Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

Week - 07

Lecture – 42

Again, I cannot get into the details part of it, but I will just tell some, I will basically motivate and give the general framework, ok. So, we are looking for the properties near the phase transition, ok. We are not looking at property of ferromagnet or paramagnet, you go close to the phase transition point and study the behavior, ok. Remember that Landau's theory does not, does not work, it does not work according to the experiments. I am exactly going to show you where the problem was, ok. So, one thing I did mention that near the phase transition the system looks like a fractal, right.

I did mention this before. So, fractal means at that point this spins are talking in bunches, of course they are talking to the neighbor, neighbor interaction, but there is lot of interactions in groups. So, we think that system like in a in a in typical stat mech we assume that this gas particles are uncorrelated, but it turns out during the transition there is strong correlations which builds up, ok. And there are big big blocks, so there are blocks of spins which are trying to align up, another block is trying to align down, then another block here and when you go closer and closer these blocks become bigger and bigger and there are correlations between these two blocks.

In fact it turns out near the phase transition or at the phase transition the correlation length is infinite, ok. So, remember the Green's function with r non-zero or m non-zero e to the power e to the power minus m r divided by r, ok. That is what we define, we derived the Green's function, these with m not equal to 0 for the equation, in fact phi-4 equation. Now what happens when m becomes 0 or r becomes 0? So, the exponential part is become 1. Now what is the correlation length according to this equation, this Green's function? Correlation length is 1 by m, right, that m has dimension of 1 by length.

When m goes to 0 what happens to correlation length? It becomes infinite. So, that is what is happening near the phase transition, it is infinite correlation length and the structures of all sizes are there, like fractal, I give the example of fractal triangle within triangle within triangle, so that is the property near the phase transition, ok. So, it turns out near the phase transition most of the properties or most of the functions are power laws. So, it is a Green's function is 1 by r, ok. So, that is what we get and much before all this theory by Wilson there was a person called Widom, ok Widom.

He had postulated the free energy should look in certain, should be of certain form, ok. So, the free energy I think I skipped one per one slide, ok, the slide has disappeared, ok. So, free energy by rescaling we get this b to the power minus d, ok, one slide has disappeared under translation. So, free energy will be look, so free energy of the coarse grained system b less, f less is it is of this form, b is a that coarse graining parameter or scaling parameter. Now, T is that difference in temperature and h is the magnetic field now.

We need h parameter now, ok, because during the phase transition always turn on the magnetic field, turn it off. So, remember what we do for during the transition is we turn on the magnetic field, keep it slightly then turn it off, ok. So, long range order or ferromagnetic order exists without external magnetic field, h is a external magnetic field, ok. So, I will have to set h to go to 0 when I want to measure the magnetization. If magnetization is 0 then I will say well it is paramagnet, if it is non-zero without h then it is ferromagnet, ok.

So, now the question is I want to derive kind of power law of free energy which has a power law form. So, what do I do? So, I will assume that t and b are related, ok. So, t prime and t in fact, we derive some of this stuff, you know under R(g) what happens to t prime to b transformation, ok. So, yt is exponent, ok, this is what is the coming from the denomination group. In fact, Vidham had guessed it that it should be of this form without doing R(g).

So, t prime is b to the power yt. In fact, yt is our R(g) that exponent, ok. So, this in terms of b though if I do in terms of l then I get y of l which I had shown you before, y of l is 2 minus epsilon by 3, ok, not in the b, but in terms of l, ok. So, this is t prime is b to the power, now what I can do is I can get b, I can get rid of b by in terms of t. So, what should I do? So, b equal to t prime by t to the power 1 by yt, ok.

So, that is what is done here. So, we say that, so t prime is, I fix t prime, ok at some point. So, this is t to the power minus 1 by yt. So, I just substitute it here. So, b equal to t to the power minus 1 by t, I just do it here, so I get this.

So, I got rid of b, ok, b is not such interesting parameter, temperature is interesting parameter, magnetic field is interesting parameter, I do not want to see what is coarse graining, all that I do not want to worry about. And similarly, we can get rid of this b by this formula as h divided by, so I can just do it here, replace that b, then you get that. So, now I have this free energy equation, a free energy function has only two variables, no b, T and h. In fact, it turns out this part is a function, but the power law form is here, this is a power law, t to the power d by yt, D is a dimension. So, in fact, since this is a fractal property, lot of system properties of phase transition can be derived by these two parameters yt and yh, ok, that was Widom's idea, ok.

