Quantum Entanglement: Fundamentals, measures and application

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Week-02

Lec 9: Problem solving session-2

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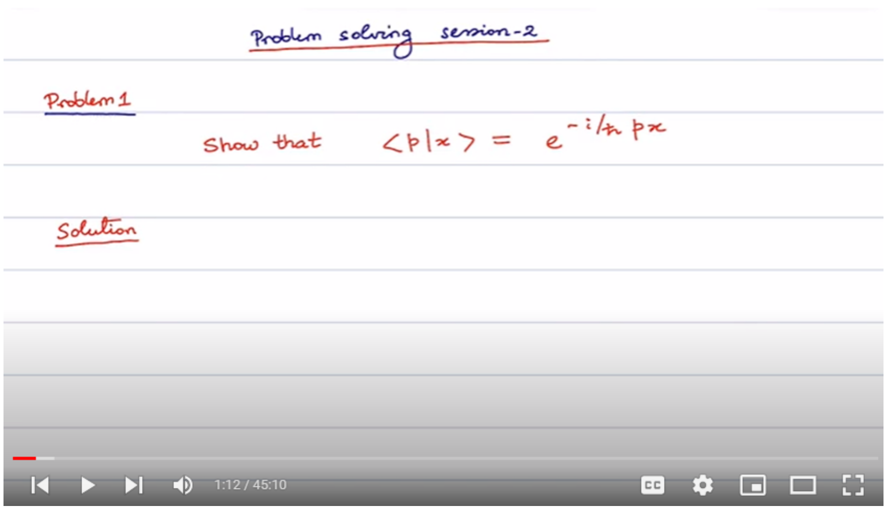

So as a first problem let us work out this. You have to show that the scalar product of P and X is equal to e to the power minus i by h cross P x. I think in the lecture class I put a plus sign here instead of minus I put plus but it should be minus. Okay so let us solve it.

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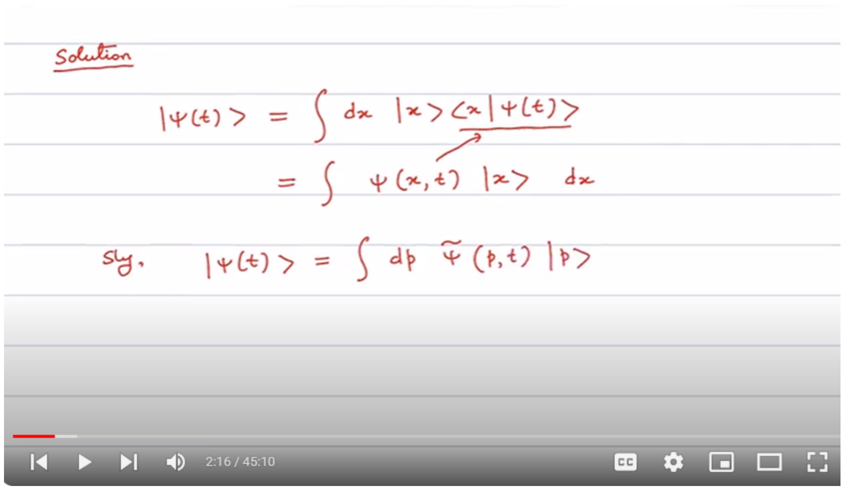

Let me show you this particular relation. In fact if you recall that we have encountered this when we expanded the state vector psi of t, ket psi of t in the continuous basis and we wrote it like this say in the position space basis we can express the state vector psi of t as this and because this is just a number and this is the so called wave function in the or probability amplitude this I can write as psi of x of t this guy and then I have this ket x and d x right. Similarly if you recall in the momentum basis I can express this state vector psi of t as integration dP the momentum wave function psi tilde of P t P.

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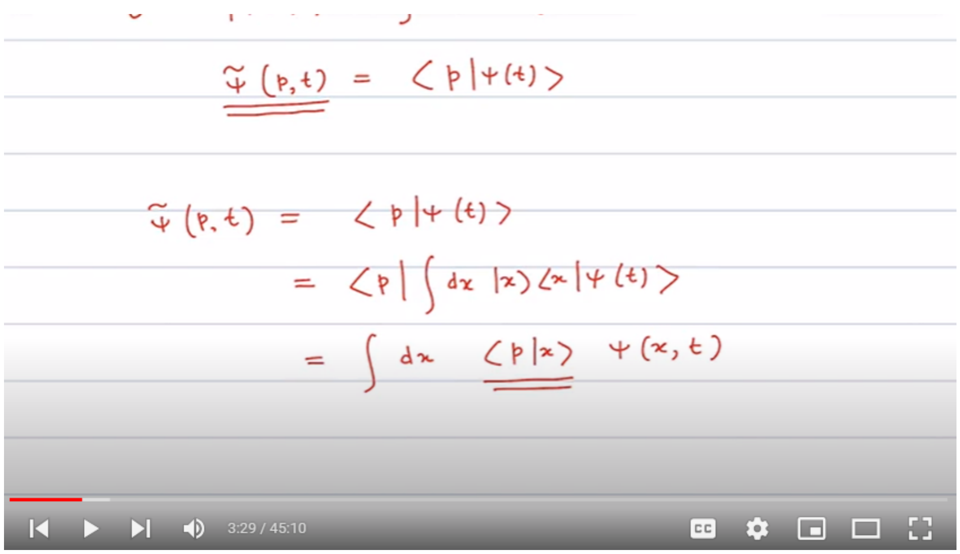

Okay now here psi tilde of P t is the momentum wave function or probability amplitude in the momentum space. Now there is a connection between this wave function momentum wave function and the position wave function and that connection we can establish and while we do that let me show you let me write psi tilde of P t is equal to P of psi of t and I can write it as this I can always put the identity here d x x x psi of t and this I can write as integration d x P of x and I have psi of x t this is the quantity now we have to evaluate or we have to find out.

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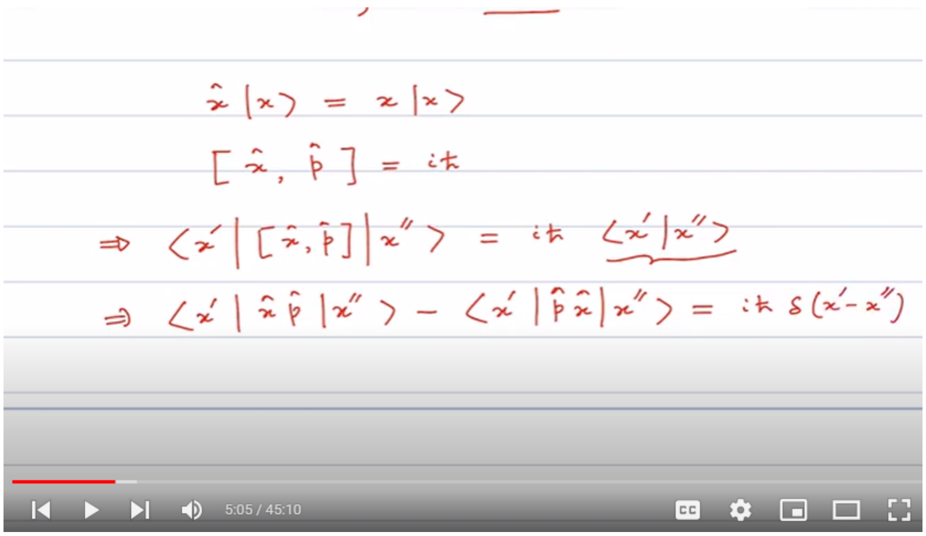

So to do that to do that let me write the eigenvalue equation in the momentum for the position operator so the eigenvalue equation for the position operator is this in the position basis this is what you have and also we know this commutation relation between the position operator and the momentum operator x P is equal to i h cross right this is the so-called uncertainty relation now let me do one thing let me from both sides let me take the inner products in fact let me multiply this expression on the left by x dash and on the right by x double dash then I am going to have i h cross x this x double dash and this I can write as x dash x P x double dash minus x dash P x this is a basically I'm expanding the commutation relation here and on the right hand side I have i h cross this quantity is the Kronecker dirac delta that is delta x dash minus x double dash okay.

