Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Week - 01

Lecture – 03

So, this is the real x, any imaginary x, so pole is shifted up, right, x equal to i epsilon, epsilon is positive, okay, so pole is shifted up. Now nobody is an object, okay, I can do the integral along a line because I am not, the pole is singularity is not on the real line, or not on that x axis. Now we take epsilon going to zero limit, so this is what we will do, so this Cauchy's prescription, okay, so this is what I have drawn it here. So basically, well, this origin is still a problem, so when epsilon goes to, tries to hit the origin, we change our contour, so I go like that, that is what I am going to do, okay. So, I am going to define few more things, just hold, okay, so this is what, this is what I am going to do the integral. So, we redefine our integral, we rewrite our integral in the following manner.

So, the first one is, first one goes from A to minus infinity, A is somewhere here, okay, so A to minus delta, so this is the first part, this one A, okay, A is here, then minus delta to delta is this integral, I am avoiding that origin, okay, though I have shifted the pole, but basically pole can go very close to the origin, so it just go below. So this is B and the C is delta to B, okay, this is C, okay, so we rewrite like that. Now we will again do bit of juggling around, so A to minus delta and delta to B, A and C together, we put it here and let delta go to zero. So my delta is going very close to the zero origin, okay, so this integral is now defined, okay, I am avoiding the singularity, okay, and this is called principal value in fact, okay, and this part, I am going to compute this part by complex integration, okay, this.

So right now I generalize the X to a complex variable and I am going to do this second part using complex integration. So the A plus C is called principal value P, okay, P is, P stands for principal value, okay, this principal value and which is I am going to show in the next slide that it is a meaningful number. It is a limit of course, I am avoiding the singularity but I go when delta is tending to zero, okay, and this part I am going to compute separately, is that clear? So that is the definition of principal value, principal value avoids the singularity from both left and right. Of course you may say what is the guarantee that principal value is a finite number, is it converging or not, okay. So there is simple argument which is a nice argument again by Cauchy.

So look this, let us get this picture, now I can expand this fx around origin, singularity is at x equal to zero, so expand fx around origin, so it is f0 plus x f prime 0, okay, we need only to first order. Now f0 by x, okay, so we basically get f0 by x plus f prime 0, okay, f0 by x dx plus f prime 0 dx. Now problem is near the origin, now near the origin you can see that f0 is a number, right, and f0 is a finite number, we assume that f0 is convergent, f0 has no singularity or fx has no singularity, okay. So f0 by x is a odd function in x, right, f0 is a constant. So this integral around minus delta to delta, this one in this singularity regime, they almost cancel each other, you understand, dx by x, so this in fact, this part you expect some problem, right, there is a singularity, but I know that this, this part is basically going to cancel each other from left and right.

So this principle value, when you take the limit delta going to zero is going to give us a meaningful number, is convergent, so cancel at both sides for small delta and my answer is finite, okay. And in fact, we will see that you will find this integral very, very often in field theory, okay. So singularity is not a problem and we can get a finite number. Of course, in field theory we want to get a finite number for mass, charge, coupling constant, so we need that this prescription. Is that clear, the definition of principle value? Okay.

Now still we need to do the other integral, one which is going from the bottom, right, I mean the complex, complex integration. So this is what I had in my previous slide, this one, minus delta to delta. So I have to take two limit, delta going to zero and epsilon going to zero. So from the top epsilon was sitting here and my contour was this one, this was my contour, no, this one, this one. I think I have this picture here, no, I don't have a picture, okay.

So let's do this integral. The answer is i pi times f0, okay. How do I get this? So it's quite straightforward. So I am going to draw this again. So this origin, this is my epsilon and this is minus delta and this is plus delta.

I put a semicircle at the bottom. Now i x is a complex number now. So we write x as R or epsilon, no, delta, delta e to the power i theta, right. Delta is the radius and theta is varying from what to what? This is zero and this is pi, okay. Or so I am going from, well, basically I am going from the bottom pi to zero, right.

So let's write this. f x i expand it as f0 plus f prime next term, but other part will be going to zero. So fx we write as f0 plus x f prime zero. We divide by x, divide by x. So this part when I take dx, the second term is zero, right, where delta going to zero, f prime is zero is finite.

So all other terms will be zero except the similarity. So f0 by x is a problem term, correct. So this is what I have written f0 divide by x minus i epsilon dx. Now I am telling epsilon go to zero limit. So we write this part, so let me write it nicely here, minus delta over delta dx will be delta e to the power i theta, but theta is a variable now, right.

So i d theta, fine. And x minus i epsilon, epsilon is going to zero. So this is approximately delta e to the power i theta. This epsilon is going to zero, so it is just x. So let's substitute it here, delta e to the power i theta and above is delta e to the power i theta i d theta.

I have to do at the semicircle. So these guys cancel and I get i d theta. Now semicircle, I go from pi to zero. Now will it be positive or negative? It's a counterclockwise, so positive and d theta goes from pi to zero or well minus pi to zero you may call it, okay, so minus pi to zero. So this i times pi, okay.

This is how a standard Cauchy integral, that's what we do, normally you might recall this. So if often we do it for the closed contour, but here we need to do only semi-contour and f0 is sitting outside, right, this f0. So my integral, this integral for that function, this one is i pi times f0, okay. So my integral, this integral from A to B including the singularity is written as principal value plus, plus this part, okay, i pi delta. So this coming from singularity, okay, this is the contribution from singularity.

In fact now you can basically memorize this that if I see integral of this sort, then I just had to, just had to write principal value plus i pi f0, is that okay? Now of course we can make the change, sometimes instead of minus we get a plus here. My singularity is not above but below, right, because x plus i epsilon, singularity will be below and then derivation is exactly same, you have to close from the top, right, for this particular case. So you can derive it in a very, very similar manner and you can derive other formulas, okay. So, shorthand, instead of writing the dx we can just write shorthand, this one, x, I mean, I am just avoiding the integral, okay. So you write 1 divided by x minus epsilon is principal value of 1 by x plus i pi delta.

