Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

Prof. Mahendra K. Verma

Department of Physics

Indian Institute of Technology, Kanpur

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Lecture – 20

We have infinite energy in the ground state or the vacuum state, do not call it ground state, this is vacuum state. So, that will come in the next slide, just give me one minute. So, so this space with number of particles, this is not like Hilbert space, right. Hilbert space for an operator linear, in fact Hilbert space is mathematical construction, it is not well quantum mechanics borrowed it from Hilbert from the mathematicians. But given operator L linear operator L, we can define Hilbert space based on the eigen states for operator L, right. I mean this is math formalism.

Now, Hilbert space for oscillator we have ground state, state 1, state 2 and like that. But now we got lots of oscillators, so we have in fact bigger space than Hilbert space, right. In fact, it is kind of direct product you can think of, of many many Hilbert spaces. So, this called Fox space, this is Russian faces or mathematician I mean.

So, Fox space is direct sum of tensor products of Hilbert faces for each particle number. So, this taken from Wikipedia and the state psi is a basically is in a Fox space. And this Hamiltonian which I wrote in the last slide, so n k particles with energy omega k living in that way number k and this is my state for the field. So, you can think of space where whether you cannot really think of real space, this is not a real space. This is some abstract space where there particles.

Let me make this remark, when we do the experiment or any of the calculations, we will assume that this abstract space, and their particles in there. So, we send one particle, this particle goes in interacts inside and comes out, where may be same particle would not come out, some particle will come out. So, it will have some signature of what is there inside. We cannot look at all of it, is impossible. So, this is scattering.

So, we have basically in a collider we are sending one electron, it does something and something comes out, from that we are trying to deduce what is there inside that state. So, this is the formalism. In fact, we send in one particle or we can send several particles and something comes out and from that we deduce what is there in that state. So, that is how we calculate, that is the normal procedure for this. And in field theory we do not work with multi particle wave function, because the particle numbers are changing is simply impractical.

So, vacuum energy, so as I said this integral is infinite, this straight forward nothing complicated. So, if you have infinity then what do I do? So, one thing is to say well this infinity, but we will work will we keep that as a background and anything above that infinity is what we will measure. So, that is what is we compute. So, we compute this. So, infinity part is kept aside.

In fact, that leads to lot of mathematical integrations called Wick's theorem and so on. How to make sure that this infinity is not coming in your calculation? So, we measure something which is on top of that infinity and that is what is our energy and this our energy without half. So, that is about what I want you to say about the second quantization for bosons. So, I think let it be on, let us see some interesting questions we can give it recorded. Is it finite? This one will be finite, but well I mean again, so this will be finite.

What we can say the psi the state I said with n 1, n 2, n 3 like that this is the Eigen state of Hamiltonian. So, we are not talking about quantization really well quantization is for the operator. The commutation relation phi pi commutation is i h bar that is a commutation, but you cannot say that this is a super quantization. So, you can say that this is a state in that box. Now, we have Hamiltonian for that the state is a Eigen state which of course, it gives you the energy as this one plus half.

So, that way it is you are correct, but normally you do not say we are quantizing each of the oscillator. You are doing it all together no no, so that, so this is a good question we do not think like that in field theory. I do not want to, so let us, so we do not think like that. So, field theory is not that you are thinking one oscillator with, so if you think like that then each oscillator will have different different level of energy. So, oscillator 1 with k will have levels like that, k prime will have levels going like that.

Then and of course, this oscillator can disappear at some point by, so then this whole energy levels are gone. So, do not think in terms of single particle or some of this non-destructive particle. You think that we have wave number k, one wave number it can have some particles. In fact, then wave number k 2 k 1 k 2 it has some particles, k 3 it has some particles. So, instead of going for you cannot identify this particle I am doing the quantum mechanics know.

You think of I am doing quantum mechanics of everything with particles in present being present there. So, you have to get rid of this single particle picture for field theory. Particle is not sacrosanct in field theory. The state psi which has all these particles that is where you start with. So, the particles can disappear then what do you do with that? So, I wrote that know for fields for classical what do you mean? Now, there is uncertain relation.

So, that half came from a k a k dagger a a k a k dagger is not a k prime is not commuting. That half in fact, exactly same now this was one h bar well omega k delta k minus k prime. In fact, I think it there is no omega k this was well the pre factors depends on the normalization. This is not 0 and the half are single oscillator in particle quantum mechanics comes because of this relation. But, there k was not there no k there is no k k prime the single oscillator.