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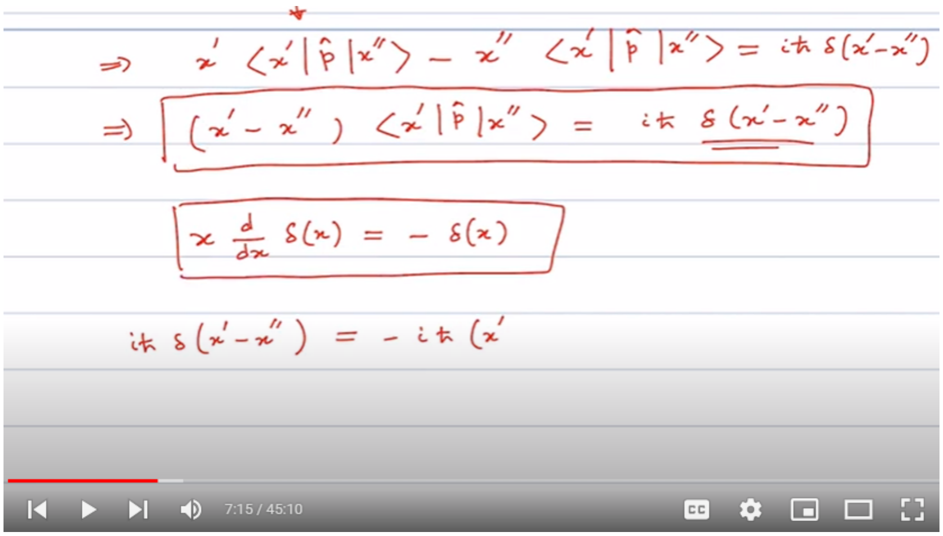

Now you know using the eigenvalue equation I can write it as d if this operates on this then it will pop up x dash so therefore if the position operator x this operate on the bra ket bra x dash then I will get x dash so and I will be left out with x dash P x double dash this is what I get here okay minus similarly I will have from here if this x dash operator position operator operate on the ket x double dash I will get x double dash and I have here x dash P x double dash and on the right hand side I am already having i h cross delta x dash minus x double dash and from here I can write x dash minus x double dash I can take it this side and this would be my common expression x dash P x double dash is equal to i h cross delta x dash minus x double dash all right so this is an important expression here I have. Now I can apply some property of the direct delta function one identity related to direct delta function you can look at any mathematical physics book or any engineering mathematics book you will get this identity x d of dx delta x is equal to minus delta x if I use this expression identity and apply it here then I can write i h cross delta x dash minus x double dash I can write it as minus i h cross just I'm applying the identity this identity I am applying here.

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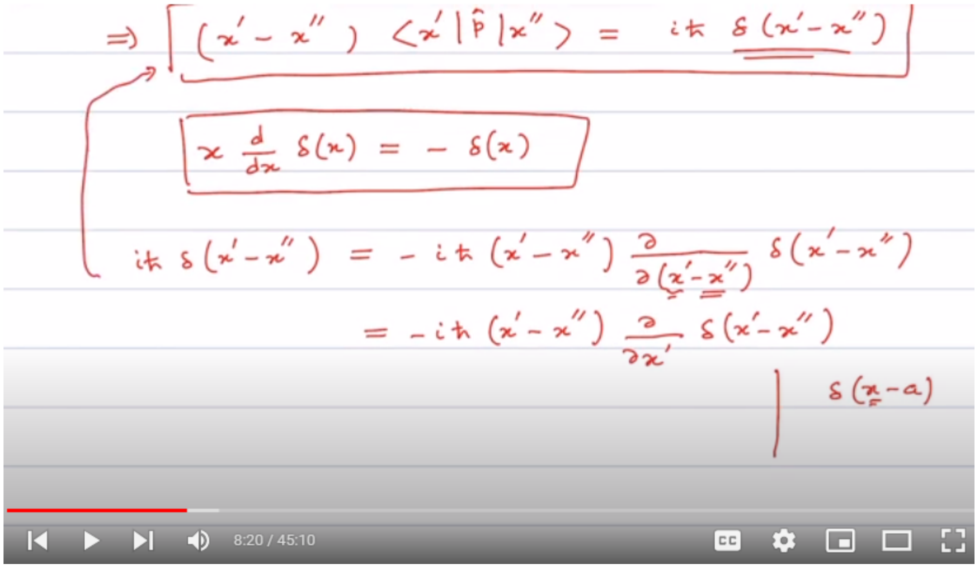

So I will get i h cross x dash minus x double dash delta delta x dash minus x double dash delta x dash minus x double dash this is what I will get and from here I will get minus i h cross x dash minus x double dash delta of delta x dash delta of x dash minus x double dash here you see I am now not you know this is discarded because this is basically a constant or it's a fixed point just like if you remember the direct delta function it has this structure delta x minus a right where a is a fixed point and x is the variable so here here this x is the variable quantity here x dash so therefore that's why I can write it in this form.

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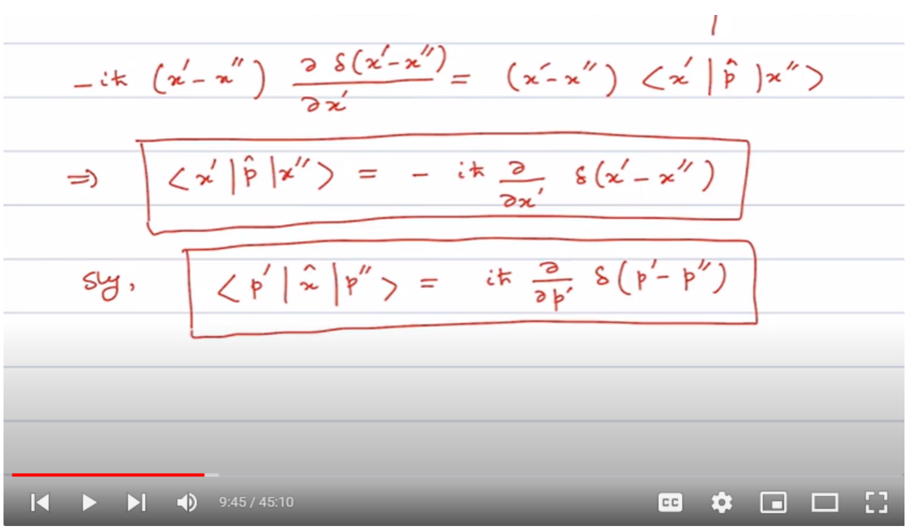

So utilizing this expression here I can now write as minus i h cross x dash minus x double dash delta of delta x dash minus x double dash delta of x dash is equal to on the right hand side I will have x minus x double dash x dash minus x double dash and I have here x dash p x double dash okay so this implies I can write x dash p x double dash is equal to minus i h cross delta x dash delta of x dash minus x double dash so this is a very important result we have and similarly we can you can actually show that you will get another expression relate in the momentum basis that would be p dash x dash p double dash is equal to plus i h cross delta p dash delta p dash minus p double dash so these two expressions are now going to be useful in proving our result.

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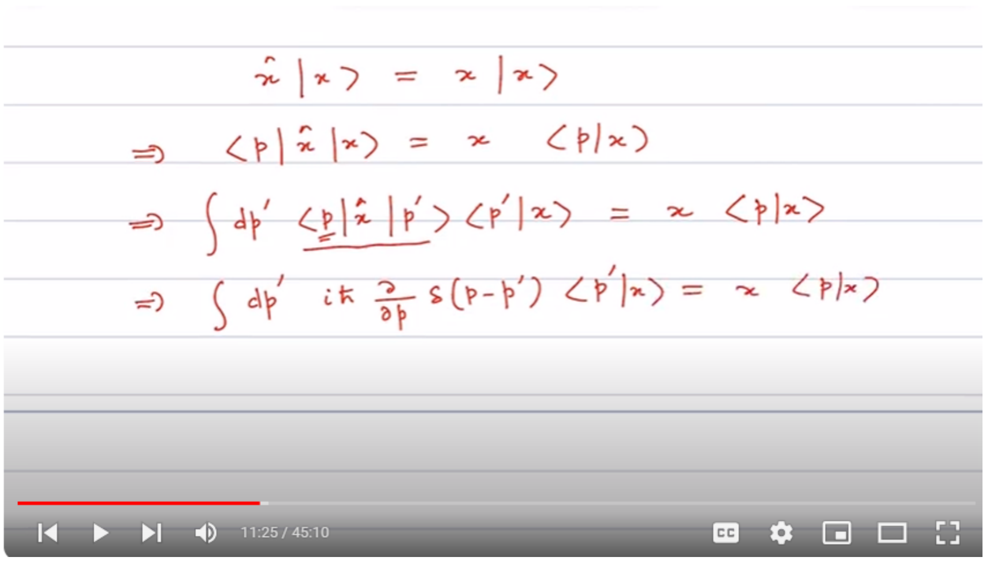

So let me once again go back to the eigenvalue equation x dash is equal to x operator position operator operating on the ket x is equal to x x is a number here and this we have a ket x right so let me multiply both sides by this bra p so I have this is what I will have and here x is a number so I will have this guy now let me use the identity here I can just use see how I do it let me put this identity dp dash let me put here and p x operator then let me send do is this p this p this that's the identity I have put here and then I am here x and this is equal to x the right hand side is left untouched I have used the identity only on the left hand side and from here I can write dp dash and this expression I already know this is what I have so if I use this here I will get i h cross i h cross delta p dash okay in fact because I have here this p not p dash so this would be p here okay delta p delta p minus p dash p dash x is equal to x p of x okay.

