Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

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Lecture – 33

So, this one phi k phi minus k with free field, so this denoted by a line. So, this k is here, wave number k and minus k and this is a Green's function in just a line. These are our Fourier space and in this expectation value, in fact, this is does not require too much of work. From what we did for the Gaussian integral, you will get this to be, in fact, I did it for earlier know. So, quadratic will be just the temperature by E k. In fact, just write this k square plus m square.

This is, let us just keep it simple k square plus m square. This E k know, this is this E k and beta is 1. I derived it know, so that Green's function is equal to that. We have done this before E k, 1 by E k.

Now, four point correlation, so we will have terms like phi, phi, phi, phi. S interaction to 0th to first order will be like what? S interaction to first order, no 0th to first order. What is S interaction? It is sum product of 4 phi's and u by 6. This is a sum of phi k 1 phi k 2 like that. This is my S interaction to first order.

Now, I need to compute with 0, well put expectation outside 0 with free field as a basic background. So, that is what we are writing. There are four fields k 1, k 2, k 3, k 4 and this is called vortex and this written as, well I have written as u by 6. This is a vortex, the four of them and this is u k, k 1, k 2, k 3, k 4. Notations are bit changing here and there, but you know this part.

So, this is our quartic interaction. When you do the calculation, you will see how to compute them. Now, we are getting to the real calculation. Now, so my Z by Z naught which I think I have, so it is a linear, either side is a linear series, Taylor series. So, 1 minus, so there is a minus sign sitting there, you know there is a minus sign which I did not worry to write, minus beta h minus beta.

So, there is a minus sign, so there is a minus sign. Now, this one I need to compute and get a number. Now, S interaction there are four fields phi k 1, phi k 2, phi k 3, k 4 and I need to get a number. So, I need to do this integrals k 1, k 2, k 3, k 4. So, the idea is to do the Wick's theorem.

So, that Wick's theorem is playing a role here. So, we have phi k, this four phi's. So, I write this product of two phi's and two phi's and the three of them, right, A B C D, A B C D, A C B D and A D B C. So, this is two point correlation. So these two guys, they have different wave numbers, this k 1 and k 2, right? You merge them together.

When you merge them together, then the wave number must be equal and opposite to merge them to get a number. I have phi k 1, phi k 2. Now, if I want to get a number out of this, then I say k 2 equal to minus k 1. Then this becomes mod phi k square and you integrate with k. So, this is putting this k's and merging them.

So, this is the one of the terms. So, I had four sums, four k's, k 1, k 2, k 3, k 4. So, I write this as sum of products. So, phi k 1, phi k 2, phi k 3, phi k 4. Now, here k 1, so this is non-zero only when k 2 equal to minus k 1, otherwise 0.

Yes or no? Exponential i k 1 x, exponential i 2 k x integrate with x is 0 unless k 1 plus k 2 is 0. So, they must be having the same equal and opposite wave numbers. So, this is, so a delta function will come and one of them will go away. Here a delta function will come and one of them will go away. So, you will get two integrals.

So, this is basically, this one is two integrals d k 1, d k 3 and two phi's will give us k 1 squared plus m squared and k 2 squared. k 3 squared plus m squared. And this is full integral. I mean this is a vector integral d d x, three dimensional in 3D, two dimension in 2D or four dimension in 4D. So, this is a number.

So, this object is sitting there and this is called Feynman diagram. Instead of writing all this, in fact writing, in fact you should, you have to start from here and then you write this, well not write, write this is a, well first put four of them, then put two of them, then we using Wick's theorem and then do the integral, all that this is shorthand. So, k and minus k, so two of them are contracted and k prime and minus k prime, then again contracted and I am summing up. So, I should write k and k prime if you like, and k prime and k.

Mass is the same. And why this factor 3? For Wick's theorem, there are three possible ways to get the quadratic and that is factor 3. And so, and there is only one u, u is the interaction term, this is that u and I am writing e k and e k prime. So, this e k and this e k prime. 3 is coming from here and factorial 4 is, 4 is the original definition u by factorial 4 and this is how we get it.