This is with a plus sign, you will find that I have to do the integral at the top and if it is going clockwise, so that is why you get a minus sign, okay. I can easily convince yourself, I cannot get into details of this but please go through it when you are, I mean, you just spent some time on it. Now instead of x I can have x minus y. So here my singularity is at x equal to 0, right, but I can have singularity at x equal to y or other, right, okay, so this is the case, then this is just straight forward, this formula, okay. So we will encounter this kind of thing very often.

Okay, now let us do an example. Now I have to compute this integral, okay, x plus i epsilon and there is 1 plus x power 4, okay. Now singularity is at x equal to minus i epsilon, right. Now so this is f of x, okay, so what is the answer? Okay, so let us look at the principle value. So I just write, use this formula, right, principle value of this 1 by x, 1 plus x4 dx minus i pi into f0 and what is f0? This is f0, this is fx and put x equal to 0, so answer is 1.

So this part is straightforward, is minus i pi, okay. Now what is this value? Can somebody make a guess? Well, not guess, it is straightforward. Okay, this is 0, right, because this is the odd function, you know, this is even function and x is odd function, so this integral will cancel from both left and right. In fact, we do not need to compute it, this is 0. So this integral is minus i pi, okay.

So you see, I mean, you can use this formula and get the answer quite easily. Okay, the second problem is slightly more complicated and this is often done in regular mathematical physics course, sin k by k dk and we will find it right now in the today's class only, I will show you that you will encounter this in Green's function computation, okay, all the time. So we want to compute this, yeah. So what will I do? Let's write sin k as e to the power ik minus e to the power minus ik. You will get this PPTs, so no need to really write down all the steps, I will share this with you, but just focus on the idea.

So by the way, this integral is 0 to infinity. Now I would like to do minus infinity to plus infinity, that's what my objective is. Okay, now you can see from this line, I can make a change of, well, this is minus ik, right. So this part, I can make a change of variable minus k to k prime. Okay, so this becomes exponential ik prime, okay, right and this k will become minus, okay, and this minus minus will get absorbed.

So I can, this part, this with the minus, exponential minus ik part is nothing but will extend this integral from minus infinity to plus infinity, that's a negative part. Alright, it's just in fact straightforward from here. Make a change of variable to minus k equal to k prime for the second part and you will get this. So I got minus infinity to infinity 2 pi dk by 2 ik exponential ik. Great, so it is now it is of the form of Cauchy principle value form, right, because exponential ik pi k.

So now I can use that, so I am now going to do this integral in a, with using contour integral. Okay, so this is, okay, now my answer is basically at k equal to 0, sine k by k is finite number, in fact is 1, right, sine 0 by 0 is 1 limiting. So I am avoiding k equal to 0. So when I say that give me this answer, I am not including k equal to 0 because that is a singularity. So in fact my objective is to get the principle value, avoid x equal to 0.

But it turns out you can take a limit, so this is basically the function goes like that into, I mean oscillating function. Tan x by x is that function, right, I mean it goes to 0 for large x. Now I am avoiding it, this x equal to 0, but this part when delta goes to 0 is basically 0. So that is why we write 0 to infinity, but we are, we have in our mind that I am taking limit del, my lower limit to going to 0, not equal to 0. Now I need to compute the principle, this one.

Now there is a trick by contour integration. So I am going to compute my integral like this. My objective is to do this, this integral, this plus this, right, that is the principle value. Now I am going to make a close contour, this red line plus this one plus the, on the top. Now I did in the last class, my Jordan Lemma, that this integral, because I had to close from the top and this integral will be 0.

The semicircle, large one is 0. So my close contour, the close contour, this, the four parts, 1, 2, 3, 4 it turns out that the large circle is 0. But overall it should be, the whole integral should be 0, right, because singularity is only at k equal to 0, rest all the place it is analytical. So as long as I have close contour, avoiding the pole, then my integral must be 0. So this integral is 0. Now as I said, this integral has four parts, two of them are the principle value, red, this one is a basically principle value, not basically, it is the principle value.

Now we have this semicircle, small semicircle, this part and the large semicircle. The decay is missing here at that corner. Now this I said is 0. So now the whole integral is 0, that means the principle value must be nothing but negative of this, correct.

So that is what we want. And this integral, well I am going to the top, so I do exactly what I did before in the last slide, I use k equal to e to the power, delta e to the power i theta and do it this integral is minus pi by 2. Why minus because I am going clockwise, this is clockwise. At k equal to 0, the exponential ik is 1 and this 2 is coming here and it is just straightforward, is minus pi by 2. So that gives me the answer, this principle value is pi by 2 because this plus this equal to 0.

So this is pi by 2. So that means this guy must be pi by 2. This is 2 principle. This should be pi by 2, right? So 0 to infinity is pi by 2, but now I put minus infinity. So instead of 0, if I put minus infinity, then it will be pi, double, no? I mean, so this integral is even in x or even in k. So if I go from minus infinity to plus infinity, which is this one, then I get pi, double of this, fine.

So you will again see this integral, in fact, you find it in optics and condense matter physics all the time. So this is what, I mean, you can, these are derivation, well basically a byproduct of this derivation. So I share this, but you can easily derive this part. So complex analysis provides important tools for solving integrals.

It is easy. If you do it by other means, it normally takes more time and that is it. So this is a pattern now, I mean, since it is being recorded in between or if you are not understand anything, but you can ask me now. Clear to everyone? Was it done in math methods, this, this, this integral? Should be. So I will close this one. Thank you.