Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Week - 01

Lecture – 02

So, integration essentially is in field theory is done using complex analysis, you can do integration for example, integral sin x by x, you can do using many ways, but lot of these complicated integrals using complex analysis. So, I will do some of it in this next two modules and Green's function will become clear. The Green's function will be again set of lectures, well one lecture actually and then you will be all set for field theory. So, integrations are appear all the time like mass renormalization integration is there everywhere. So, and we need to know how to solve it, some of it is done analytically and I am interested to do this computationally as well numerically. Of course, people do it like in quantum chromodynamics, integrals are done numerically, that is quantum chromodynamics, some of it is the perturbation it does not work.

So, people resort to numerical integration. Anyway, so we will try to do analytically some of the simple ones. So they are done using contour integral and that is what I will do. So some basics, analytical function, analytical function in complex variables f of z, if f of z is analytical, then is derivative in the z plane.

So, z is real of z and imaginary of z, z is a complex variable. So it is a 2D plane and in this plane if I take any point, then its derivative is unique. Now I will not derive the condition and so on this called Cauchy criteria, but just assume well that is the definition of analytical function and if a function is analytical, then it has many many interesting properties. One such property is integral of f z over a closed contour, any closed contour, if function is analytical in a region, in fact it does not need to be analytical everywhere, if a function is analytical in this let us say in this region, take any closed contour f z dz integral is 0. You know this but just to repeat that, if f z is analytical then this is also a interesting formula.

So, z minus z naught, so z naught is a pole. So f z by z, so this whole thing is not analytical, at z equal to z naught there is a singularity, but this integral is solved is 2 pi i f z naught. Derivation is also nice, simple, but you can just refer to it in a complex analysis textbook. So, we need this often, in fact this is a how we will integrate line number 3. This is one, let me I can derive this one actually, in fact I wanted to show this.

So, how do we derive this third line? So this is z naught. So we can, well I think let us skip it, I think we will save time, this I think everybody knows this, so I will skip this. So this formula we will use it all the time, the one. Now sometimes we get this over which is more than 1, n plus 1. So then the integral for the again closed contour, this means closed contour, this closed contour is 2 pi i by n factorial f and z naught.

So Jordan's Lemma, we need this again often. So what is Jordan's Lemma? So f z is analytical in the upper half plane except at for a finite number of poles in the upper half. So this is a z plane, but in the upper half it is analytical in the upper half except there are finite number of poles. It, there could be pole here, there could be pole here, there are some poles, finite number of them. So it is analytical in the upper half, means derivative is defined and if you avoid the pole and take a closed contour integral to 0, all the properties.

Also, maximum of f z, so z going to 0. So imagine that I have a very far-off point from the origin. So z going to infinity, you know that this is, this is the condition. Maximum f z, f z is also goes to 0. So if you take large z from the origin, well z is the distance from, mod z is distance, you go far-off from the origin, then the f z is 0.

Why do I need this? In fact, I am going to close this contour like that and I can get the value, the integral on the line. That is the beauty of this. So you want to get a trick to close the contour. So if that is, these are the two conditions, then for m greater than 0, these are result. If these two conditions are true, for m greater than 0, then this integral.

So you remember, I said my condition is f z goes to 0, mod f z. But then we find that, well, I say max f z. So for large f z, I can make a, basically a curve and maximum f z on that curve is, goes to 0. But then we can get this thing, that exponential i m z, f z goes to 0 as r goes to 0. So these are result, for m positive.

So why is it happening? So m positive, so i m, so then z will be real part of z plus i imaginary part of z. This part, exponential. So you can see that, the one which is damping, so these already going to 0, this part is already going to 0. Well, modulus is going to 0. The proof is also nice, proof you can look at book by Churchill in detail, but it is nice, but I am just giving a gist of it.

So this part, well, its mod is 0, but not the function. But essentially you say, well, this part is small, but then is damped further by exponential i times i, this i and this i, because minus m imaginary part of z. Imaginary part of z is positive, because in the upper half, so is damped further. So then this integral, goes to 0, as r goes to infinity, r is the radius from the origin. So on this curve, contour which is a semicircle, we will get 0 contribution for the contour integral.

