Tapestry of Field theory: Classical & Quantum, Equilibrium & Nonequilibrium Perspectives

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Alright, so classical field theory we will do today. So, examples some let us look at some examples to keep what we are going to discuss in this waves on a string, some height, but we will assume right now that it is a small amplitude wave. So, that is what we will assume. A Navier-Stokes equation it is a non-linear, but we will do it later. A 5-4 theory which I am going to tell you what is 5-4 theory, Schrodinger equation which all of you know classical thing and Klein Gordon equation. So, these are some examples some of it we will do in detail, some we will do it not in so, one detail well the other.

So, we will do that. So, let us look at example basically let us think of string first. So, field is we I am denoting by phi and phi is function of x and t. So, earlier particle has position and velocity.

So, there are two coordinates for particles, but here there is a field which has a value for given x and given time. String is an example or temperature in this room that is an example, velocity of particles, velocity of in this room not particle, velocity in this room is a example. So, let us take example of string. So, I will not derive in detail, but we will motivate it. So, kinetic energy how do I define for particle is half m e square right mass is a mass of the particle.

Normally think of point particle or sphere you know, but a string is a extended object. So, we define basically we are going to use the same formula half mv square, but m will be the kind of we need to think of total mass, but v will be different at different places. So, we have to just say density. So, if this is string then we think of small element here. So, rho d x and then velocity of that element which is d phi by d t is moving all in transverse direction.

There is no motion in that direction. So, it does transverse direction. So, d phi by d t is the velocity. So, these are local kinetic energy or kinetic energy density multiplied by d x well this half is convention. So, these are kinetic energy which is function of phi and density.

Similarly, we can define potential energy. So, potential energy is coming from the curvature right string when it is undulated like that it has a undulation. So, this is the potential energy this multiplied by elastic constant kappa. So, that is the potential energy of the string. So, we have 2 d energy and total energy will be t plus v, but Lagrangian is defined in a similar way as t minus v.

So, Lagrangian is the same thing you know kinetic energy minus potential energy, but instead of particle we have for field. So, this is t minus v d. Now, action will be Lagrangian will be integral t minus sorry t minus v which is total energy minus potential energy. Now, that will be integral d x and this object rho phi dot square minus and dot dot dot dot. So, what is action? Action is L d t.

So, this is what I wrote action. Now, please keep in mind that for particle we define the initial condition and final condition. Now, here initial condition will be initial condition of the string it could be starting like this with certain velocity. So, phi is 0, but it has velocity is a starting point or it could be like that is initial condition and final condition is somewhere it goes to. So, this is we have to phi x t equal to 0 this initial condition and phi x t equal to t final is a final condition.

So, in Lagrange in action principle we define both initial condition and final condition and then we see what is the evolution from initial to final and what is the equation of motion that is what we do it. So, we can do similar analysis for fields. So, we need to give initial condition and final condition these are the constraint I mean I am not making anything telling anything mysterious. For particle we have initial condition and final condition and in between particle follow the classical path. Now, if you see here we have dt and so thus dt now t and v are function of x we integrated t is integral dx.

So, you see that that dt dx will come for particle we only have integral dt, but here we have dt dx and some function of phi partial with x a partial of phi with x d phi by dx and d phi by dt. So, these are the three independent variables phi partial x phi partial t phi. So, this called Lagrangian density. So, you integrate Lagrangian density with dx then I get the total Lagrangian and integrate that with time then I get action. So, here for this system is like string this is the Lagrangian density this is not the total Lagrangian, but this is function of x well this function of phi this one and this it could be function xt as well, but normally we will have function of phi and this.

Now, let us I am quickly going to derive principle of least action well equation of motion using principle of least action. So, for particles we have principle of least action which is the particle follows a trajectory for which action is minimal or extremum which will not be the extremum. So, here also L dt is extremum. So, we want this object to be extremum given initial condition 0 to final. Now, of course, this function is reasonably complicated.

So, I need to make now is that clear that is principle of least action and that is applicable for fields as well all sorts of fields this is a general theory like Euler Lagrange equation for particle similar thing we are going to derive for fields. So, I want to make delta s by delta phi to be 0 this is a as I said is a variation calculus is not d by dt or. So, if you recall how I was varying the field. So, let me just show you that part. So, if you recall what we did.