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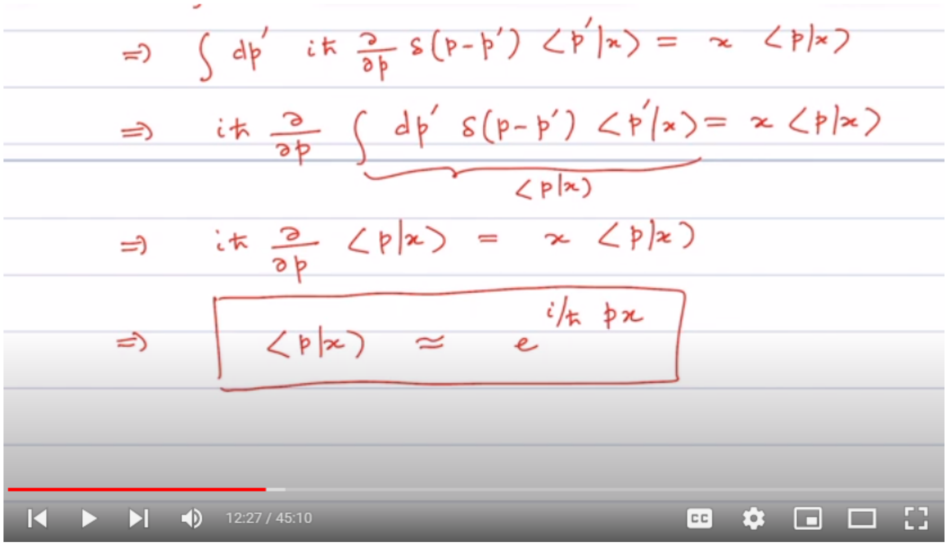

This I can further write as i h cross delta p because p dash is the variable here so I can take this integration inside and I have delta p minus p dash p dash x is equal to x p of x so from here I get this differential equation applying the property of the dirac delta function I have i h crosse delta p p x I have applied the property of the dirac delta function right this is going to give me simply p x so this would be equal to x p x so this differential equation you can easily solve and you will simply get p x x would be equal to e to the power i by h cross p x so this is what we were asked to prove.

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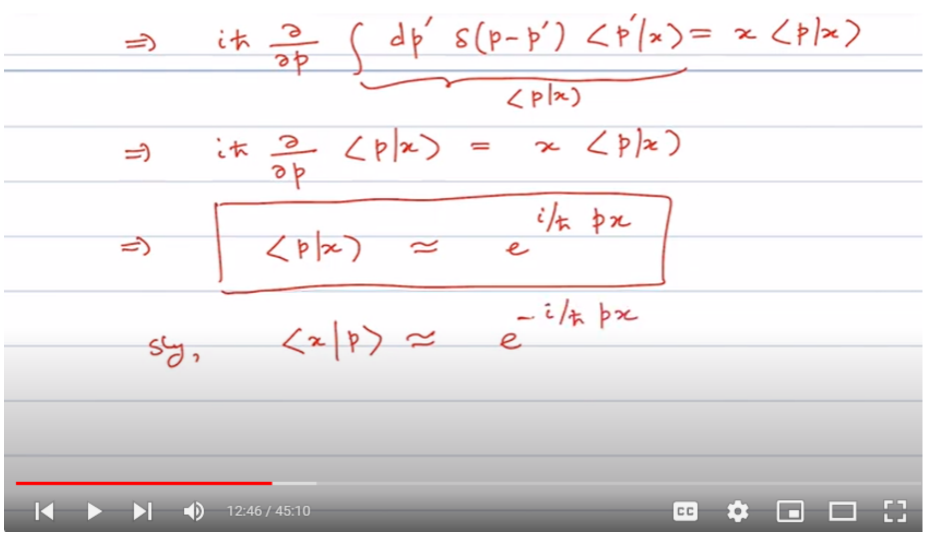

Similarly you can prove that x p would be equal to e to the power it would be some factor may be there but it would be overall proportional to e to the power minus i by h cross p x so this is how you can prove these relations.

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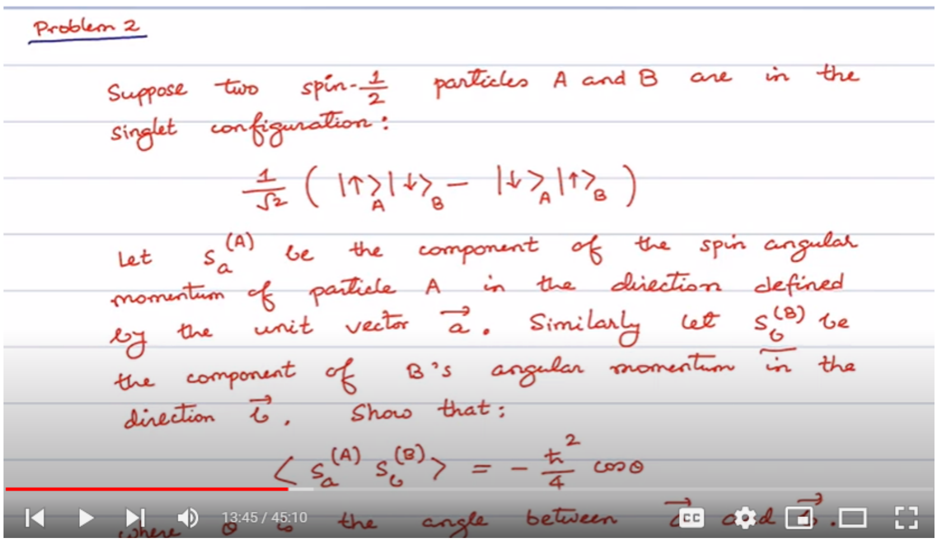

Okay now let us work out this problem suppose two spin half particles a and b are in the singlet configuration let sa be the component of the spin angular momentum for particle of particle a in the direction defined by the unit vector a similarly let sb be the component of b's angular momentum in the direction b you have to show that the product of the spins for both particle a and b is given by this expression minus h cross square h cross square by 4 cos theta where theta is the angle between a and b this is an important problem and useful in the context of Bell's inequality that we have discussed in the class and this is completely a quantum mechanical result so let us show this prove it.

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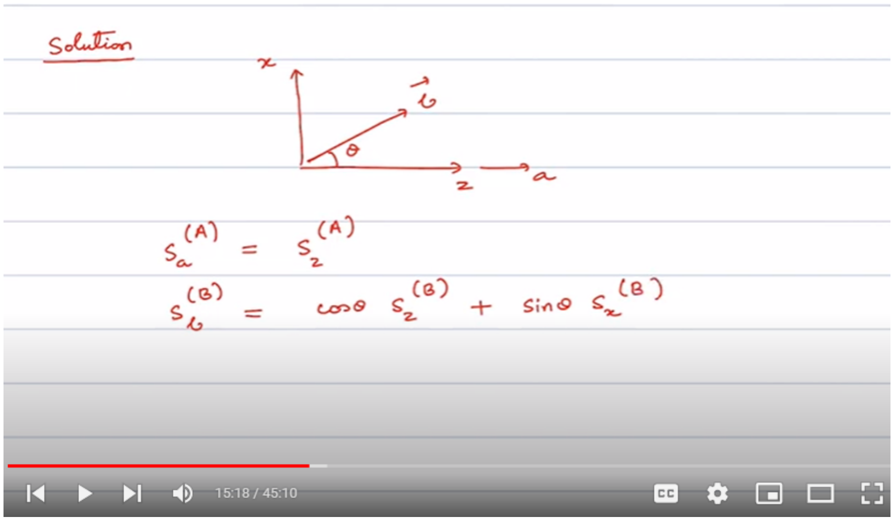

This is what it is so basically what we are having is a suppose my z direction is along this and x direction is along this direction let me assume that my the detector direction a is along z and the detector direction the other direction it is along this direction b is in the x-z plane and it is directed along this and the angle between a and b is theta okay so in that case i will have this spin direction you see sa for the particle a capital a it is because a i am taking along z direction small a so it will be sza for particle a and for the particle b i have for particle capital b this this its detector direction is along direction small b and which is lying in the x-z plane so i can write it as cos theta as z b i think it is easily understandable and the other one would be sine theta s x b right so this is what i have.