Lot of things happened in Cornell University, ok, so Fisher, Wilson, Widom, all of them were there in Cornell in 70s, this is a big revolution, so this really caused lot of flutter in physics. In any case, so this is what is the free energy form. Now, with this free energy we can get some, I am going to demonstrate by one function, magnetization, ok. Magnetization is, so this you might have studied, I am sure you have studied this, near phase transition it goes as T minus Tc to the power beta and anyone knows what is the beta exponent for Landau theory? So, if I am near the phase transition, I decrease the temperature below critical temperature, then magnetization grows, ok, at Tc it is 0, but when it goes below Tc then magnetization grows, but does it abruptly jump to some finite value? It does not, it grows as T minus T to the power half, beta is half. In fact, you might have seen this thing T, Tc and it goes, magnetization is that curve, ok.

This is a very important curve, this is from Landau theory and this is a parabola which is exponent half, ok. Now, let us try to derive it from, let us connect this to our free energy. So, what is magnetization in terms of free energy? Is delta f by delta h, right. Derivative free energy with h is free, is magnetization. If I take the derivative what will I get? So, T, I am not taking derivative with temperature, T will remain as is, yt, but it will be f prime and h is there, so I take the derivative, there is going to be T to the power yh by yt, ok.

Now, f prime is function this one h divided b.. Now, we set h equal to 0. So, this f prime is a number. So, my M is basically T to the power d minus yh divided by yt, ok. So, what is beta? Beta is d minus yh by yt. So, if I know yh and yt then I can compute beta, ok and they will come from R(g).

Got the idea now how it connects with Rg and this is finally done by Wilson, Wilson and Fisher. I did it for one of them, but you can do it for more functions. So, these are the lot of functions in stat mech, heat capacity, magnetization, magnetic susceptibility, field dependence and chi. So, there are many exponents alpha, beta, gamma, delta, nu and all that, ok. And eta is coming here, ok this eta.

Eta is safe to set it to 0. There are more advanced theories we will give eta, but we will right now we are safe to keep it to 0. You can see that for r much less than zeta I get a power law. Zeta is a kind of a system big size and below that size I get a power law Green's function, ok the way I just mentioned and above that it is there is a exponential part, ok. Now I cannot I mean by the way what I did can be done for other functions and you can easily do this stuff.

So, free energy second derivative, first derivative, with temperature, with h and we will get all that, ok. Susceptibility is the second derivative, you know, a free energy with h squared I believe, right delta M by delta h. So, you can do that. So, by all that operation we get all these exponents in terms of yt and yh, ok. Now we already computed yt and remember that I said that yu is irrelevant.

I said now that non-linear term is the system goes towards the fixed point and eigenvalue is negative for yu. So, the non-linear term is not contributing to the exponent. Of course, it contributes to the physics, but it is not computing contributing the exponent, but yh is yh and what is yh? I derived yh much earlier, yh is t minus, ok, actually the slide is disappeared, ok. So, yh is d by 2 plus 1, ok. So, this is derived from free energy rescaling, ok.

So, free energy the integral of free energy of the scale variable and before R(g) after R(g) must be the same and that gives you the D by 2 plus 1, ok. Now we got all this. Now this is the final slide, ok. So, if I want to go to D equal to 3, let us first do equal to D equal to 4, ok. D equal to 4 epsilon is 0, ok, epsilon is 0.

Let us focus on beta which is what I discussed. So, D equal to 4 yh is d by 2 plus 1. So, what is these 4? So, yh is 4 by 2 plus 1 is 3, 4 by 2 plus 1 is 3 and so D is 4, 4 minus 3 is 1, right. So, this is Gaussian, 4 minus 3 is 1 divided by yt, yt is 2, no, for Gaussian fixed point.

yr is same as yt, 2. So, beta is half, ok. So, these Landau fixed point, beta correspond to Landau fixed point, beta is half, ok. Now let us do it for 3D with non-Gaussian fixed point. So, for non-Gaussian fixed point, so I am moving for D equal to 3, ok. So, you should not put this capital D as 3, that is a trick.

So, you should put this capital D as 4 only, but epsilon is 1, ok, that is the idea. So, this remains d is 4, but I choose epsilon is 3, ok, this is perturbation order. So, this small d is 3, no. So, this d is 4, I keep it 4, minus y H is 3, but yt, what is yt? y t was 2 minus epsilon by 3, ok. What is epsilon? My space dimension is 3 now, so epsilon is 1.

