Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Week - 08

Lecture – 44

This is field theory, you know I did mention that in QED the charge of the electron changes with distance, ok, mass of the electron changes with distance. So, this is what we will do now, ok. I will follow one method which is follow up of what we have done so far, ok. I like this method and it gives the same result, ok. I will mention about other method a bit later but let us just do this method. So, we will normalize both charge and mass.

Now so, so far I was working with free energy, you know free energy for spin system. Now, we are going to change the gears and we will work with 5-4 theory with m is the mass. This in fact I have written this before. So, m square phi square plus grad phi square.

Now and plus this is a non-linear term lambda phi 4 by 4 factorial, sorry 24. Now if you recall this was r naught, there is a pre-factor c naught there for free energy and lambda was called G, ok. So, it is exactly same Lagrangian or C-infinite energy. So, we can use lot of the results which we did before, ok. So, I am going to follow that method which I did with equations, right.

I did it in the class. So, I renormalized m square, r naught as well as lambda. If you recall this, these two are renormalized. c naught was not changed, c naught remained the same. In fact that is why c naught is not a factor here.

We just removed that stuff. So, these two parameters m square and lambda will be renormalized following very similar procedure, ok. Let me say this, we will do only coarse graining, but no rescaling. That is the difference. So coarse graining, when you coarse grain, I have system at very small scale, right.

This is the system. You can think of these are field variables, ok. Now when I coarse grain I get bigger systems, I mean bigger packets, for spin bigger packets or for fields I have I am looking at the system at a larger scale, ok. I coarse grained further, then what happens? Well, I mean there are let us put 9 of them or 16 of them, but now I coarse grain these that bigger blocks. So, what happens to these blocks? The block sizes are increasing, ok.

So, if I just coarse grain, not rescale, my blocks are becoming bigger and bigger, ok. Wilson what he did or what Wilson did was he after coarse graining he compressed it. So, as a result what happened, these big blocks become a same size as this earlier block. So, system size did not increase. All the thing is what we had the parameters or system variables were coarse grained plus rescaled variables.

But for today's lecture I will not rescale. So, my system size will grow and grow and grow and I am going to look at what happens to m squared under this coarse graining, ok. So, rescaling is not required, ok. So, let us just follow that trick. I mean I think that is pretty simple after that.

So, the equation I can derive equation of motion from this. So, I will get this, this equation of motion. Now, we will set this to equilibrium. Equilibrium means where d by d t of phi is 0. So, I am going to re-normalize, I am going to work with this equation, ok.

Now, we work with Fourier space. So, what happens in Fourier space? Laplacian becomes k squared phi k hat and this becomes a non-linear term, right. Non-linear term is convolution. So, lambda by 6 where the factorial 4 with 24 and the 4 came from after differentiation. So, that becomes one 6 and this one you take a stuff, functional derivative, you get Laplacian, ok.

Now, this becomes the convolution. So, phi k 1, phi k 2, phi k 3, this k 1 plus k 2 plus k 3 equal to k, right. That we have done it before k 1 plus k 2, k 3 equal to k. So, like before I am not going to belabor on this point. Like before we will separate the variables.

For coarse graining I said let us use a small variable and big variable where I said that if this is scale then we had k greater and k less, ok. So, when I coarse grain I average over these variables and I keep this, retain this variable, ok. So, this was coarse grained that way. Please recall that discussion which we did before. So, we will do a differential equation.

So, the equation is this, ok. I wrote that before I mean in the previous slide. I did not put a hat here, but ok this is and these are non-linear term which has become a convolution and delta function. So, we divide phi k into two parts, phi less which is in the for lower wave numbers. So, these wave number k and this larger wave number and we are going to coarse grain on this and this was lambda and lambda 0.