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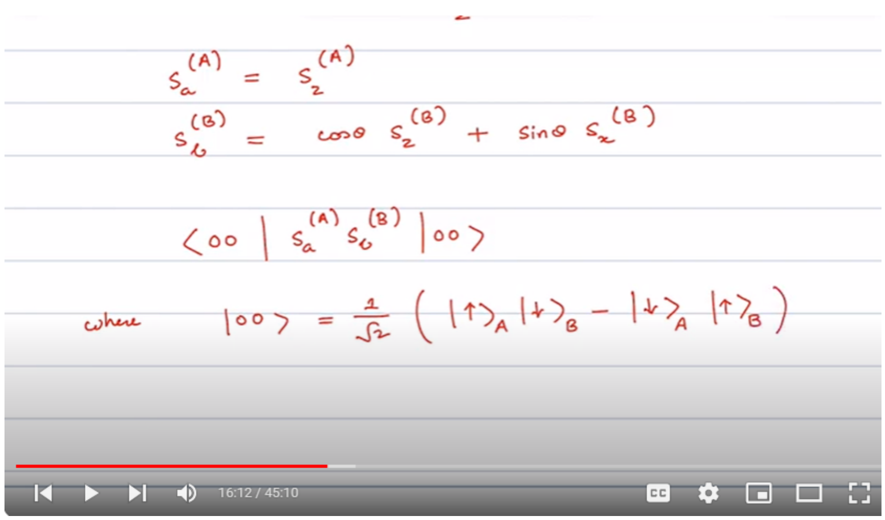

Now our goal is to calculate the expectation value of the product of the spin operators sa and sb and we have to calculate the products with respect to the singlet state and let me denote the singlet state by 0 0 ket 0 0 where this ket 0 0 as it is given as 1 by root 2 up spin a is in the up state then b is in the down state because the total angular momenta has to be 0 that's why it is 0 0 state or if spin of the particle a is in the down direction the spin of the particle b has to be in the up direction this is basically an entangled state .

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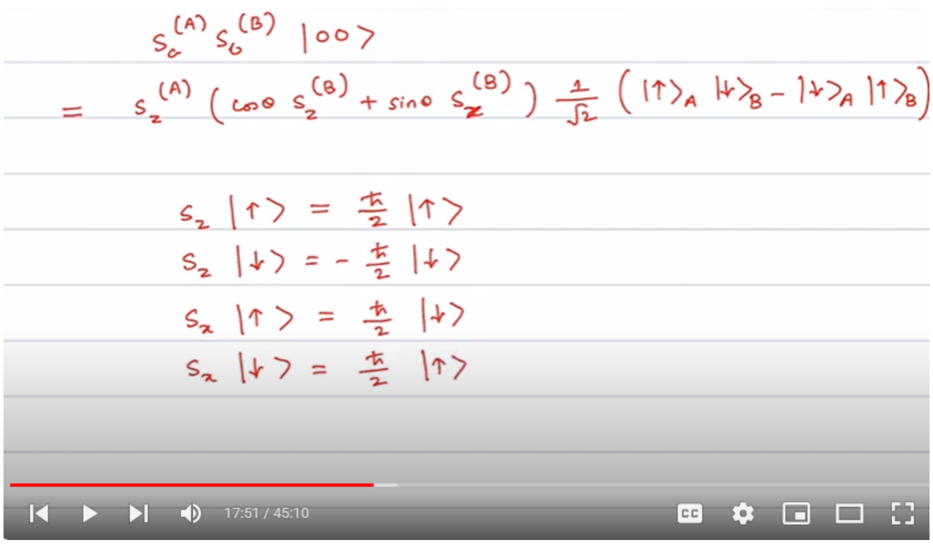

So to work it out first let us calculate saa sbb first let us work it out okay. To work that out let me already i have taken a to b along z direction so let me write it like this and sb is cos theta as z b plus sine theta as z b s x b okay s x b and this is operating on ket 0 0 which is 1 by root 2 up a down b minus down a up b okay so this is what i have. Now before i proceed further we need to note that in the sz representation or in the basis of the sz operator i have this results you can just consult any quantum mechanics book in the sz representation if the sz operator operates on the up spin then you will get h cross by 2 this eigenvalue equation you will get if it operates on the down spin you are going to get minus h cross by 2 down and if sx operates on the up spin it basically flips the spin and it is going to make it down and if sx operates on the down spin and it is going to flip it and it will make it in the up state okay.

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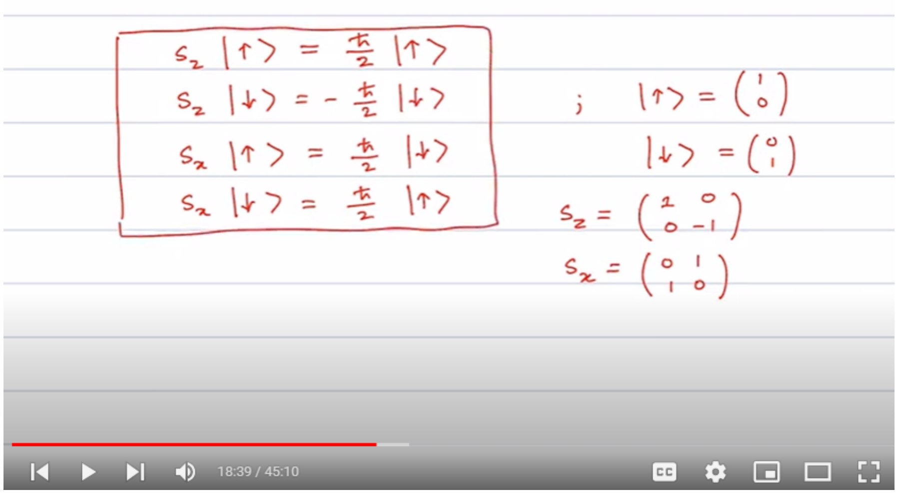

So this is what you have to utilize it and this is very easy to prove these relations just you have to note that up spin is we can represent it by this column vector and the down spin is represented by this column vector 0 1 and sz in the matrix representation it will be 1 0 0 minus 1 and sx is equal to 0 1 1 0 so if you can if you utilize this matrix representation then you can easily verify that these results that i have written here is correct so please do that if you are not convinced.

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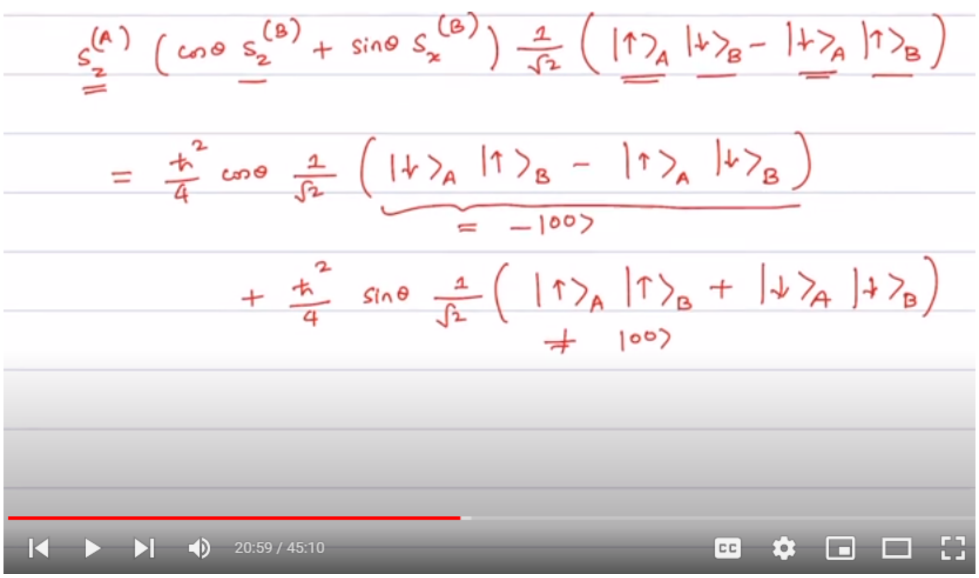

Now utilizing this i can work out my expression here so first let me work it out so i have sz a cos theta sz b plus sine theta as x b and it is going to operate on 1 by root 2 up a spin up a spin down b minus down spin a and up spin b now as s z a operates on only the spin component having this part or this part and similarly s z b operates on the spin component referring to the b part only on the ket referring to that is denoted by this b so if i open it up and i ask you to do that if you do this finally what you are going to get let me write it down it's very simple to do that you will get h cross square by 4 cos theta let me write it more clearly you will get h cross square by 4 cos theta 1 by root 2 down a up b minus up a down b which is basically this expression if you can see this is nothing but the opposite of this singlet state ket 0 0 right anyway this is equal to this and you have another part also that would be h cross square by 4 sine theta 1 by root 2 you will have up a up b plus down a and down b all right this is obviously not equal to ket 0 0 it is something else it is orthogonal to ket 0 0.