So that is a nice thing, know that if I want the integral along this line, my arrow is in the wrong place. If I want to integrate, get the answer here, like for example, I want sin x by x integral dx. Then I write sin x e to power i x by x dx, but x, you replace x by z. So this will be e to power i z by z dz. Now you see, 1 by z goes to 0, for r goes to infinity.

So this condition is satisfied for this z, 1 by z. It has a pole at z equal to 0, but that is okay, we can allow finite number of poles. But then this one I do i z, so m is equal to 1, so m equal to 1. So if I close it from the top, I am safe. I can basically get answer for this by closing at the top and this is 0 anyway, and that is how that is done.

We will do this later in the course, I am not, later in tomorrow, next lecture, I am going to do sin z by z. So Jordan Lemma plays an important role for getting the close contour. This is for positive m. What if m is negative? Then you should close from the bottom. You can easily see that what I was trying to argue here, I should close from the bottom and then the condition is reversed, you know condition will be basically reversed.

If the analytic in the lower half plane except for a finite number of poles, it just reverse it instead of top plane, you say bottom plane. But m will be negative for, m negative will give, will go to 0. So I put a minus sign here or if you want, you say m less than 0 and exponential i m z, that is also fine. So if it is minus then for m greater than 0. So you have to be careful when you apply Jordan's Lemma and you can use it to solve lot of integrals and here it is a bottom curve.

So let us do one example. This integral I want to solve. So this integral is from minus infinity to infinity on omega plane. This is the omega like that. It is going like that. Now this integral looks daunting, you know, I mean definitely I do not know, I mean you can do by some parts or something like that.

But contour integral is very nice, you can easily do this. So what shall we do? Is minus I omega t, so this t is also there, know. So for t greater than 0, time greater than 0. Now first let us identify the pole. Where is the pole? As function f z, I want to make f z as this function, minus i omega plus kappa k squared.

Kappa and k are constant for this integral. Omega is a variable. So where is the pole? This function is identical everywhere except at the pole and pole is I omega plus kappa k squared is 0. So omega equal to minus I omega equal to minus kappa k squared. So minus minus is going to be plus and I goes there, so it is the minus I.

The pole is here in the imaginary line, minus I kappa k squared. The pole is here. So I can close from the top as well as from the bottom. Now let us see the interesting stuff. Now I am going to use Jordan Lemma.

If t positive, then is minus I omega t. So omega t is positive and there is a minus sign. So I want this Jordan Lemma to be used. So where should I, which will, bottom one or top one, which will give a non-zero value, zero value for that like this or like this, which of them will give zero contribution.

We go back to our thing. In fact, you can see it from here. Minus i omega, let us call it alpha, omega t is alpha. Alpha can be complex, omega is complex. So one thing, we expand, extend omega to be complex. My integral is on the real line for real omega, but I will assume omega to be complex.

So this alpha will become now complex. So minus, now alpha is complex, so alpha real plus I alpha imaginary. So look at the imaginary part, that is the damping part, no, minus I alpha really oscillating part. That is not going to damp your integral. The one which is damping is a imaginary part, I times I.

So I squared is minus 1, so plus alpha imaginary and exponential we are putting, this exponential stuff. So exponential of this, then I alpha real. So exponential plus alpha real. So alpha real, sorry exponential plus alpha imaginary, this one. So alpha imaginary must be negative to damp.

So for omega t positive, I want to close it from the bottom to get a zero contribution from the semicircle, yes or no? So we close from the bottom, let us call it contour A. So I am going to convert my, this is integral which is of my interest I. So I plus contour integral t, not contour, semicircle is not closed, bottom semicircle, bottom, same thing exponential minus I omega t divided by minus omega plus kappa k square d omega equal to the full integral, contour integral of exponential minus omega t divided by, is that clear? So left hand side is this line integral plus semicircle, so that makes it a closed circle. Now so this is closed contour. Now I know that this left hand side is zero by Jordan Lemma.

So my integral I which is of, which I want equal to the closed contour. Now what is the closed contour, value of the closed contour? 2 pi I. So 2 pi I, by the way I also make, I want to make, this one goes to zero modulus of mod Fz, for Jordan Lemma we need the mod Fz goes to zero as mod g goes to infinity, max of this. So this one is omega squared plus kappa squared k4. So omega squared, kappa k squared can be finite value but with this omega squared and at the bottom omega squared will be omega imaginary squared and so that will go to zero.

