Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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So, we have Hamiltonian right, this will also become operator. So, I am going to use that this potential. So, here so it is easier to write in momentum space of course, you can write this is a in real space pi will be pi of x t I think is pi dagger pi x t. So, we can do this operators and integrate d delta function x minus x prime d cube all that can be done in real space, but it is easier to write them in Fourier space and in fact, we work mostly in Fourier space. I think so let me just make a detour why Fourier space are important and I think let us let us take the detour later. So, let us let us continue right now I will tell you about Fourier importance of Fourier space in little bit later.

So, this pi square we derived in classical fields it was pi k pi minus k right you just remember few slides back for classical it was pi of k pi of minus k or pi k pi star k I put mod square no. So, pi k whether please remember for classical fields pi k is a complex field pi x was real, but pi k is complex here pi k and pi pi k dagger now this this minus k is replaced by dagger operator adjoint operator which is complex conjugate and transpose. So, you need to do that because these are operators. So, to make sense we have some matrix in the this pi dagger k is transpose and complex conjugate.

So, this pi square is replaced in Fourier space by pi dagger which is adjoint operator n pi k and similarly, for so this sum is for both sum is for this one as well as that one and this grad phi square and phi will give this is a pre factor no remember phi k dagger phi k. So, earlier it was basically phi k phi of minus k now this phi minus k is replaced by adjoint operators. Now, that is what I have written it here and this integral is sum is integral in 3 d is that, but in higher dimension is going to be 2 pi to the power d and this one we write as omega square is function of k. So, we write as omega k square. Now, you can see that this is basically this Hamiltonian is looking like a lot of oscillators know what was the Hamiltonian for oscillator e square by 2 m plus x square.

The Fourier space well this is a oscillator potential p square x square. Now, you can think of the first thing here is a p square and this is x square well you should put omega square half half m omega square k square. So, you make m equal to 1 and this is omega square this for single oscillator no oscillator no sum this Hamiltonian for oscillator. Now, for many many oscillators I need to sum. So, this is what I have summed.

So, you can think of each of this phi k s and phi k is a momentum n position of one oscillator will have phi k s position and phi k is a oscillator is a momentum the other one has phi k prime phi k prime and many many of them. So, this is a combination of many many oscillators and it is not surprising because what was this field for in fact, we derived it for a string which was oscillating. So, string oscillation we can think of superposition of many many Fourier modes which are oscillating right that is a beauty of Fourier. Fourier lets you decompose into many many components and for oscillating field each component is oscillating. So, that is why we have lots of oscillators giving you this this oscillation of the field and that is one advantage of Fourier.

So, we are able to decompose this field equation into Fourier in the Fourier basis where they just look like sum of n oscillators. Now, is I said is a free field a free field is linear field right now I have no non-linear interaction. So, these oscillators are all uncoupled they are not talking with each other they are oscillating on their own that is why we have nice phi square plus x square when do we have non-linearity we can put x cube here that is a non-linearity you may know some non-linear physics know x cube is a non-linear operator and that will couple two fields in Fourier space two two oscillators and this one we will derive it with later. So, you can couple this oscillator by non-linearity, but right now we will assume it to be independent oscillators which are oscillating. Now, this is a classical picture, but quantum picture will will also be independent oscillators, but then we will have something more.

So, is I said in classical fields we have phi star k was phi of minus k this you can derive it. Now, this phi of minus k is replaced by adjoint operator phi k dagger phi k dagger is transpose complex conjugate. So, that we need to take care of, but I assure you that we do not need to construct this phi field these operators in this course at least we will well some cases we will assume something, but we will can just deal with particles and carry out our work. So, the Hamiltonian is this and this is our frequency square. Now, we can define the way we did for oscillators creation and annihilation operators just pay attention to this part I mean this is just look at it very carefully.

So, we create we will construct a new operator destruction operator which is with the plus sign. So, plus is destruction operator I made a mistake that time. So, omega k phi k plus i pi k x plus i p. So, this i sign here, but for getting a proper normalization omega k is put in here. So, p plus this p plus omega well I mean this is well I am not sure this is how we construct for bosonic scalar field.

This is creation operator which I am going to explain why it is creation operator. Now, a dagger so we can take a adjoint operator of this. So, if I take a dagger what will I get a dagger k. So, this is sum of two operators. So, you take adjoint of each of the operators.

So, this will become 2 omega k no problem omega k phi k dagger. So, this is straight forward no I mean a 2 matrix is sum of 2 matrices and you take the adjoint. So, first one a second one i because minus i because of complex conjugate minus pi k dagger and this is what I got here. So, this adjoint operator dagger. So, you have two operators a k and a k dagger straight forward I mean this is a construction I did nothing more I just constructed given the field operators pi and phi I created this operators.

Now, it turns out we can prove it this can be proven. So, we have commutation relations for phi and pi phi and pi do not commute, but pi pi commutes phi phi commute. Now, you can show that a with a will commute a dagger with a dagger will commute, but a and a dagger do not commute it turns out a and a k dagger gives you delta function. So, this is the this is what it is. So, there is no i here this is the relation.

Now, the delta will come because we are dealing with infinite space otherwise for finite volume we will get Kronecker delta function I am doing with dealing with infinite box. So, it is a Dirac delta function you everybody knows the difference. So, this what we will get. Now, creation and annihilation operators now let us look at now this slide is a most important one. So, this is my Hamiltonian how do I derive it? So, again I am skipping the details is kind of may be boring.