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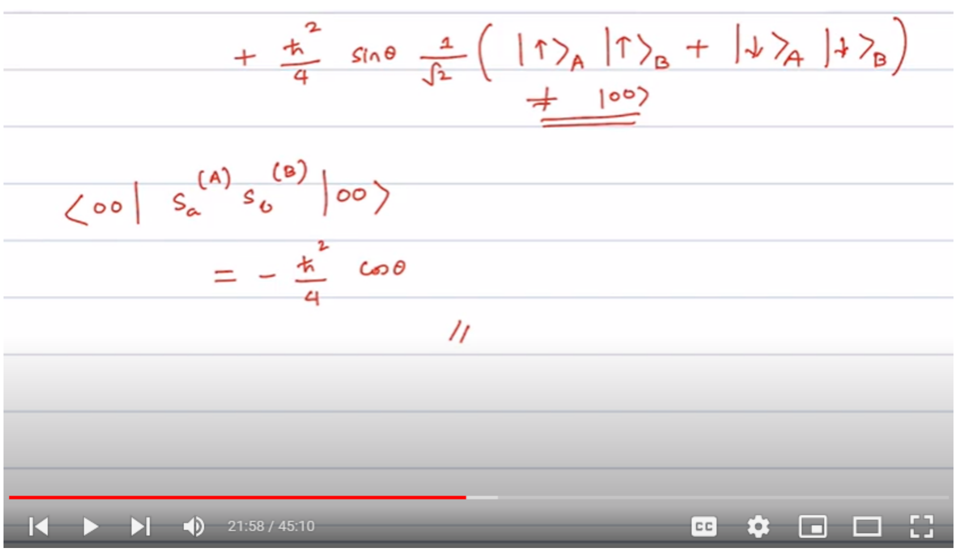

So therefore if i now take the inner product with ket 0 0 s a a s b b ket 0 0 this is already i have this is my ket 0 0 right so because of the orthogonality and this guy is orthogonal to ket 0 0 so this will lend up me in and this is minus is there so minus h cross square by 4 cos theta okay this whole thing is ket 0 0 with a minus sign so therefore minus h cross square by 4 cos theta is what i am going to get and hence the relation i am able to prove it this is a quantum mechanical result okay this we have discussed in the context of Bell’s inequality as i have already told you.

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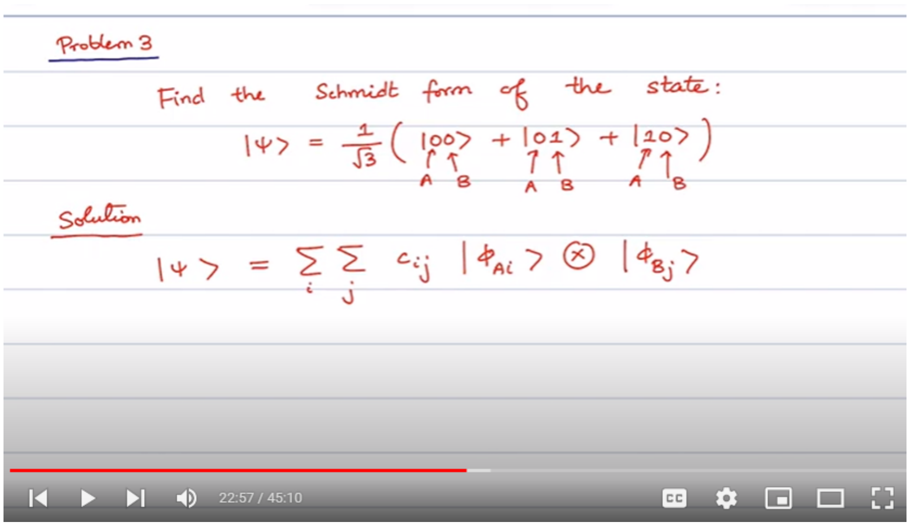

Let us now work out this problem you have to find the Schmidt form of the state ket psi is equal to 1 by root 3 ket 0 0 plus ket 0 1 plus k 1 0 let us do it as you know i can express this particular ket psi in this form as well say summation over i j its coefficient c i j phi a i direct product with phi b j and here the first ket refers to system or particle a the second ket refers to particle b and so on here 0 refers to a this one refers to b here this refers to a this refers to b.

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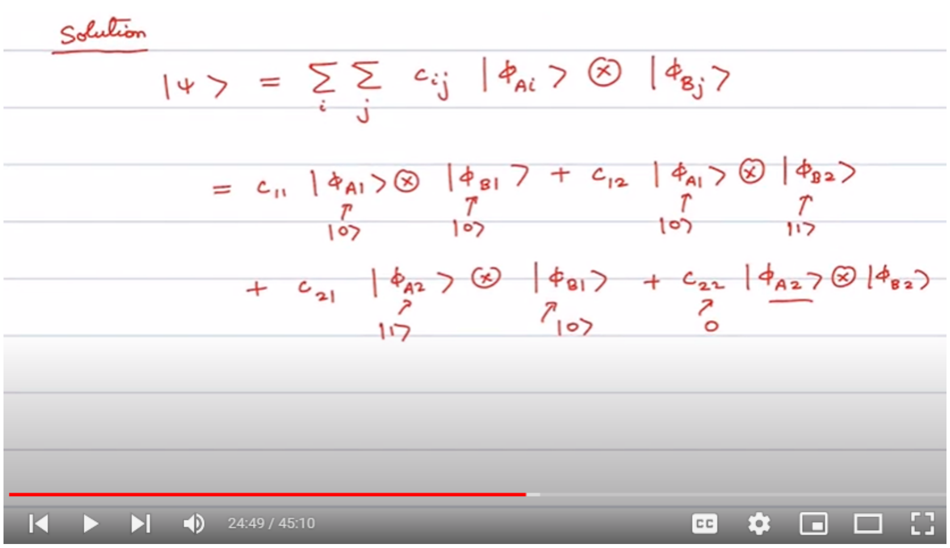

So i if i just open it up let i will what i will get is this and uh okay let me write here c 11 would be phi a 1 direct product with phi b 1 and this phi a 1 as you get phi a 1 is as you can see this ket 0 and phi b 1 as you can see it is ket 0 and you will get other expression like this c 12 that would be phi a 1 direct product with phi b 2 and again you see phi a 1 is ket 0 phi b 2 is ket 1 and you will get c 21 would be phi a 2 direct product with phi b 1 and phi a 2 is ket 1 phi b 1 is ket 0 i hope you can see it and the other terms would be phi set c 22 let me write this also c 22 would be phi you know it would be uh it would be a 2 direct product with phi b 2 but already c 22 is is going to be 0 because there is no phi a 2 that is not there right phi a 2 is 0 this is not there it is null and similarly phi b 2 is it is though it is there but because phi a 2 is is a null uh vector null state so therefore c 22 would be equal to 0.

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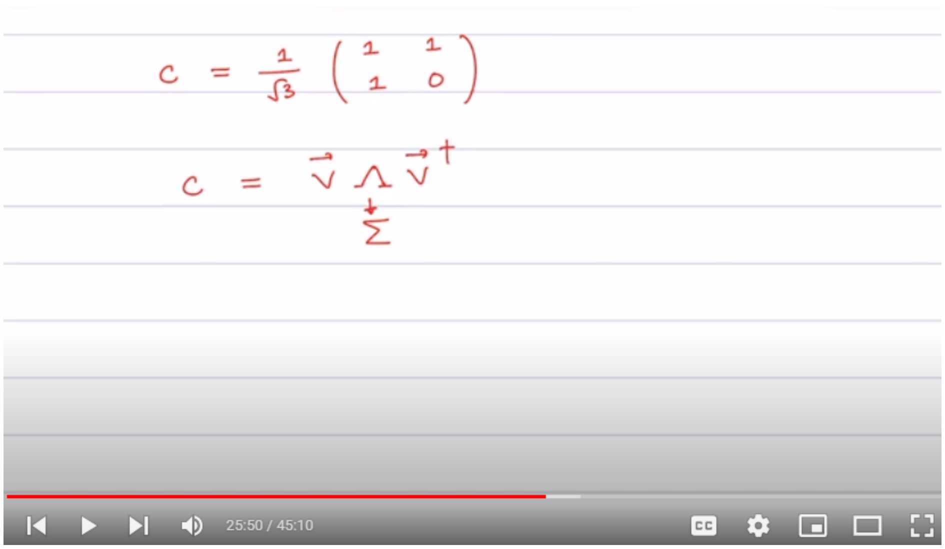

Utilizing this i can now write the c matrix c matrix would be if you look at this ket here from this you can make out that the c matrix would be 1 by root 3 1 1 1 0 so it's a square matrix and as you can see it is easily diagonalizable so therefore the singular value decomposition here boils down to getting the decomposition only so c i can write it as the v vector and this diagonal matrix sigma lambda and then i have v dagger or i can just say this is nothing but in singular value decomposition this is going to be my sigma so first let me diagonalize this matrix and to that let me find out the eigenvalues and eigenvectors of this matrix c so eigenvalues eigenvalues of this matrix c.

