Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, we did field theory for quantum field theory in fact so far. So, that was the whole formalism is called second quantization. I described about the second quantization, it was field is quantized with the field operators and there is a commutation relation of field operators. Now, we are going to go to different way to do a field theory. So, this is called using functional. So, the functional will come is connected to final path integral and partition function.

So, this field theory which I am going to describe now is used in statistical field theory as well as in quantum field theory. So, this is in fact more popular technique. I will just going to give you overview not get too detailed, but you should be able to do calculation. In fact, we are going to do we spend some time on statistical physics.

Then we will go to again quantum field theory. So, statistical field theory means we need to do some stat mech review very simple well what we need that is what I am going to review and it will be mathematical. So, in statistical physics we have system with we will assume it has Hamiltonian and it is in a heat bath. So, system could be an atom or could be set of oscillators you know so which is in a heat bath right. So, or metal electron gas you know so all that is system.

So, system is in a heat bath which is a heat bath is at a constant temperature and the system is in equilibrium. We assume the system is in equilibrium. So, one more thing that we use often use this parameter of beta which is 1 by k B T, k B is a Boltzmann constant T is a temperature. So, beta has dimension of 1 by temperature. In fact, we will go to offset to 1.

When you need it you put the beta and we say we use that system is under equilibrium. So, there is no energy transfer system is in equilibrium that is assumption. Equilibrium field theory or equilibrium stat mech. So, partition function so we have Hamiltonian so it has a it is an operator. So, it has eigenvalue e lambda and eigenvectors lambda and assume there n of them.

Hamiltonian is symmetric so it eigenvalues are real. So, given the eigenvalues we can compute the probability. So, this is a Boltzmann distribution, canonical distribution. So, given energy e lambda so this system which could be atom. So, we can compute what is the probability of the atom to be in energy eigenstate lambda.

So, that is given by exponential minus beta e by partition function Z. Everybody knows this I mean this is a standard Boltzmann distribution. So, that is how we do it and Z is the partition function which is a very important role in in this stuff which is sum over e to the power minus beta e lambda and sum over lambda. Remember beta is 1 by temperature. Now so given partition function we can compute almost all quantities.

Free energy, specific heat, whatever you want partition function has that information. This course is not on stat mech this course is on some aspect of stat mech which is relevant to field theory. So, I am not going to go into that, but we will have to compute averages of some functions. For example, there is a for spins, Ising spin we have average magnetization. So, we have some quantity phi and so I can compute the expectation value of phi.

So, how do I do it? So, we have the probability P lambda. So, this is an this is an operator. So, spin will be an operator in in general. So, we write in terms of operator and this is how we write the expectation value of phi. But this will also go into quantum statistical mechanics.

Now, P lambda is 1 by Z e to the power minus beta e lambda. That is now we can write e to the power minus beta e this these two together e to the power minus h beta h lambda. So, beta is a number. So, this will give us e to the power minus beta e lambda. So, I can push this exponential this part into the along with the field phi.

So, phi operator and this e to the power minus beta h has been pushed inside. h is also an operator. Now this is in a nicer form. So, we write e to the power minus beta h is a density matrix. So, this in fact is written as a density matrix.

And we can multiply density matrix with phi this operator phi and take the trace. This is a trace in energy basis. So, this is a formalism, but this formula will become very useful in near future. So, we will find this e to the power minus beta h coming all the time. So, this is density matrix.

So, this is straight forward algebra. So, now how to compute phi? There is a nice trick which is used all at all places in field theory. So, I want to compute that expectation value of phi. So, the idea is to add this source term J k phi k. Now I am in fact I am going to

Fourier basis.

We can also work in real basis, but J k right now is a Fourier component. So, this is k k and sometimes I may just use that as a scalar not operator. Now if I write down the field equation. So, remember we have delta h delta phi k all that thing comes to. So, the right hand side will be J k.

So, when I take the derivative of this with phi k I will get J k. So, my operator L acting on phi my solution right hand side will be J is a source term not like for either potential computing potential this is my Laplacian operator this is phi and this will be source delta function source. So, J is a source term. Now with this it turns out that our computation of phi becomes very mechanical I mean you can just apply this mechanically. So, let us see how it how it goes please pay attention to this part.

So, this is what I wrote partition function is trace exponential minus beta h. Now my Hamiltonian is both all of it. So, this is the partition function. So, my Hamiltonian I added a source term source term in Hamiltonian will be J k phi k, but in the differential equation it will be just J k. I hope this part is clear because I need to take this stuff.