I hope this is clear. So, we need these kind of integrals in future. Now, this is for, now this is some more discussion. So, idea is when you have interaction, then how do we compute the quantities, either correlations or free energy, you know. Given z I can compute free energy, you know, log of z is free energy minus log z. So, that will also involve integrals, because S interaction is sitting there and I cannot compute partition function with S interaction.

So, some of it is a theorem. So, I am just going to state the theorem for completeness. So, if you do this z by z naught, I wrote, there is no correlation sitting there x, but this is infinite series. Infinite series will involve lots of terms. So, this one we did in the, So this is a number.

So, I demonstrated in the last slide that to first order, I need to get a number by doing this. This is called vacuum diagram. You get one more. So, you had one minus that, that was the first term. You get more terms, some of it we will show you bit later, that you get terms like that.

So, it has a two vortex. There could be some terms plus, terms like this. So, there are two vortices, but four integrals, 1, 2, 3, 4, disconnected. This is a short hand. If I want to write in terms of expressions, it is going to be huge, lengthy and that is Feynman's idea that you can represent these sums, rather this full formulas, that means, well full formula in terms of the diagrams.

And that is why it is become very handy and very useful and very popular. He is becoming immortal, Feynman diagrams. Schwinger computed them by just writing all the Green's function and that is very long. So, this 1 plus all vacuum diagrams, so this could be diagrams like that. There could be three of them or four of them like that.

It turns out the theorem called cluster theorem, we can write down, if I do the logarithm of this, take a log of this, to compute free energy I need to take log anyway. If I take a log of that with a minus sign, this is all connected vacuum diagram. What does it mean? So, this is a connected vacuum diagram, but this unconnected vacuum diagram. There are two unconnected diagrams, two separate, there is no connection. In fact, there are four of them, but this is connected.

In fact, this should have been added like that. I need a number, so I cannot have open lines, open lines are not allowed. You can have more complicated closed lines like this and like that, so this is possible. So, then this the theorem which I will not prove it, if I take a, so this log of that is sum of all connected vacuum diagram. So, this one will involve terms like this, terms like that, terms like this, but not this.

There are huge number of terms, but we keep, we want to minimize them. I mean, you do not know, you will get, in fact, it is really, you can get really crazy, because maybe you can write down these diagrams, but then what are the pre-factors? And it is a pretty complicated, well, it is a mess actually. So, it turns out this theorem says that this is equal to all connected vacuum diagrams. So, this covered, this is shaded picture. So, we do not have to compute the unconnected vacuum diagrams and this is called Linked cluster theorem and we use it for, in fact, this is what is used for computing the various quantities.

In some sense, you can say that it is basically, if I have this diagram, if I do the exponential of that, you can treat it as a number. If I exponentiate this, then what will I get? Well, just put 1 plus this. I get infinite series, no? So, I will get 1 plus this 1 plus 2 of them plus 3 of them. So, when I take a log, I do not need to compute all this, I just need to compute that. So, I think very crude way to say why I am taking log and I am using only the, I am ignoring this unconnected diagrams is because of this.

When I take, if I take exponential of this, then I will get all these other diagrams, which we do not want to compute. It is already too messy, but you want to create less, less work. Now, so this proof is by Replica trick and the book I am taking is by the book of the book of this from this Kopietz. This is functional renormalization, the title is introduction to functional renormalization groups. If you want to dig in, you can, you can look at this book.

This proof is quite complicated. This is Replica trick is by Parisi, who got the Nobel prize last year. Now, we can compute correlation functions. So, this phi k is, if I want to compute this, then I have these legs are outside. So, remember in the z by z naught, I had this loops are closed, I get a number.

Here, I will have legs out sitting outside. So, two point correlation will have two legs outside phi k and phi minus k and these are summed. It will become clear when you do the real calculation. In fact, this loop is same as what I wrote 1 by k square plus m square integral d k, but, ok. So, two point correlation will have two lines and four point correlation, there are two phi's and will have four lines, ok. Four lines is one and it could be like that, four lines.