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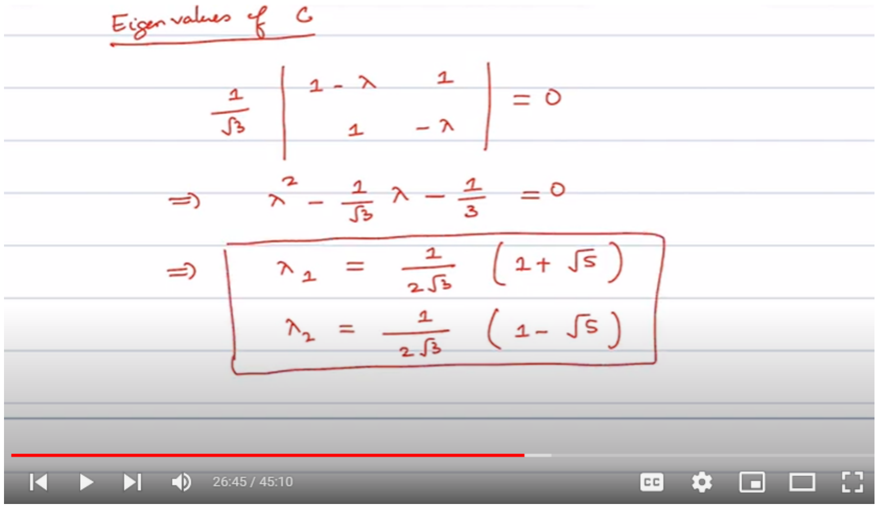

To do that i just have to find out the characteristic equation that would be 1 by root 3 1 minus lambda 1 1 minus lambda is equal to 0 and if i do that this is going to give me the equation lambda square minus 1 by root 3 lambda minus 1 by 3 is equal to 0 and this is going to give me the eigenvalues lambda 1 would be equal to 1 by 2 root 3 1 plus root 5 and lambda 2 would be equal to 1 by 2 root 3 1 minus root 5 so these are my eigenvalues okay.

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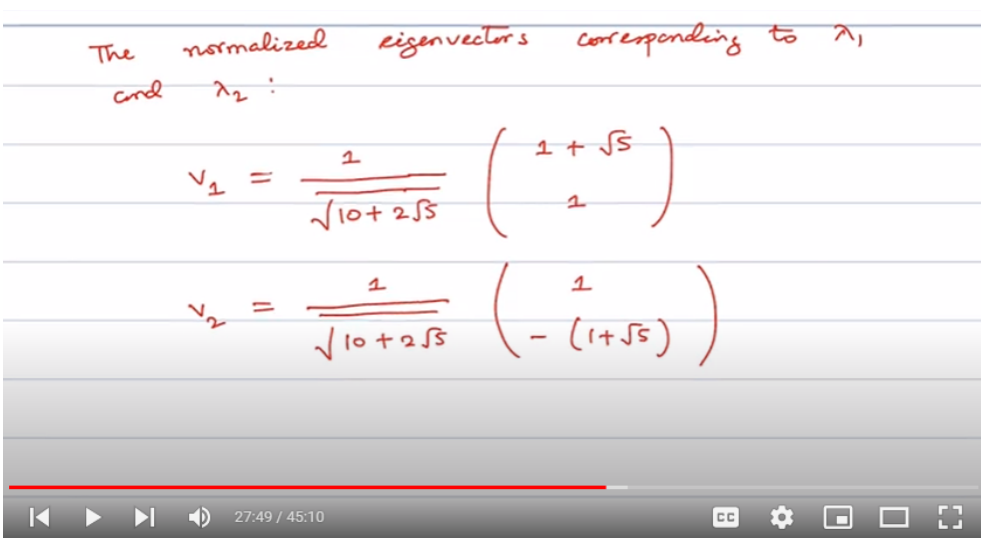

What about the corresponding eigenvectors the normalized eigenvectors and it's very trivial to work it out anybody knowing can work it out the normalized eigenvectors normalized eigenvectors corresponding to corresponding to lambda 1 and lambda 2 you please work it out it's very trivial you will get for lambda 1 it would be v 1 is equal to 1 by square root of 10 plus 2 root 5 and here i will have 1 first row 1 plus root 5 and in the second row i have 1 and v 2 would be equal to 1 by this is the eigenvector corresponding to lambda 2 that would be 10 plus 2 root 5 and here you will have 1 minus 1 plus root 5 so these are the eigenvectors.

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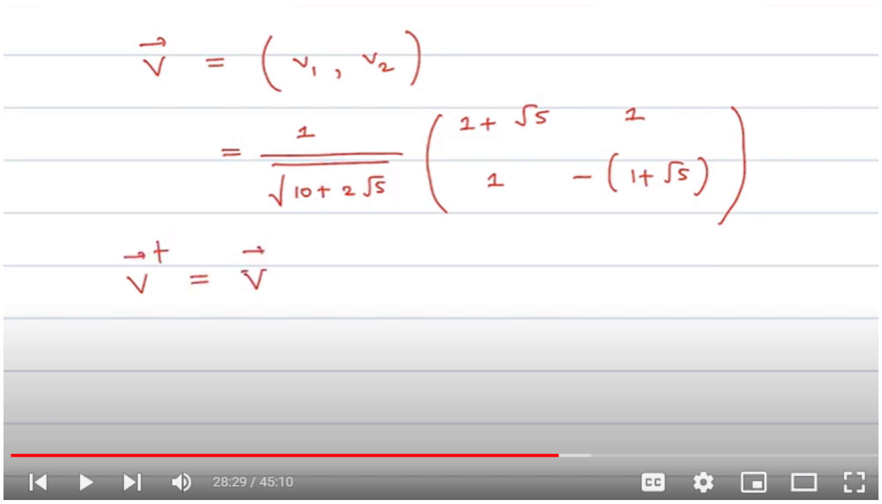

So therefore this v vector that it be formed by this v 1 and v 2 okay and if i can write it let me just write the whole expression whole v vector in the matrix form that would be 10 plus 2 square root of 5 and the elements here would be diagonal would be 1 plus root 5 1 1 minus 1 plus root 5 so this is my v vector and as you can see immediately that v dagger is actually going to be the same with v okay.

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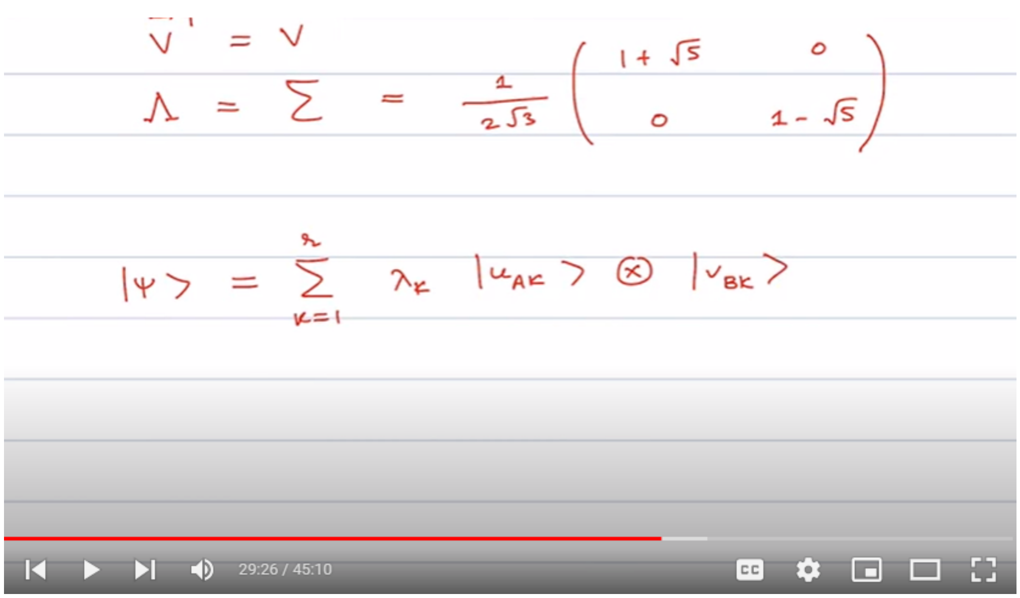

And this diagonal matrix which is lambda basically i can also denote it by sigma as that is what i have used in singular value decomposition this would be sigma would be equal to 1 by 2 root 3 1 plus square root of 5 0 0 1 minus square root of 5 okay so this is what i am going to have. Now what about the Schmidt decomposition or the that form so as we have seen in the lecture class we can write this ket psi in the Schmidt form as follows we are going to have k is equal to 1 to r lambda k u a k direct product with v b k okay.