Lambda 0 is the wave number of the bigger sphere and lambda is the wave number of the smaller sphere. So, I am basically coarse graining on this in d dimensions, ok and d is will be 4 for the reasons which is obvious that at 4 and above the non-linear term is small in terms of perturbation. So, I can do perturbation that is idea that why we use 4, why not 3. It turns out in 3 dimensions the non-linear term is huge and I cannot do perturbative expansion, ok. So, this is exactly same what we did before.

In fact, I have taken the same slides put it here, ok and so this convolution by the way this term is well I am going to look for phi less, ok. So, I am going to write equation in this regime. So, I am looking for equation for phi less. So, this part convolution has lot of terms, right. So, it could be phi less less, ok.

I mean this is a product, no. So, I can get k by many combination of k 1 plus k 2 plus k 3. So, this also you should know k vector k 1 like that k 1, k 2, k 3. So, here all k 1, k 2, k 3 are less than small, ok. So, they are basically in that band, but it is possible that k 1 is huge, ok and k 2 is also huge, k 3 is also huge, right.

So, that will come here, but sum is k, but k 1, k 2, k 3 magnitude wise can be large. So, that is what you got, but there are other possibilities we can have two large and one small, ok or two small and one large. These are four possibilities in fact for this, ok. Only condition is that k equal to k 1 plus k 2 plus k 3, vectorial, ok. Now, I will do ensemble average assuming that these field variables are Gaussian, phi greater is Gaussian that is assumption we will make, ok.

The same assumption is done in all field theories that we assume the field values to be near equilibrium, near equilibrium or equilibrium is Gaussian, ok. So, what happens to three, I mean all greater what happens to them? Because it is Gaussian, odd order it should be 0, right, Wick's theorem, all order is 0. Next, what happens to, by the way this one will remain as is, this should not change because this is less, less, less, I am not averaging over less variables, this is r as is, ok. So, these two are taken care of. What about this? Or let us look at this, what happens to this? Two less which is not to be averaged, and one greater.

So, less will not average and one greater. So, basically averaging will fall here and should be 0 because average phi is 0, ok. So, that is also 0. But the next one is non-zero term, well this is same as that and the, as I said, ok, so I missed one. So, the one term which is not 0 is this term because there are two phi greater, two phi greater can be non-zero, right.

I mean this Wick's theorem says even order things will be non-zero. So, that is what we are going to compute, ok. So, this is the term, two greater and one less. So, I just rewrote that, ok. So, I think you should see this equation very carefully.

This part I replaced, this goes there and which other term goes there? Two greater and one less, this goes there. So, this cube, the triple order term correlations has combination of two terms. One is this guy which is, looks very similar to the original equation, right. But this new term which is two greater and one less and two greater and one less I take it to the right hand side, ok. Is this clear? No, I just took it right hand side and I am going to compute this using field theory, ok.

And so, Feynman diagram we write like that, two greater and one less, these are five variables. And in the next slide I have, ok. So, this is exactly, this is right hand side and I am, so less will come out, less is not to be average, this is correlation function, right. This correlation function k1 plus k2 will be 0, right or k1 equal to minus k2. So, this is exactly, this is a second order correlation function, ok.

So, that is written as Feynman diagram, this is loop diagram, this is called loop. So, this is CFK, correlation function. And correlation function for phi-4 theory is what? Without, well I mean this is linear term, ok. So, it is perturbative. So, I can use the linear correlation, this k2 plus m2, ok.

Well, of course I need to integrate it, dk. So, this is a correlation function. If your space is that and of course you will get that one, ok. So, this is the coming from here. But it is possible that I can go to higher orders, one order higher.

And so this is what I wrote, integral. By the way, this integral is from lambda to lambda naught. I am doing the sum, I am averaging only the large wave numbers, not averaging all of it. So, this integral is from lambda to lambda naught, ok. So, it is done over this shell.

This is procedure of Wilson, ok. It is very famous procedure and that is what we do. Now, let us look at T2. So, we can go, so I have two greater than one less, in fact this is the Buhler Feynman diagram, ok Feynman that is a very nice way to do it. So, one less I keep here, one less. One greater I am going to, one greater I will keep here.