Now I want to get phi k out I want to get phi k out what should I do? This is a very interesting trick take the derivative of this function with relative to J k right. So, what if I take a derivative with relative to J k what will I get of exponential minus beta h naught minus J k phi k. If you want you can put a k prime and these are more all k's. So, exponential will be exponential and now J k will act inside. So, what will I what will I get? I will get phi k and exponential and that is what I want phi k to be filtered out.

So, that is a great trick. So, we just take the derivative of the partition function with J k. So, or well I am here is J i and well of course, J i want remember this is what I want right. This is what I want or well operator I am not putting operator here phi k average over well basically I need to integrate this expectancy means I need to integrate over all modes ok. So, that means I need to make J k J to go to 0 all J k should go to 0 all J should go to 0 ok. So, this part is kept as a source, but then you turn it off and this is my expectation only of phi i is that clear to everyone? By the way you can do as a sum for Eigenvalues or we can also do integral if it is a continuum.

So, we will come to writing the exact stuff in very soon. So, this is basically sum over all possible configurations that is what we write know. So, you can use either in a set of basis functions or we can just sum over all configuration that is a partition function. I this is not a course on stat mech, but the sum over all configurations beta H beta E with this energy, but your sum over all configurations and all configuration either you can expand in terms of basis functions or in terms of all possible fields and so on. Now, this is for single well phi average phi, but I may have a correlation phi i phi J ok.

So, I have phi i 1 phi i 2 and like that. So, what should I do? So, when I take derivative once I get phi k phi k prime know this phi k prime. If I take another derivative J k double prime what will I get? So, this is sum. So, phi k double prime is sitting inside or J k double prime. So, this will give us phi k double prime right.

So, for any phi what you get I just take a derivative and there are let us say n n phi variables here then I do n n derivatives ok. And of course, I need to divide by z ok, this z is there ok. So, this is a trick to compute partition function. Now, this is for stat mech. Stat mech normally we do not have non-commuting operators classical stat mech.

So, these operators will be like you can think of Ising spin S i S j. So, spin sitting at site i, spin sitting at site j I do the average a real space. So, there I will say that my Ising spin in Hamiltonian is in the real space, but I put a source at every site j i and so, this is like magnetic field at every point. And I want S i then I need to do the derivative z versus magnetic field at site i, that will give me S i out and then set magnetic field to 0 alright. So, is that clear everyone? So, we will going to use this this trick quite often ok.

Now, this is for stat mech where as I said we will use classical right now we are going to do classical in next 4-5 lectures. But what if we are doing quantum field theory? So, quantum field theory will have field operators like electric field or fermion field. So, then these operators are field operators and these are the field operators. So, I am taken this from this book Lister and Blundell gifted a mature book quantum field theory ok. Now, I also need to put a time ordered product.

Remember we had done in the last class. So, is sensible only when you do the time ordered product which is combination of two. So, creation operator for a particle is first then destruction operator we cannot destroy before creating. So, that kind of thing is there. So, time ordered product exponential now for statistical physics we have minus beta H is the probability. But this one comes from Feynman path integral ok, which I am going to do it after this lecture.

So, for Feynman path integral what is the probability for a given field configuration? So, e to power minus i S by h bar that is a probability. For what you are studied in classical well not quantum mechanics for particles. So, particle goes from here to there then what is the probability for choosing this path? Probability for choosing this part is e to power i S by h bar. S is action computed for this path correct. Similarly, probability can compute for fields. So, field goes from so, imagine there is a string which is going from initial condition to the final condition.

But it can go from initial to final in many many ways. So, this is so, this initial condition and final condition is here. But in between it can follow many well this is in fact forms a surface you can think of the surface in 1D and it lots of surfaces are possible. So, you compute this action of this surface and probability for following one particular path is e to power i S by h bar and S is computed as integral d 4 x Hamiltonian ok.

So, this is term is sitting here. So, probability in classical statistical physics it will be this is one the one it this for statistical physics and this is for quantum field theory. So, if you want to follow this path or follow this scheme for computing Green's function or correlation then we need to compute in the following manner. So, this is time order product of this functions field operators field operators and probability. A probability is so, what is the probability for statistical field theory? It e to power minus beta e by z I need to divide by z addition function. So, here this is the z operator sum over all configuration e to power minus beta e.

So, these are vacuum states and I am summing over this e to power minus beta e right z was sum e to power minus beta e. So, this replaced by z probability which is e to power minus this this operator and 0. Is that motivation clear? In fact, this just one to one map one to one correspondence in the formula. The slight difference this Hamiltonian I integrate over real space. Now, it is a Hamiltonian for statistical physics I am going to compute this Hamiltonian for the field in d cube x.