So, I am going to write this dt dx is dx mu. So, in 1 d it is going to be dt multiplied by dx and in three dimensional fields then we have dt dx dy dz this is our dx mu basically inspired by relativity this is space time this is space time, but in 1 d it is just dt dx. I am not going to do relativity right now well in between we will make some connection, but we will basically stick to non relativistic fields. So, this is dx mu right now you just think about dt dx. Now, how do I get this delta s? So, I said phi you vary the phi by a delta function right remember we had done this thing.

So, we are I am varying this phi by epsilon delta x minus x prime. So, we said where there is a field and you just give it a small kink at x prime and integrate with dx. So, I am going to basically get something as function of x prime and then do division by epsilon and take that stuff. So, this is the interval with dx. So, Lagrangian which is function of these phi which is perturbed phi and then subtract this with Lagrangian with original phi and its derivatives.

When I subtract of I need to do the expansion of this right Taylor expansion because we will assume these two quantities to be small. Following exactly the same procedure like what we did for particles we get this. So, first this term minus this term will give you that one because basically you will get contribution from here and the second one will get contribution like this. So, I have to take a derivative with relative to this and then multiply by that. But now I have partial well derivative of delta function and delta function I do not know to deal with it.

You recall any function delta x minus x prime f x prime dx prime is f x sorry what I am saying delta right. So, I want without delta function without derivative. So, what do I do here? I use part by parts. So, by parts if I do it then derivative will come here and delta will be outside and that is the reason and we get a minus sign right there will be minus sign. So, that is why the minus sign and we get this.

Under the assumption what there will be another term called surface term right. The surface term know when I do the derivative of parts I will get a surface term which will be delta L delta mu phi and delta function. First function integral of second. So, this is a surface term which is at 0 and T final minus this term the one which is the next term minus this term integral with integral the integral. But this term is 0 surface term because at delta function is 0 at beginning and at the end.

So, we get this and I want the action to be a extremal. So, I get 0 and this is Euler Lagrange equation. It is very similar to what we do for particles right. For particle what do I write partial L delta x equal to d by d t of delta L by delta x dot right that is what we write. So, instead of x I have to say phi right this x is replaced by phi and d by d t is replaced by this derivatives.

But there are four derivatives in 3 D and in 2 1 D times time one time and one space there will be two derivatives. So, let me write this in in. So, this is there will be two derivatives. So, let us write that one will be d by d x delta Lagrange density del x phi and there will be other one which is d by d t minus sign time derivative. Now, I take Lagrange density del of del t phi.

Remember Lagrange density is function of phi d phi by d x and d phi by d t. So, I need to have both these two terms. So, here there was only one independent variable time, and here I have two independent well two basically very independent variable for phi. So, I need to do this x and t. So, if you know this well then you can also write down for fields and you need to know this equation.

In field theory we need to know this equation. This is a classical fields and quantum field. So, we need this we need to derive equations from given Lagrangian is that clear to everyone. So, I want to emphasize on this that we need this put a 0.

So, now let us apply this for string. I know the Lagrangian. So, I should be able to derive the equation. So, this is what I had written in the last slide right. So, this for 1D non-relativistic. Relativistic has only some minor changes you need to keep a covariant, contravariant that combination, but let us not think about that right now.

So, my Lagrangian was this right for a string. So, this is function of this d phi d t and this is d phi d x squared. So, when I substitute what will I get? What is partial L partial L partial phi? 0 right. This is not function of phi ok.

So, partial L partial phi is 0. What is partial L partial d x phi? This term I look at it here this is the term right. d phi d x is the function. So, when I take derivative what will I get? Half kappa and derivative of this with that. So, it is going to be 2 d x phi ok.

So, 2 and half will cancel. This kappa d phi d x. Similarly, what is this one? d t phi is just for this is a minus sign I am sorry this is a minus sign here. This is a L has a this is a minus sign. So, I will get that d L partial L partial of d phi d t. So, this will act on that I will get rho d t phi.

Substitute it here I get first term will give us there is a minus sign out there. So, minus minus becomes plus. So, that becomes plus kappa I have to take a derivative kappa is constant. So, I take a derivative with x. So, I will get d x d x phi double derivative phi with x.

Second term minus and derivative of this with t rho is constant density is constant we will assume that rho d t phi is 0 ok. This means double derivative in time. So, this is the equation of motion for for a string ok that is the equation ok. And of course, we know the solution it is sin k x minus omega t and like that.