So, let us call it A, A is here. This B I keep it here, but this C I am going to expand, ok. How do I expand this C? So, C is phi greater, no? Phi greater. This is C, phi greater of K, ok. So, now, well what is this value? So, first order of course I can, I did that before. Two, two of them become a correlation function, but I say well I go to next order.

So, remember equation for phi which is m squared plus K squared phi equal to non-linear term. Non-linear I take it to the right hand side. So, lambda by 6 phi, phi, phi, 3 phi, no? I am not writing k1, k2, k3. So, phi greater is nothing but this is Green's function inverse, right? This is what Green's function is important. Green's function inverse, this is GK inverse, G naught, linear part of the Green's function.

So, we take it to the right hand side. Inverse I multiply G both sides, it is an operator, G naught of K. I just take and this is lambda by 6, phi, phi, phi, ok. So, this is what how I expand and compute the function to next order. Is that clear to everyone? So this is very similar to what you mean you are doing quantum mechanics, Born approximation, no? So, you compute this wave function to next order.

You might have done this. It is pretty similar idea. So, this is the Green's function, ok. This is the Green's function G of K, G naught of K, and I have 3 legs, ok. The 3 phi's, so 2 phi's I keep it less, this is less, less, 1 greater. You understand now? The 3 phi, so these are 2 less.

So, this I call it C, D, E. So, this is C, this is D and E will come from this side E, ok. Now, this B and E will come together. So, this is phi of greater of K, 1 and this phi greater of some other wave number is S1 let us say. Now, when I average them what will I get? Phi greater K1, phi greater S1? I will get a correlation function, right? Yes or no? But K1 plus S1 must be 0. This is the condition, no? The wave numbers must be equal and opposite, ok.

So, this is what we get here. So, 3 less you keep it out. This less, this less and this less are here. 3 legs, this is called 3 external legs are outside and I got this Green's function is coming from here, this g k1, this Green's function, right? Here, this is the Green's function and this is the correlation function, this correlation function here, ok. It turns out for equilibrium system the Green's function and correlation function are same. Well, I should call it g naught, linear Green's function. What is this one? This m squared plus k1 squared and this m squared plus k2 squared, k2 squared, ok.

And the integral is here G k1, C k2, but we did that in slightly in the past. k1 and k2 are larger and k is small, so we can make some approximation which is always done in field theory. Essentially, we basically we say well k1 and k2 are roughly same and we get k1 m squared plus k1 squared, ok. So, here k1 and k2 are not same, right? k1 is here, k2 is not here, but under the assumption that the waves to be disintegrated are much bigger than the waves to be for wave number k, we can get this. So, this I think I did describe it in my earlier lecture, but this is my integral.

Now, the overlapping with this, but this is the integral lambda, lambda 0 d k1 by 2 pi 2 to the power d m squared plus K squared. The earlier integral was there is no square, earlier integral was only single power, but this is 2 power, ok, correct? So, I expanded the system up to second order, ok. Now, let us look at all the equation, put all put this all together. So, these were original stuff, right? This is what I wrote, less less less. In fact, this equation if it is equal to 0, then I say well I get back the original equation, but it turns out you do not get back the original equation if there is some more terms.

So, what are the two new terms? One term was this, right? Well, I am sorry, this should be m squared and this should be node C naught, ok. This is copied from a previous slides, ok. So, m squared plus K squared, ok. And the second term is the quadratic term which is this term.

Oh, I forgot to, no no. There is a multiplication that integral dK by 2 power d 1 over K squared plus m squared square lambda to lambda naught, ok. This part is missing here, this is a term, ok. So, these are new terms which have come. Now, let us compare how, so these terms we should do something about them, so that my equation, the new equation looks similar to old equation. Now, what should I do to the first term, this one? This is a integral, this is a number, right? And is multiplied by phi less.