In 3D correct? So, we have spins and I compute the spin interaction Hamiltonian in a 3D space, but this is in 4D space space time ok. So, that is a difference. I will briefly mention it in later lecture of today, but that is that is a critical difference, but we do not have time for getting into the surgeries of this. So, that is a part which I will mention it and we will we just go ahead. But I hope you can see the correspondence of statistical field theory and quantum field theory.

There of course, differences like this is a time order product, these are operators, we have Feynman path integral compared to e to the power minus beta s, but there are lot of similarities these like partition function. Now, to get this phi 1, phi 2, phi 3, phi 4 we can again compute by the derivative trick and s operator. Now, I wrote this is s operator. So, but anyway I do not want to discuss too much of computing this way. By the way we can do lot of these calculations with second quantization.

I did show you how to do this scattering. Well, I did not calculate it fully, but I did show you how to compute this phi hat x 0 phi dot x phi y 0 right. So, this is a creation operator and this is a destruction operator and we did in the last class itself right time order product. So, in fact Feynman did this way in his first work, but later than people made this these kind of tricks to do the in fact mechanize the computation, we can compute it in a somewhat standard manner ok. This is a functional form or using path integral, these are all using path integral or I mean path integral is has connection with partition function calculation here ok. So, this is a brief review of stat mech and so please remember my partition function is critical.

I will use this e power minus beta h sum over all configurations and then using partition function I can compute field operators ok. We need these aspects a field correlations ok. I mean whether this formula is called Gell-Mann-Lowe theorem, it was in 1950s very early many years back and please keep in mind one important point is the system is in equilibrium. Here there is no time well there is no frequency dependence, its system is in equilibrium and I am going to show you bit later.

So, here you see that now actually you can see it here. A Green's function is same as correlation function right, this is correlation function phi x 1, phi x 2, phi x n operators time ordered. So, Green's function and correlation function are equal under equilibrium, they are not equal when in non equilibrium situations ok. So, we assume know when the system is in equilibrium and I am going to prove it for a special case in a while. For two point correlation is same as Green's function. I will show that g k is 1 over k squared plus m squared for for the field we are considering Klein Gordon equation without time part.

This is the Green's function for that right, I mean we did we discussed that Laplacian plus m squared, this is my operator. For that the Green's function is this ok, yes or no? Yes and this is same as phi k phi of minus k. This I will show you exclusively ok, but general proof is I mean Gell-Mann Low theorem. So, the proof general proof is more complicated, but we can show that for free field this will work ok. So, any questions on this this part? g is a source for every so here.

So, I have n variables phi k will go from 0 to n minus 1 ok. For every phi k I keep a source. So, imagine n Fourier modes and I keep source for every Fourier mode. So, if you look at this ok, let us think in real space. So, we have n points in real space. Now, we are field at n points and n points I keep a charge particles a source.

So, that is a source do not for potential this for Laplacian source is charge particle. So, that will be source for you. In Fourier space also we can have for every Fourier mode I have a charge a charge Fourier mode rho of k. So, is it clear? Now, by the way this j k is the function ok, which can have different values at different times. It is like a magnetic field in fact, the easiest is for Ising spin you can think of magnetic field at every side.

And magnetic field can take different values at different times and different place. But keep in mind that I need to set with j to 0. So, these are trick actually this is field is put I take the derivative and then set to 0. So, here I mean I need to set to 0, because I do not want effect of this otherwise my correlation function will depend on j and I do not want that.

So, this is just brought in and let go. Whether this is a trick you can use in general. So, imagine that I want to do a pretty complicated integral sin square beta x x dx. Well this is not too complicated, but let us say I have this integral. So, what do I do? So, this x sitting outside.

So, now, this is my f of x f of beta x. So, I can do this sin square beta x dx this can be easily computed then take derivative of this with beta. So, that will pull out x. So, and then you set beta to 0 if you like no beta 0 will become problem, but had it been ok. So, here I cannot put it, but sometimes you can just set beta to 0, but this is a nice trick no when you can compute this without doing parts and, but sometimes this is very useful I mean there is a factor sitting in front and we can easily do this.

Yes this Feynman used it quite heavily. For example, we have this x square e power minus beta x square dx. So, I can do use this trick by this derivative with beta and. So, I just need Gaussian integral I do not need to worry about x square and then I can do it. If I have x power 2 n then what do I do? I do the n in n derivatives which is easy no when integral of this with this function only beta take n derivatives and it works. So, it is a nice trick which is used now in heavily.