So, 2 minus 1 by 3, which is 5 by 3, yt is 5 by 3. So, 4 minus 3 divided by yt, yt, was it yt is 2 minus epsilon by 3? Yes, it is correct, 5 by 3, so 3 by 5, no, that is not correct. So, beta is half minus epsilon by 6 by this, this formula, ok. I think I am making a mistake here. So, a beta from here, ok, this bit of algebra problem here, but if you can derive this formula from here, so beta is half minus epsilon by 6, ok.

So, 4, ok, this done again using epsilon small, but set epsilon 1. So, if I put epsilon equal to 1, then half minus one sixth is, it is 0.33. So, half minus one sixth is 6 divided by 3 minus 1, it is one third.

So, this is one third, ok. If I do this, this calculation was done to first order, ok. You can do this calculation to higher order, one more loop, ok. Now, that is more difficult calculation.

If I do to higher order, then I get 0.340. This is more work, I mean this is what more difficult work, which I will not do in this course. So, then we get 0.340. If I use numerical solution of Ising spin, 3D Ising spin, then I get 0.327.

This is a numerical work. So, epsilon second order, epsilon squared order is reaching close to this answer, ok. Experiments are away from half, but it is close to 0.32, 0.33. These are not, I mean the numbers vary. So, this is not like I can get up to 6 digits, nothing of that sort works for phase transitions. But we are finding that we are getting closer and closer to the numerical or experimental value. So, that means this theory of phi-4 theory and exponents coming from this calculation is correct. We are going in the right direction. People have done this calculation of two fourth order, ok.

So, there are more Feynman diagrams and more calculation, but it can be done and that is what we get, ok. For 2D, this epsilon going to 2 is not, is a, your parameter, small parameter must be small, right. epsilon 1 is already questionable. People will not believe you that you say epsilon equal to 1. So, but epsilon equal to this exact calculation, this by, ok, this calculation is exact, ok.

So, we get this one eighth Onsagar, ok, Onsagar calculated this. So, 2D Ising spin is a difficult problem, but solved by Onsagar and he gets beta is one eighth, ok. These numbers you can see they are coming from mathematical model, ok.

So, they are not 0.32, 0.33 like that and, but 3D Ising spin has no exact solution, ok. Nobody solved it yet and so, these are the exponents. Now, you can look at other exponent like alpha is 0, but the real number is 0.110 and my, the field theory calculations are not so close, but ok, it is getting there, ok.

So, this is the triumph of phi-4 theory, ok. It is able to get the correction to some degree what is required to explain the experimental results, ok. So, this table is from our book Field Theory for a Mature, Gifted a Mature, ok. So, Wilson-Fisher fixed point gives a good agreement with the experimental results, ok. The summary, so, we performed Perturbative RG for phi-4 theory first order. Some system parameters renormalized like R renormalized and R is a relevant variable, U is not a relevant variable for RG, renormalization group, but

H is a relevant variable, ok.

By the way, H it does not get renormalized, but H is a relevant variable because of scaling, under scaling H gets this d by 2 plus 1. Now micro, so, what does it tell you? Ok, one more important point which was another very important triumph of this. This exponents which I showed you the table, these exponents for paramagnet to ferromagnet transition, we get numbers of that sort, also for liquid to vapour transition, ok, water to vapour transition, but at what critical temperature. So, if you look at the phase diagram for water, temperature, pressure, ok, I hope I got the axis right, ok. So, if you draw this line, water vapour line, so, high pressure is water and vapour, ok.

So, if I cross here, then I need a latent heat, no. So, this line if I cross from water to vapour, I need to supply some energy, but at some point this line stops, right, this line stops. And we at this point latent heat is 0, you do not require any latent heat to go from water to vapour. If I am here, then this phase, this at this region, water and vapour are interchangeable, you will at that stuff, there is no difference, there is no phase difference between water and vapour, ok. So, the Wilson theory works for water vapour transition at this fixed point, ok.

And this is called second order transition. Why is it called second order? So, the first, so, free energy delta F by delta T is entropy, right. So, here entropy changes at this value, at this value because water and vapour have different entropy, right, more or different order. But at this point what happens, at this point? Entropy is latent heat is 0, so entropy is not changing. So, the first order, so this entropy is continuous across that fixed point, across that point, transition point. It turns out delta S by delta T which is connected with specific heat is discontinuous, ok.