So, which, where should I combine it to? I should just add it here, no, because both these terms have phi less k in the front. So I should just combine it there. Now, this is a number, so what should it, what will happen to the pre-factors, m squared plus k squared? Which term will it get added to? m squared, right. So, this term is correction to m squared, ok.

So, this is called mass correction delta m squared. It comes straight from the renormalization. Is that clear? I mean this in fact the idea is very simple, there is nothing very complicated, ok. So, this is the correction to delta m squared. Of course, I need to compute the integral and then I will get even better equations, but this is a correction to mass. What about this term, second term? Well, these are so constant, right? So, where should it, what, what should it, where should I combine this term? Now, you can look at this equation, this part, ok.

Ok. So, this is second part. This looks same as this, right? I have a multiplied by a constant. I get a new term which is just constant and this correction is lambda squared by 4 and this integral. So, what should it correct? So, this getting added to lambda by 6. So, my lambda by 6 is getting a correction with lambda by 6 less, ok, the new one is lambda by 6 minus lambda squared by 4 and this integral, right.

So, my new lambda is less than old lambda and how much less? So, this is the new term. It is this factor, lambda squared by 4, ok. So, these are the correction to what? Lambda is, is a coefficient of the non-linear term, ok and this is called coupling constant, ok. So, this I forgot to write, coupling constant.

So, this is similar to electric charge, ok. So, this is a coupling, you know. So, this is a coupling constant which is coefficient of the non-linear term, ok. So, coupling constant is also getting corrected, ok. So, we just rewrite them, we just write here, ok. So, m squared less is m squared plus this correction, this correction and the lambda by 6 is this.

I had written it before, it is just that. So, we just rewrite this part, this one as lambda by 6 equal to, lambda less equal to, I mean 6 will go up and you get 3 by 2, ok. So, this is my integral which, this is the correction to lambda. It turns out if I go to the left, both mass, this integral is finite and positive. So, when you go to lower and lower way numbers, the mass is decreasing and my charge is also decreasing.

Lower way number means larger scale, ok for this phi-4 theory. You cannot say that this will happen for all theories, but for phi-4 theory both mass and charges are decreasing when you coarse grain. Coarse graining means I am looking at a larger scale, alright. Now, let us do the integral, that is what is left now. So, this will, so I will first worry about the coupling constant lambda.

So, what do I do? So, this equation for lambda less is that, right. Now, let us do the integral which I did not discuss in detail when I did this Wilson R G, but I should have, but let us compute it now with more care for, with more care, ok. So, this correction is, I just write this part correction as delta lambda, ok. Now, this d k integral is written as, so this is a sphere, no? So, we write as this integral d k by 2 pi to the power D is written as a, is a shell area, right. In 3D what do I do? 4 pi k square d k, right, that is what we do. This 4 pi is the radius of the, area of the sphere, ok, and multiple of a k square which is coming from the pre-factor.

So, in d dimension what is going to, what do I expect? So, omega D, in fact omega D also includes 2 pi to the power D, ok. So, this is what I have written it here. This part is the surface area and 2 pi D is a, this, this stuff, omega D integral k to the power D minus 1 d k, ok. That is a formula for the volume, this volume element, omega d k to the power D minus 1, ok. So, that is what we have written here, k to the power D minus 1 d k, omega D is outside, and this one I will assume m is much less than k, ok, mass is in fact, mass is in fact we will see that it is close, is a value for lower number, in from m going to, this k going to 0.

I also should tell you that k is like momentum, right, h bar k is p in quantum mechanics and this is all language of quantum field theory, ok. So, h bar k is, so when as k will be also called momentum, you do not have h bar here, but ok, h bar can always be plug in, so k is momentum. So, when momentum is 0, which means I am looking at the system, the free system or at large scales, so that is m. So, m is mass less than k, k going to 0 is m, ok. Now, so we can write this part is k to the power 4, no, k square, square, ok, so this is written here.