See the four lines of phi k 1, phi k 2, phi k 3, phi k 4 will have four lines and the rules what should the pre factor and so on. So, we will do simple calculation and I will show you how you get the numbers. So, example I showed you how to get three, right, factor three. Now, we can also compute these objects, these again that link cluster theorem. So, this is, this one can be written as sum of all open diagrams, these open diagrams multiplied by 1 plus all vacuum diagrams.

It has these set of many, many diagrams, but, if I divide this by our partition function, then this, these guys will disappear. And so this Green's function after division, when I, well of course, Green's fiction that definition is by after division, no. So, I need to divide by the partition function and this does not include that, this part is not included. Here only all open diagrams without vacuum parts, ok. So, in the previous slide for two point correlation we have one, other vacuum diagram will be this multiplied by this.

So, this diagram is not included. We need only this, all open diagrams without vacuum parts. So, this is vacuum part not required, ok. So, this is a very handy theorem which gives you answer with less work. So, I want to just give you a little glimpse of what is this perturbation about. The problem is that in this infinite series, the terms which were expanding, they need not be small, ok.

They are typically not small. Still we get meaningful answer and that is a mystery which is not clear to the best minds. We have some basically is more like engineering tricks which we apply, ok. So, let me just show you what is happening here. So, I want to expand this 1 divided by 1 minus a.

So, I can do a Taylor series. So, this is a plus a square and dot dot dot dot dot, right. This is a Taylor series, ok. This is valid only when mod a is less than 1. Otherwise this is series is not valid.

Now, let us solve this equation x equal to 1 plus a x. Assuming a is less than 1, then I will do perturbatively, ok. So, how do I do perturbatively? This is standard trick. So, x 0, what is x 0? Zero th order.

What is the zero th order? 1, right. I ignore first order. I keep the lowest order term x 1. In fact, the idea is in this equation you put instead of this x you put x 0. This is the first order or you call epsilon order, you know, you might have seen epsilon order.

It is 1 plus a. Next order x 2. So, I put x here, I put x 1. So, I get 1 plus a plus a square. I keep going. In fact, I can see what if I just keep many, many terms and I make a guess that yeah, maybe this x is looking like this, right. So, I can say well x is essentially I can make a guess that it is that, correct.

But this valid only for a small, correct. But you see this I can solve this analytically. What is the formula? x equal to well I mean this straight forward 1 minus a. So, we did all this labor but ok, somebody is well why are you doing all this you can just compute like that. But this formula is valid for any a, right, valid for any a.

But this series is valid only for small a, a less than 1. So, what do you do in field theory? Well, we say that well let us assume this to be small, a x to be small and do perturbatively. And then say well maybe this is also valid when a is large and we are hoping that our result, this result is true when a is large as well, ok. And so that is basically I would say this my take on what is being done because the term which you has a so in fact you can see it here the phi 4 which is phi k 1, phi k 2, phi k 3, phi k 4. When phi is phi may be small but there are integrals, no? This is integral and this integral could be huge number.

So, it is not justified. In fact, there are lot of objections. Dirac was very very uncomfortable with this. In fact, he started some of these ideas, but Dirac said this is not good. This is not I mean even engineers do not do this and if we faces or mathematicians are doing it. So, but it works to some degree and it is been very successful like QED there are very good predictions for lot of atomic physics phenomena. So, this works and I think that is why there is a faith that this works.

But by the way some places it does not work. I think I will explain that bit later that QED is ok because the fine structure constant which is expansion parameter non-dimensional expansion parameter e squared by h bar c this is less than 1. Anyone knows the answer what is e square by h bar c what number non-dimensional number? 1 by 137. It is 1 it is less than 1, but it does not mean that your expansion parameter is less than 1. Well expansion term I am doing this it is expansion S interaction term is not small typically, but for QCD the interaction parameter g is order 10.

So, QCD people do not even do this perturbation they will say will need non-perturbative techniques which are not well there very few non-perturbative techniques. So, that is where the things are, but we are happy with whatever we get, but it is not a settled problem and is not a very happy situation that thing seems to work and we believe it, but there are mathematical issues. With that we will close for today. Thank you.