Here we have k now for every oscillator we have this a k and a k dagger do not commute and that half is coming precisely from this this relation is a purely quantum. This is not classical do not say this property of the operator this property oscillator like two matrices either they commute or they do not commute. It has nothing to do with which vector is going to act on well now it is a deep question and it is interesting question. So, fractal space is so everybody knows what fractal is somewhat. So, one example of fractal is that we have triangle one triangle in between we remove one third one quarter of the triangle remove this.

So, we get out of four triangles we get three now the three triangles you do the same thing to these guys. Now, you have nine triangles and keep doing it. So, this is a fractal object and the amount of space is less than two because it is not there are lot of holes. So, it is a fractal dimension I think log four by log three or something of that sort. Now, it is possible to construct some well field theory will not have we cannot deal with this discontinuity and so on.

But people construct fields by so in 2D so let us say we have a box finite box. So, we will have number of modes let us keep it finite n x into n y number of Fourier modes I keep it finite let us say some cut off. So, I say along x direction there is n x modes and y direction n y modes if you want you can keep it infinite no problem. But now we can make a scheme where you can cut down one third wave number every time. So, just think of 1D well instead of 2D you can just think of 1D you have you have 1D line.

Now, in Fourier space we will have k equal to 0 1 2 3 4 like that. So, say well I remove k equal to 1 k equal to 5 k equal to 9 and I can remove that. So, that will that is analogous to what is done here. Fourier space also you can remove k numbers at different scales, but quantizing and all that we will come to non integer dimensions later it is possible to do field theory in higher dimensions and the d is there and it has a very important role. You might have heard that in a four dimension four space dimension the fluctuations are

Gaussian like I heard of this line statement or no we will come to that it turns out in four dimensions physics is simpler compared to three dimension.

So, we will we will do that later. So, dimension of space will play a role anything else? So, I think let us do one more simple slide and then we will break for today. If I exceed 5 minutes nobody is a class. So, let us finish this slide otherwise it will be it is a deeper slide I mean redo some of it, but let us start it. So, we did work with a basically field which was oscillating field which we mapped it to many oscillators and that had nice set of properties, but that unique to that field the same quantization does not work for all fields.

So, depend on Lagrangian we will have to create ways to quantize it or create ways to deal with deal with multi particle field theory. So, QED has a different field theory, quantum chromodynamics a different field theory, Schrodinger equation different field theory. So, that is important point. So, let us do some more well I am going to now develop the formalism well some more formalism that was not not a general formalism that was somewhat confined to a bosonic field which was for a scalar particle it did not have components also. So, free particle in many body physics is more like a condensed matter physics that is what I am going to talk about right now.

The Schrodinger equation free particles. So, again we will start with simple. So, this is a Lagrangian know I wrote before and we can derive this equation of motion. You can treat this is a classical field. When you quantize this field then second quantization will occur.

For particle wave function this is classical totally you can solve this in a computer psi is unique at a given time. So, this not of course, this is like string of a movement of a motion of a string. Its interpretation leads to quantum mechanics position is average and so on. We set psi v equal to 0.

So, that is a free field. So, free particle I am so going to a free field you may call it. So, my Hamiltonian is this. Now, this is a different Hamiltonian what we had in the previous set of slides. So, total Hamiltonian will be integral. So, this is integral and Green's function will be this.

This I derived it I mean you can just this I omega coming from here and this is that v 0. So, it is v keep v equal to 0 non interacting. Now, so this we had done before at least this part. So, this is in real space.

Now, we can do it in Fourier space. Now some of it you should know it by heart how to derive that. So, this Laplace grad phi square will give us k square. So, this k square is here this phi phi square will give you phi k phi k dagger phi k operators. So, and this 1 by 2 m is here and some more all case. Now, there is no momentum here by the way see this pi is gone we just have phi.

So, here I do not need to deal with momentum operator. If we know momentum is grad phi. So, this you see this whole structure of Hamiltonian is different and quantization is different. So, this I think you should keep this in mind. Now, this is my Hamiltonian and I will try to quantize it and work with it.