So, the second order quantity is discontinuous and that is why it is called second order. These are called first order transition. When there is a latent heat is involved, first order. It turns out the properties of phase transition at this point is same as paramagnetic to ferromagnetic transition at this point. This point is exactly same as that point, ok, that is a very important result, ok.

And the exponents for these two systems are same. And why is it same? Because they are governed by large scale variables. So, the phi-4 theory works for both these systems. You are not worried about the details of the system, ok. I am not worried about what happens interaction of spin at small scale, does not matter.

Or spin is in a triangle or is in a square, does not matter, ok. So, details of the physics does not matter. It depends on how the big blocks are interacting with each other. That is why blocking or coarse graining is important for this, ok. So, when coarse, keep coarse, graining, then you are not looking at individual behaviour, but looking at a global behaviour, a block behaviour, ok. So, in a population, well I do not want to make their analogy, but yes, ok.

So, individual Indians may have different behaviour, but Indian as a whole, you can say that there is certain property of in the nation, ok. So, that kind of property is taking, is giving those those exponents, ok. So, details do not matter, that is what is comes from this. It also has a very important implication which according to me I like that.

These property cannot be explained by microscopic physics. Phase transition is unique feature which is not explainable by microscopic physics. So, saying you know that reductionist approach will work for all, if I know, if I understand electrons, protons and quarks, I understand everything, not true. This is unique, this is one of the systems which cannot be explained by understanding electrons. However, great theory you may have for electrons, it will not work.

This requires this coarse-grained picture 5 4 theory to explain, ok. And so, this is called universal theory like, we have universal theory of gravitation, right, gravity force between this block and this block between two planets, between two stars or galaxy, there is a same theory, right, same law. Same way, theory of phase transition with 5 4 theory is a theory of phase transition of second order, ok, is a universal theory, ok. So, it does not matter whether I solve Ising spin or liquid vapor transition or Heisenberg spin, if it is second order then it should follow this property, ok. And so, I, this is one of the remarkable theories of last century, ok. The first order theory has a different behavior, ok, because it is a different theory.

So, in fact, first order transition is given mostly by phi-6 theories, ok, that shows hysteresis and other stuff, ok. But that is a different property, it is not same as phi-4 theory. So, physically when you go to larger and larger scales, how the system is behaving differently, ok. So, that is a, so this is like under rotation, you know, we rotate the object by 10 degrees if it is not sphere then you know it is changing. Similarly, system change when I go from one scale to next scale, ok, and next to next, ok.

So, it is known, right, my body if I go deeper and deeper I will see different different things, right. But it turns out during the phase transition I can get a fixed point, what does it mean? System is remaining unchanged when I go to one scale to the next scale, ok. It is like fractal, indeed it is like a fractal, where system is unchanged when you go to scale to scale to scale to scale. Of course, in this room if you look at random gas, you keep going to scale to scale it will remain same. But it is a trivial system where it is a zero correlation, you go to any scale it is the same.

But that is non-trivial system where this correlations or the interactions are look roughly the same when you go to bigger and bigger scales. So, a small scale spins of this blocks are talking, many blocks. So, you go to bigger scale then spins of bigger sizes are talking in the same manner like at a smaller scale. So, that is the property of fixed point, ok. Now, so the question is if you just know the fixed point that does not tell me the answer, ok.

How to approach the fixed point that is where you get magnetization. The beta exponent is how you approach the fixed point, right, that is delta f by delta m. Now, m is t minus tc to the power, so when you go to t c then how does magnetization change. So, that comes from the eigenvectors. But it is beautiful know that eigenvector is giving us that property. It was bit of work to compute that those eigenvectors, it required field theory, perturbation Feynman diagrams but once we had the eigenvector it was I mean this is just piece of cake after that right.

I mean of course, it is a greatness of those those guys who discovered it. But those two by two matrix, ok, that has all the properties of phase transition. It is not as obvious as Newton's law for gravity but it is that. Now, when you say mass renormalization of electrons, ok, that is very similar idea which we will do it in the next class. I will do phi-4 theory because a phi-4 theory we can do mass renormalization but in QED it is more complicated.

Now, I need to I think I will not get deeper into that. I mean Dirac matrices and it is very complicated algebra but you can get the idea. Here I do not rescale back. So, the second operation will be avoided and now see how this R is changing with scale and that will give us mass renormalization, ok. And so mass changes with when how I look at it, at what scale I look at it, at what energy I look at it.

So, this operation is gives you that. So, this scaling know I think all of you are kind of familiar with rotation operation, translation operation but this is scaling operation. Thank you.