Now, this integral will be, can be easily computed, this is straightforward. So, what is this? k d minus phi and that integral is this, ok. Now, I am doing from lambda to lambda 0, so pull lambda 0 out and it is b is, I should write b equal, b is lambda naught by lambda, is greater than 1, ok. Now, you can just do the algebra, this will come like that, ok. Now, please recall that I want D to be 4 or near 4, ok. So, this guy becomes 1, right, D is close to 4, D is not equal to 4, close to 4, you just keep it close to 4 and this guy becomes what? I, in fact, it is conventional in field theory that use b is exponential l, I did describe it before, equations are nicer with l.

So, when I put that this object 1 minus this stuff becomes l, ok. This algebra I will not do it, this is straightforward, exponentially you expanded 1 plus x plus dot dot dot dot dot is that, ok. Now, looks great, no? This is very simple and lambda d, what is lambda d for D equal to 4? Lambda 4, sorry, yeah, no, this is omega 4, ok, no lambda, omega 4. Omega 4 is 2 pi squared d 4 divided by gamma 2 and 2 pi to the power 4 is 16 pi 4, right. So, this pi 4 will cancel with that, gamma 2 is 2, ok, this gamma function 2 2 cancels, so 1 by 16 pi squared, so this is omega 4, ok.

So, I will rewrite this delta lambda, delta lambda is a correction. So, we write this is a d lambda by dl, right, it can go to the left, no? So, lambda less minus lambda is d lambda, so d lambda by dl is that, ok, straightforward. Now, we can write L as log of b also, right, L as log of b from here. So, we write this part as d lambda by log of d log b and this is called beta function, this standard in field theory is called beta function, ok. So, far so good, but can we solve for lambda from this equation? Right, it is a simple differential equation is a, in fact not linear but it is a simple differential equation 1D.

So, we solve it, ok, before I solve it I need to, I need to make a one more change. So, here in Wilson theory I am going from large wave number to small wave number, or I am going from small scale to large scale, but in field theory normally we go from low energy to high energy. So, I am looking at electron, I am proving it from far and I am going closer and closer. So, going closer what is it, what does it mean, going closer? I am not coarse graining, I am doing other way round, I am going closer. So, I am making some of course the intuitive argument, but going closer means I am doing reverse of coarse graining.

So, l is going from other way round. So, here lambda increases, decreases when you go to small momentum, momentum is scale, ok, I am going right now, that is why it is easier to talk in momentum language. I am going to lower wave numbers, no, here lower wave numbers, lower wave number means I am going to larger scales, but I need to do other way round for field theory. So, I go to higher momentum, if I go to higher momentum then L will change sign, right. So, this full algebra I did not want to do it for other way round, but just simple intuitive argument, the lambda increases if you go to larger momentum, the sign will change.

So, this becomes a plus sign, ok, that is it, great, ok. Now, I can solve this equation, it is bit of algebra, but so d lambda by lambda square will give you this, ok. So, just believe me this is what we get, these are, these are momentum for lambda 0, lower limit, lower limit momentum is m, momentum is k, ok, that is this is dimension of k and at the upper limit momentum is p, just simple integral, ok, I hope I do not need to do this stuff, ok. And I can get equation for lambda which is this, this is my equation. So, my charge when I go from lower momentum to higher momentum this is the equation for the charge which changes with momentum, ok. Now, how does it look? Now, can we relate it to something which we might have studied in other courses, ok.

So, let us go to quantum electrodynamics, I am not going to derive this equation which was done by Feynman, Schwinger, Tomonaga, ok, they derived this equation, this object for QED, ok. And of course, that algebra is very complicated, phi-4 theory is simpler, but for quantum electrodynamics you need to work with Dirac equation and then poly spins and so on, it is pretty complicated, but I just take the result. So, this equation was derived by Feynman, for QED and the equation is not so difficult, the equation is this, beta beta function for QED is this.