So, it is not going 1 by 1. So, indistinguishable particles this one is the most important. Now, this is a very very important slide you should pay attention to this one. So, in classical physics we assume the particles to be distinguishable. They may be particle of identical mass, identical looking particles, but since the trajectory of the particle is we can determine from the initial condition we can follow it. And so you say this particle is basically the left particle and that particle is a right particle.

So, this is classical physics. In quantum physics of course, we trajectories are not defined. So, then there is no way to say this particle A particle B particle A is following this trajectory particle B is following trajectory. So, remember I had said we have a box and I shoot 2 particles that is a 2 3 1 particle. So, let us think of shooting 2 particles A and B and something will come out imagine the 2 particles come out. So, I send in 2 electrons and 2 electrons come out imagine that it was really well I mean think of classical physics where well not classical classical very simple interaction not field sharing interaction there is a potential scatterer there is a big charge particle sitting there and we send like that.

But I just shot it, but I do not know what is what happens inside, but when it comes out can we say the left guy is A and right guy is B you cannot right quantum mechanics does not allow us to say this could be A or B or this could be B and this could be A. In field theory it becomes more apparent that this is a big thing in between we do not even know it was if the same particle which was sent in which is coming out it could be very different particle. It could be electrons coming out 2 sent in 2 coming out, but it could be very I mean there is no way to say that was same electron which was sent in. So, these are indistinguishable particles.

So, that means we can create particle well. So, this is for scattering you can think of scenario where I create 2 particles now we already familiar with creation operators know. So, imagine that in the vacuum this state 0 vacuum state I create 2 particles particle 1 and particle 2 same 2 electrons 2 electrons I create level 1 and level 2, but level 1 was before and level 2 was later before later. Well before later also becomes a problem sometimes, but order in which I do it or you can do 2 first and 1 later. It turns out there is no way you can distinguish. So, this is inherent in the field theory that I create this 2 particles first P 2.

So, you go from the right to left now operator will act on a state psi. So, I create P 2 first and P 1 later or P 1 first and P 2 later these are creation operators. So, these 2 operators will will do the same thing on state. So, this acting on psi will give you new state psi prime or phi.

Well now the phi state has 2 additional particles. Now this acting on psi will also give phi or it could give you minus phi because we do not worry about the sign we what we measure is the probabilities. So, sign is not important of course, you may say well it could be some arbitrary phase, but that is not considered it is considered either plus or minus. So, these 2 operators even you commute it when you change the order this pre factor can be either 1 or minus 1 that is only allowed. There are of course, people are talking about more complicated stuff, but we will not get any on since on this connected to this, but we will assume either plus 1 or minus 1. So, plus 1 is called bosons when you commute this 2 creation operators.

So, for bosons this creation operators will commute in fact, I showed you creation operators commuted, but what if they do not commute and get a minus sign this is minus sign it need not commute. This possibility in nature that it can come with a minus sign and that is fermion. This is the simple formulation to create both this set of particles and I think maybe I should stop here and then we can work out. So, bosons I kind of done it right now I will just recap it in the next lecture, but fermion we can derive from this simple postulate that the number of particles in a given state can be either 0 or 1 cannot be 2 particles.

So, this is just a simple framework we adopt. So, this operators are very very important and once you set the rules for the operators everything follows. So, I think it is better to stop right now and I think we will do things carefully and we do not do rush. In quantum mechanics well I am I made life simple here that I put some some simple one scattered. In general this particle or in condensate matter physics too. If you send something here what comes out is very different you send 2 electrons and you get all this Higgs particle and some quarks the quarks do not come out, but the just bunch of particles come out.

So, here in quantum mechanics we do not know what is inside. If fully you can make make predictions and we can make some guesses. So, we will calculate some of this stuff in I will show you how this scattering is calculated. This Feynman scattering well basically coulomb scattering you know know when one particle goes and there is a coulomb particle then they scattered. You can do similar calculations for quantum mechanics too, but then it is not in terms of trajectories, but in terms of operators which I will not do in this course, but it is done with full operators and their virtual particles and all that will come. So, it, but you basically you are trying to compute from the Green's function plays a big role and you are trying to get some properties what is there inside, but this state psi or vacuum you know you do not have much control over that.

You can make this is what so, we can say when I send something this much of a change will happen that is a law, but you cannot say what is there inside that is not not not well many times not allowed to ask. You can say that well yeah if I if something goes through it comes out this will be the change that is a law. So, it is a problem yeah.