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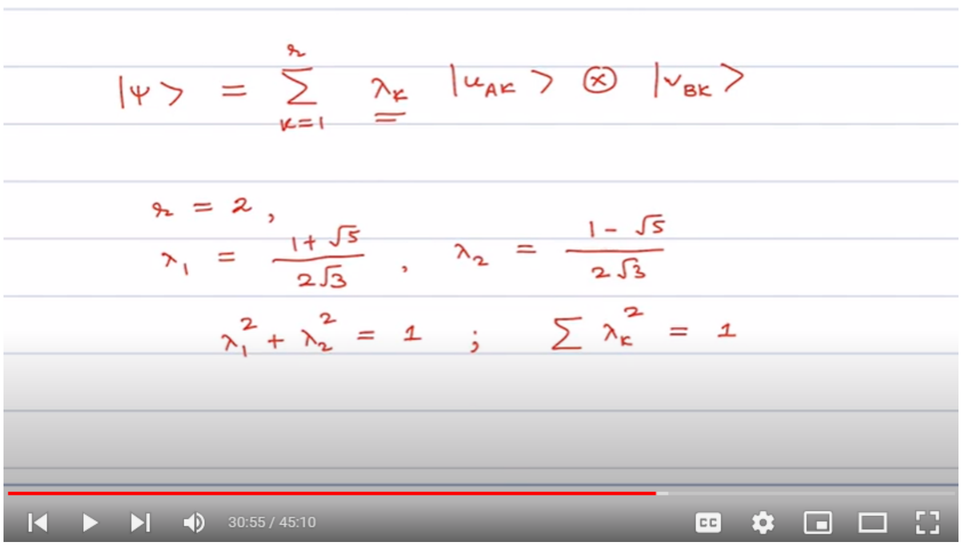

And what about r here in this particular problem here as you can see r is going to be equal to 2 why because there are two non-zero elements in the diagonal matrix in the sigma matrix as you can see here there are two non-zero elements in sigma so therefore r is equal to 2 and what about the coefficient lambda so there is going to be two coefficients lambda 1 and lambda 2 so let us not mix that with the so-called eigenvalue here this is i am talking about the coefficient Schmidt coefficient and you can easily make it out that this is going to be equal to lambda 1 would be equal to 1 plus root 5 divided by 2 root 3 this is you can get it from the sigma matrix here and lambda 2 is equal to 1 minus root 5 2 root 3 and also you can verify if you see you can easily find that lambda 1 square plus lambda 2 square is equal to 1 which is one of the co condition that this Schmidt coefficient has to satisfy as you know lambda k squares would be equal to sum of lambdas should be equal to square of lambda should be equal to 1.

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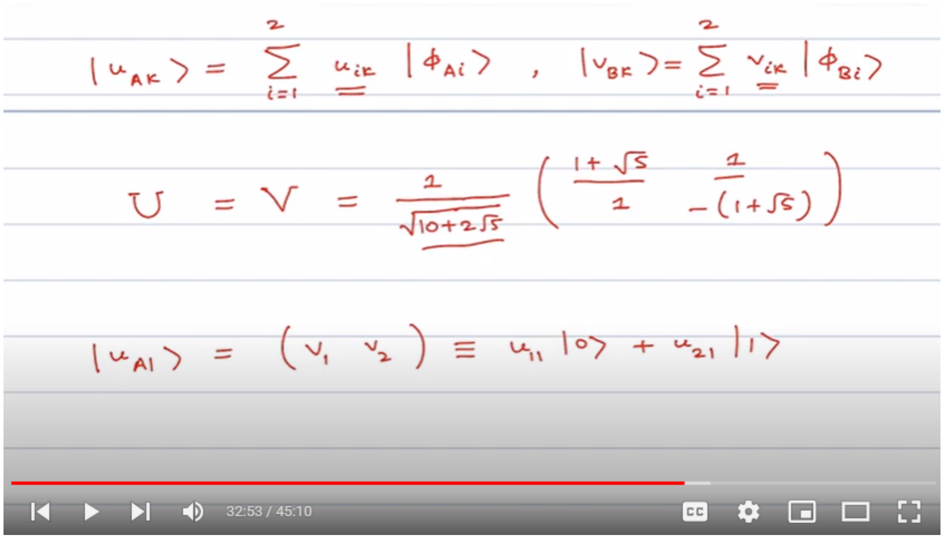

So using utilizing all this information that we are having now let me just write down what is this ket u a k which is i can write it as i is equal to 1 to 2 u ik phi a i right and we also have v bk is equal to summation i is equal to 1 to 2 v ik phi bi now you notice that u ik and v ik these are basically the matrix element of the matrix capital u and capital v and here capital u is nothing but the capital v and which we have already worked out and that was v is equal to 1 by square root of 10 plus 2 root 5 this we have worked out and the elements were 1 plus root 5 1 1 minus 1 plus root 5 so is the for capital u. So therefore i can now work out u a1 for example this u a1 small u a1 is going to be formed by this v1 v2 which is equal to if i just uh open it up i am going to have it as u11 ket 0 you please have to verify it very patiently it's very straightforward u21 ket21 and u11 here is the this matrix element multiplied by this also of course and u21 is this one right similarly for u12 and u22.

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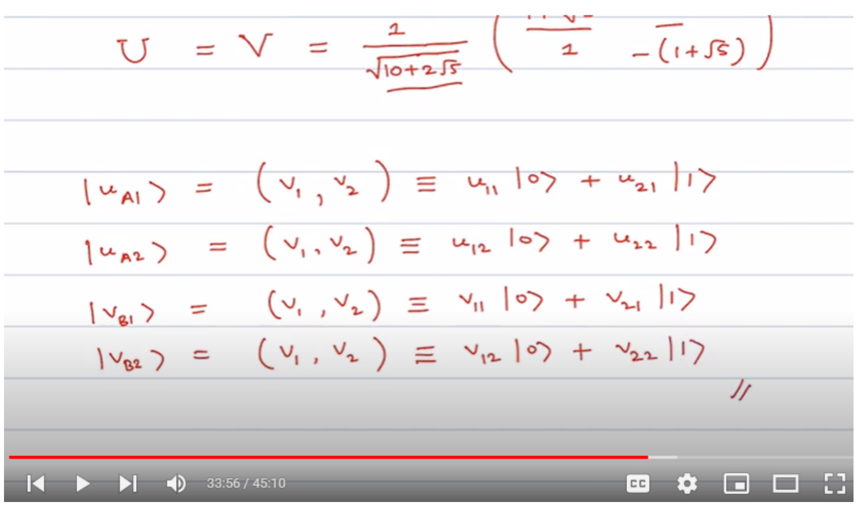

And u a2 also you can find out u a2 is again this would be given by this mid vector v1 v2 and in fact this can be written in this form that would be u12 ket 0 plus u22 ket 1 and similarly you can find out what is your v b1 this would be the Schmidt vector it will be given by this mid vector only so it would be v11 ket 0 plus v21 ket 1 and v b2 would be equal to in matrix form this is what you will have and if you write it in the this decomposition or in this form in superposition form you can write it as v12 ket 0 plus v22 ket 1 okay that's how you can work out this particular problem.