So that condition is also satisfied, that condition better be satisfied otherwise it would not work. So 2 pi I and the Fz is zero, at the, compute the function at the pole. So 2 pi I, now this is my function. Jordan Lemma, F is different than the integral, for Jordan Lemma I was using this as the function, bottom one. Here what is the function for Cauchy formula? So top one, where the pole is here, remember Cauchy formula is Fz by z minus z naught dz is 2 pi I Fz zero.

So you see, I mean the bottom is the pole, we get pole from the bottom. So my Fz for this one is the top one. So you have to juggle around a bit. So exponential minus I, so minus omega equal to, you just substitute minus omega is minus kappa k squared.

So value of, evaluate the function at the pole. So I omega is replaced by kappa k squared. So the minus sign here minus kappa k squared t. So, and there is a 2 pi, sorry, there is 2 pi coming from here, there is 2 pi from here, this 2 pi, I should put 1 by 2 pi, 1 by 2 pi. So this I, no wait, I made one mistake.

So we have to rewrite this z minus z naught. So there is a minus I sitting outside. So I have to pull out the minus I. So I have to write this as omega minus I kappa k squared and pull out minus I. So let me just do it here. This board is not very friendly, but 2 pi pull out minus I, then this is exponential minus I omega t d omega divided by omega minus I kappa k squared plus, plus, plus.

So now this is a 2 pi minus I is outside. So this is a minus I outside. There is one more issue, there is one more issue. What is the sign of this? The control integral is clockwise or counterclockwise? Is clockwise. So this integral is positive if it is counterclockwise, negative if it is clockwise. This is becoming because of the integral d theta counterclockwise is positive.

So because I am going this way, it should be minus sign. So there is a minus i there, minus sign here. So minus I minus I cancels, I get exponential minus kappa k square t. So this is a Green's function in fact for this problem, this is for diffusion problem. So I will not, I will come to that exactly for which equation this is a Green's function later, but you can see that we are getting a Green's function by contour integral.

This is simpler than doing by parts and so on. This is definitely contour integral gives you a result very quickly as long as you know how to solve it and we will do most of the integrals using this complex analysis, but we are not done yet. This is for t positive. What happens when t is negative? So let us use the next slide. I want to do for t negative.

G k omega 1 over 2 pi integral e power minus I omega t. Now omega t is negative. So to apply Jordan Lemma, I want that exponential part to, well first of all here this corresponds to m positive. See m is pre-factor m z.

So this t is like m. So t is negative means m positive. So this minus i and minus i t is m. So m positive. So Jordan Lemma says that m positive means upper half. Upper half contour, upper semicircle contour will give you zero contribution.

So this contour will give zero contribution. So and my pole is here. My pole does not change. Pole is sitting there. So I can solve this stuff now.

So this integral I, let us call it I2. I2 plus, now I am going to do upper integral. Upper integral or upper. This stuff equal to now positive contour counterclockwise. 1 by 2 pi exponential minus I omega t square d y.

Now this is zero. What about the contour integral? Zero because there is no pole. It is analytical. So this is zero. So I2 is zero.

So t less than zero for negative time, Green's function is zero. In fact you might have, if you have done Green's function in your course, you write like this. Theta function of G K,omega, is exponential minus kappa k square t for t positive, zero for t negative. Or we use theta function, theta t exponential minus kappa k square t.

Theta is a step function. It is theta t is zero and jump. So we solve one Green's function here. We start different Green's function. In fact it will come for the diffusion equation. In the Schrodinger equation there is only one change.

Kappa becomes i kappa k square. And there is a bit of change in the formula because these oscillations will come. But yeah, I think your mathematics should be strong. But you can learn this.

These are not difficult stuff. You just pay attention. Some of it you should do it yourself. All the, whatever I do in the class and some homework, you do it. You will become master of Green's function.

And that is it. Your, half of your difficulty is done. Any questions? So, I end here. Now for the module one, we will do some more integral in the next class.

I am done for today. It is seven o'clock. No questions? So, we will stop. Okay. Thank you. .