than x prime, okay.

So, that is 1D. In 3D, you will point charge and you can vary x, y, z like that, but this is what is meant by 1D. Now, how do I solve it? So, I am going to do this problem, this is very easy to do it, but let me just do it in detail, okay. I am doing it in real space, we will try to do it in Fourier space as well, okay. So, for x not equal to x prime, then the right hand side is 0, so I get d squared by dx squared G x x prime is 0.

Its solution is immediate, right. So, the solution of this is gx comma x prime. I am going to do it for, well, let us put x prime is 0, okay. I am just going to choose x prime to be 0. My force is at x equal to 0, infinite is a plate. So, it is a linear, right, the second derivative is 0, so it will be a1x plus b1 for x greater than 0.

x less than 0, solution can be different, no, because there is a delta function sitting there. So, let us choose a2x plus b2 x less than 0, okay. Now, I demand, okay, so this is a delta function, so I need to take care of that part, but I demand that g should be continuous, like you have done the Schrodinger equation for a delta potential, that is very similar problem. So, I want continuity at x equal to 0, so what will that give me? Continuity at x equal to 0. So, at x equal to 0, my solution will be b1 in the left and b2 in the right.

So, they must be equal, so b1 equal to b2, fine. Now, how do I take care of delta function? I do the integral, so this is my x equal to 0, I integrate both the sides from minus epsilon to plus epsilon, right, that is what is done, so minus epsilon to plus epsilon. So, let us do the integral, let me substitute here, dx by dx square minus epsilon to plus epsilon. So, what is the delta function integral from minus epsilon to plus epsilon? 1. And what is this integral? dx, so x prime is 0 here, so the left hand side will be d by dx g minus epsilon to plus epsilon, correct? So, there is a change in the slope and the change is 1, right, change in slope from left to right is 1, this 1 change in slope.

So, what is this slope in the left? It is a1, sorry, slope in the right, right minus left, no, that is what we do, so epsilon is positive, right minus left a1 minus a2 equal to 1. Now, of course, we have only one equation in fact, this is the constraint and we have four constants, well, we have three constants, so I have the choice, I can choose. Ideally, I should give the boundary condition, if I give the boundary condition in the left and the right, then I can uniquely determine a1, a2 and b1, b2 I do not need 2 because b2 is same as b1, but I right now, I am not specific in the boundary condition, so I can choose my way. So, I will choose a1 to be half and a2 is minus half and b1 I can choose to be 0, I just want the function to be looking manageable and simple, okay. So, the function looks like, so this x, G of x, it is x by 2, right, this is a1, a1 is half, so it is x by 2 and this part is a2 is minus half, so minus x by 2, okay and change of slope is half minus minus half is 1, so this is my function.

Well, we are assuming it goes to infinity, okay and my charge is sitting here. Electric field you can compute, of course, I do not put minus sign and so on, electric field is going both sides, gradient of potential, so electric will go in that way and electric will go in that way for positive charge, okay. So, but my objective is to illustrate the computation of Green's function, okay. So this for 1D, now Laplacian of g x x prime is delta x minus x prime, now Fourier transform, okay, I am going to do this Fourier transform, okay, let's do the Fourier transform, so I think it is good idea to do it from here, G x x prime is delta x minus x prime, because I think you need it in slightly, so I will do two steps, okay. I think some of you who know, expert of Fourier transform can immediately see the answer, but let's do it slightly carefully.

So G x, x prime is a number, what is varying is x, right, the temperature I have measured at different points, x is variable, but x prime is a number, so G of x x prime, x prime is, so integral remember I said dk by 2 pi to the power d, G of k e to the power i, this is the k is a vector, k dot x, this is my definition, right, this is the definition I showed you. Now what is delta function? Delta x minus x prime is integral dk by 2 power, this x is a vector, 2 power d e to the power i k x, k dot x, so this is the definition of delta function, right. So now substitute it in this equation, so when I substitute what will I get? So Laplacian will act on what? Laplacian acting on g, Laplacian acting on this, so Laplacian is function of x y z, it does nothing to this one, nothing to this one, it will act straight on this, so what will that be? Minus i k squared, right, because gradient gives you i k, another gradient gives you i k, so i k dot i k is minus k squared, okay. Now please convince this yourself, I cannot do the algebra in this advanced course, so this is nothing but dk by 2 power d G k by minus k squared, okay, this is the left hand side, right hand side is integral dk e to the power i k x, sorry. So derivative is dropping down, well I am apologies, minus k squared on top, not below, so i k dot i k, so remember gradient of e to the power i k x is i k, e to the power i k x, you can easily show this, and del dot del will be minus k squared.

Now right hand side is 2 to the power d e to the power i k dot x. Now e to the power i k x is a complete basis, you know, by linear algebra, so then I can equate the integrate both sides, okay, is that clear or not clear, right? So that means G k times minus k squared equal to 1, okay. So I am using the function a e to the power i k dot x equal to b e to the power i k dot x, a and b are function of k, then a must be equal to b, okay, that is because e to the power i k x forms complete orthogonal basis, orthogonal basis, okay, that is important. So that means G k is, yes, so Fourier transform of a Green's function for Poisson operator, okay, the Poisson equation, this equation, G of k is 1 over k squared in any dimension with minus sign, minus 1 by k squared. In, my G of x will vary in different dimension, where G of k is minus 1 by k squared, okay.

So if you remember this is very useful, you can see the answer, okay, we will encounter this operator in field theory and, okay, I know this easily, okay, so that is very useful to identify the answer. We will encounter more operators, in fact I am going to do quite a bit of operators in this Green's function lectures and, but this one, okay. What is the interpretation of Green's function in the set? Put your source at x prime, x prime is a fixed number coordinate, okay, it could be in three dimension, four dimension, two dimension, so in high dimension it will be a vector, but in field theory again you do not write vectors, you will find x is written just like that in 3D as well. So these are for shorthand, you do not want to write vectors, so this is what we will do all the time. And the resulting potential at x is Green's function, my source is there, is a delta function.

And in fact it turns out that again once you know this then you can also interpret path integral, okay, so I am going to do that I think today's time, today I won't be able to do it, but path integral of Feynman path integral is nothing but Green's function, I will just show you, we can easily derive it by this method. Thank you.