So, we had pi dagger pi k pi dagger let us go back to old pi dagger pi plus phi dagger phi just substitute pi in terms of a and a dagger. Like what we did for oscillator in your course you do that then you will get this a dagger a omega k and half. So, this is omega k is coming. So, this derivation I am not doing the algebra is kind of boring to do it especially on this board black board is better, but I will just telling you that if you just substitute pi with a plus a and phi a minus a you will get exactly like what you do for particle oscillators and we get this. Now, this looks very similar to what we had for linear oscillator know that is why I covered that one.

So, now here we will make a bold step. So, this integral is sum of Hamiltonian for many many oscillators. So, you said each of them is each wave number k has an oscillator and we are just summing up our many wave numbers. Now postulate that there is a vacuum state. So, vacuum state is now for every k.

So, is very similar to what you do for oscillators particle, but do not think like that you have to think that we have for every wave number I have number of particles for every wave number wave number k I have particles it could be 0 1 2 3 4 like that. So, this is wave number k equal to 0 k equal to 1. Now, in three dimension is going to be 1 1 1 2 2 2 like that this is schematic. So, I am not saying one. So, I am just labeling them 2 like that.

Now, for each wave number there are particles. Now, there could be 0 particle or there could be 1 particle. In fact, bosons so there can be two particles here there is one particle there are three particles here. So, these are possibilities in quantum field theory. So, to begin with we just postulate there is a vacuum state which has no field no particle in any other wave numbers is empty state.

But that is a state is not nothing is very very important in quantum field theory. Nothing in fact is very very this is in fact it is like ether has lot of important properties. We find that vacuum has infinite energy. So, postulate vacuum state this is like a wave function. But it is not here particles will go away.

So, it is like a state we call it state we do not call it wave function. So, we have vacuum state where there is no particle in any other wave numbers. Now, we so if there is no particle then if I apply this A A k on 0 in the state I will get 0. So, this is like a postulate. Now, so we can also define a state psi.

Now, this properties can be proven I am not proving it in here. So, you in general we have state psi where for a wave number k will be some number of particles. So, k equal to 0 has well I should put n 0 k equal to 0. So, we have n 1 n 2 n 3 n k these are for every wave number I have number of particles. So, these we define we call it state psi.

So, number of particles in each of these wave numbers can vary is allowed in fact it happens in real life. Electron positron pair can collide and give to two photons. So, this electron positron pair have gone are gone. So, this particle numbers can change in real life. So, this is taken care by this formalism.

So, we have state psi with number of particles for every levels or every wave number. Now, A k acting on psi is A k this is a destruction operator. So, it will going to act on k wave number k wave number. So, this A k will not act on anything else A k is going to look for wave number k and look at how many particles are there there n k particles. So, you are going to destroy one among them just destroys it and bring it to n k minus 1 and rest all particles remain unchanged.

So, this is let say wave number k. So, it just going to. So, if there are n k particles it will remove one of them destroy it is basically is gone in the system. So, this particle is gone. So, n k goes to n k minus 1 and pre factor square root n k will come. So, these are destruction operator, but destruction operator will be different for different wave numbers.

What I just told you about for wave number k. Similarly, we can create a we have A k dagger it has a property this is in the proof the in fact, we can follow exactly same algebra is particle quantum mechanics for oscillator and we will find this A k dagger acting on psi is going to look for again the wave number k it goes there and is going to create. So, it does not destroy it creates a new particle. So, it creates a new particle from here it has gone to there n k plus 1 and the pre factor is square root n k plus 1. Now, we in fact, we can use this two properties and you can substitute it here.

It turns out this psi is a Eigen state for H this H is a big operator no it is a big Hamiltonian which has many many sub Hamiltonians and so Hamiltonian operator acting on this state psi will give you some energy because Hamiltonian gives energy and gives back the state psi. So, if I act H on psi I will get the energy. So, that is the next slide. So, it turns out with this can again be proven I am not doing it here A k dagger A k psi gives you in fact, you can just use this two relations A k acting on psi will give you n k minus 1, but now at A dagger k on this. So, A k dagger can just act on this what will I get.

So, this n k minus 1 become n k and they will pre factor square root n k will come. So, n k n k multiplies and you get n k this for this given these two it is straight forward to get that. We can look at proof in any of the standard book Leister and Bundel which is I like this book quantum fields very for gifted amateurs or you can look at any book where you will have all the details of the proof. So, A k dagger A k will give you n k. So, A k dagger A k so psi is the Eigen state of A k dagger A k just like for particles state n was a Eigen state and it gives you number of particles.

So, it really works that this is basically sum of n oscillators. So, Hamiltonian will be following the same lines this factor half is here. So, A dagger A k will give you n k on if I acts on psi. So, now, this guy is jumping in here n k plus half omega k. I for you look at number of particles at every k I just multiply n k plus half with omega k and I sum over all k that is the energy of the system.

Now, this is new right I mean classical physics cannot derive this. Classical physics we have energy was just phi square plus phi square right kinetic energy plus potential energy phi square plus kinetic energy and grad phi square was the potential energy. This is purely quantum and this of course is H bar which I did not write it and the number of particles being destroyed created is not part of classical physics. Of course, there are their interactions so the amplitude can go down and go up, but not in terms of destruction. So, here is discrete number of particles can go by one up or one down is purely quantum.

So, and this half is very very important factor. Now, it looks like for single oscillator we are happy this is a this quantum fluctuation and we are getting half H bar omega right omega. Now, there are many many oscillators and each guy is giving you half and I need to integrate this. So, what will happen to the integral? This half H over omega, but this integral is k square d k we know this right. So, this integral is infinite. So, you get infinite energy in the when there is zero particle and there is no way out I mean this is coming straight from quantum mechanics formulation.