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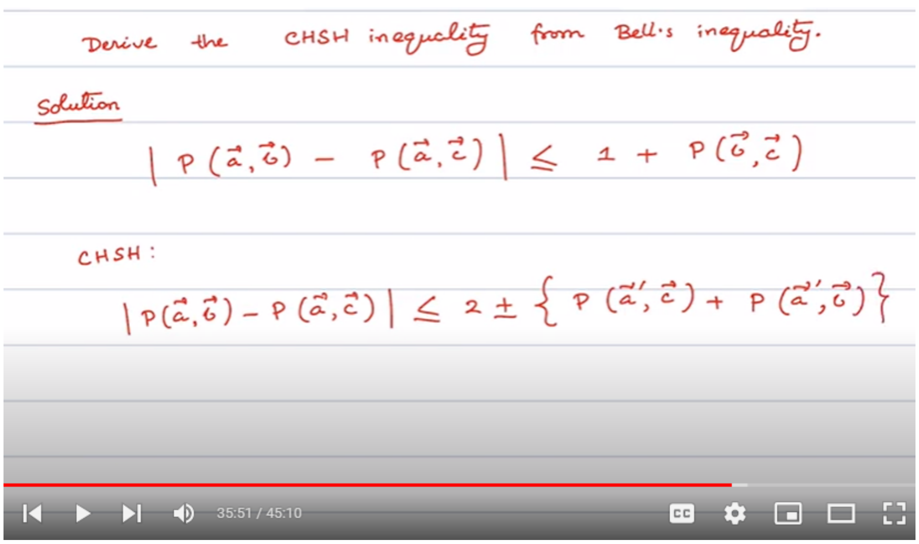

Now let us work out this final problem derive the CHSH inequality from Bell's inequality this actually i have promised you to do in the problem solving class so let me do it so to do that let me write down the Bell's inequality that we have derived in the lecture class and it was as follows it was the expectation value of or the average of the product of the spin measurements here measurements when the given set of detectors are oriented along a and b so that's what p a b refers to it again let me reiterate that p a b refers to the average or expectation value of the product of the measurements in here by measurement i am referring to spin measurements when the given set of detectors are oriented along a and b minus p ac similar meaning to p ab that modulus is less than or equal to 1 plus p bc right this is what we have derived now the so-called this is Bell's inequality and the CHSH inequality is p ab minus p ac modulus is less than or equal to 2 plus minus p a c plus p a dash b okay so this is what we have as CHSH inequality so let us now derive it.

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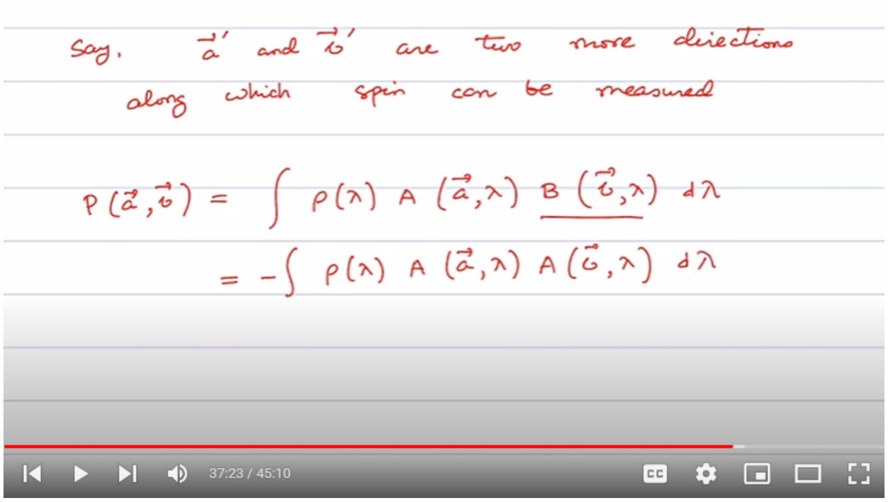

To do that let us say a dash and b dash are two more directions are two more directions along which spins can be measured along which spin can be measured can be measured so basically we are introducing to other detector directions now you have to recall that p ab we wrote it like this we have written it as the density function rho lambda that is the density of the hidden variables a is associated with the measurement corresponding to system a or particle a ab b lambda d lambda this is what we had and also from here i can write rho lambda minus of rho lambda a a lambda a now you see b i'm replacing b i'm replacing by a that's why this minus sign is coming up and we have already discussed it in the lecture class so please refer to that so this is what i'm going to get.

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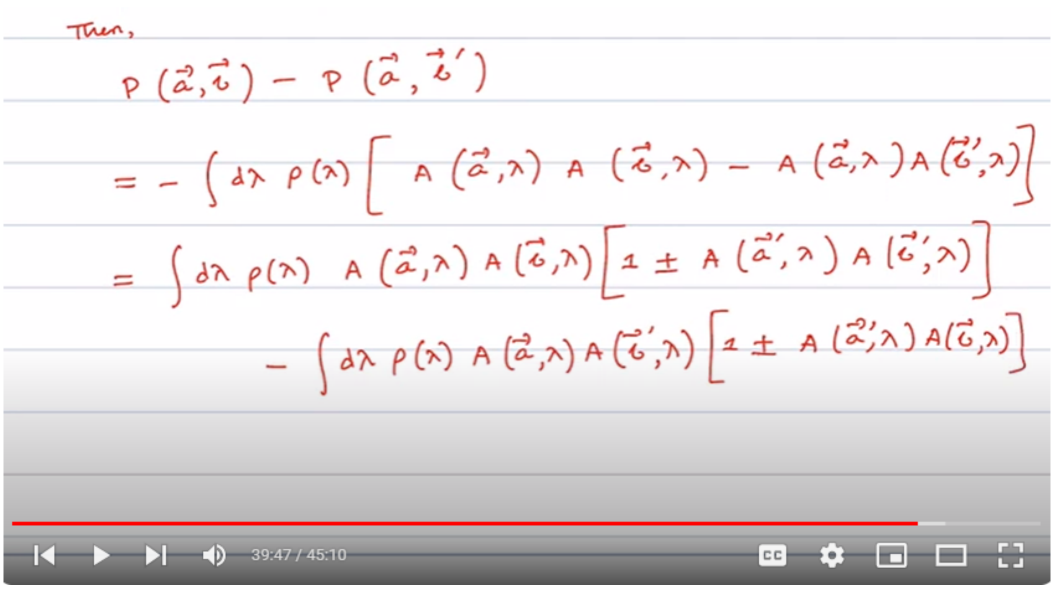

Then utilizing this i can write p ab this is simple algebra so you just have to work it out p a b dash and this would be equal to and this would be equal to utilizing this this expression here this or actually let me say specifically i'm going to use this expression if i utilize this expression i can get minus d lambda rho lambda a capital a a lambda a b lambda minus a a lambda a b dash lambda okay this is what i will get and in the similar way that we have done in the lecture class if you follow similar steps it is very straightforward to work it out i'm not going to do it here because that will be repetition of the class actually so if you can do that in the similar way just look at the lecture class you will get the d lambda rho lambda i'm giving you the procedure only a a lambda a b lambda and here i will have i'm going to take that common then i will get one plus minus a a dash lambda ab dash lambda dash is what i will have okay of course there's another term is also there that is minus d lambda rho lambda a a lambda a b dash lambda and i have here one plus minus a a dash lambda a b lambda okay so this is what i am going to have.

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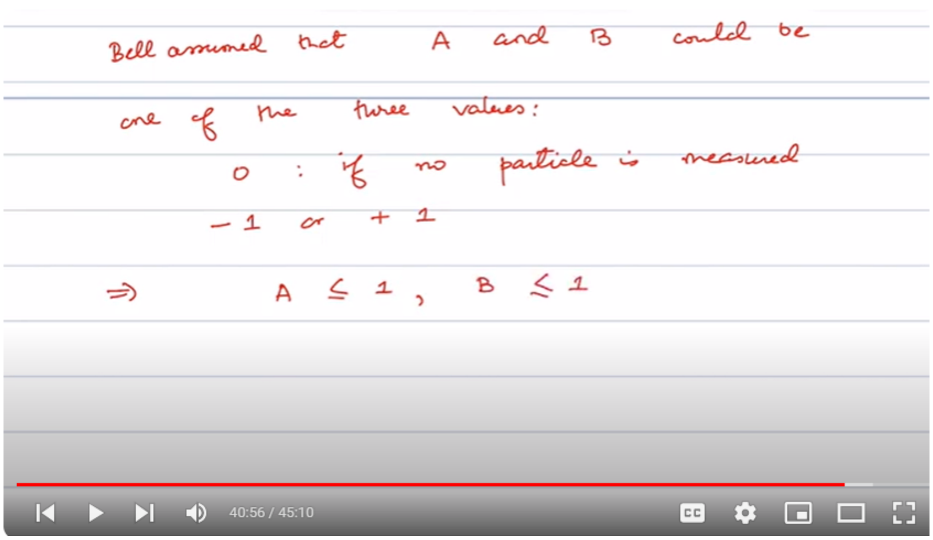

And now you see the Bell assume that a and b could have this capital a and capital b let me write about Bell's assumption Bell John Bell assume that a and capital a the measurement basically a and b could be one of the three values three values and that's very straightforward to understand one value is zero if that is the case when if no particle is measured no particle is measured then you are going to obviously get zero value or you are going to get either spin down or minus one or spin up that is plus one in the unit of h cross by two that means this implies that i can have a is less than or equal to one and b can be less than or equal to one.

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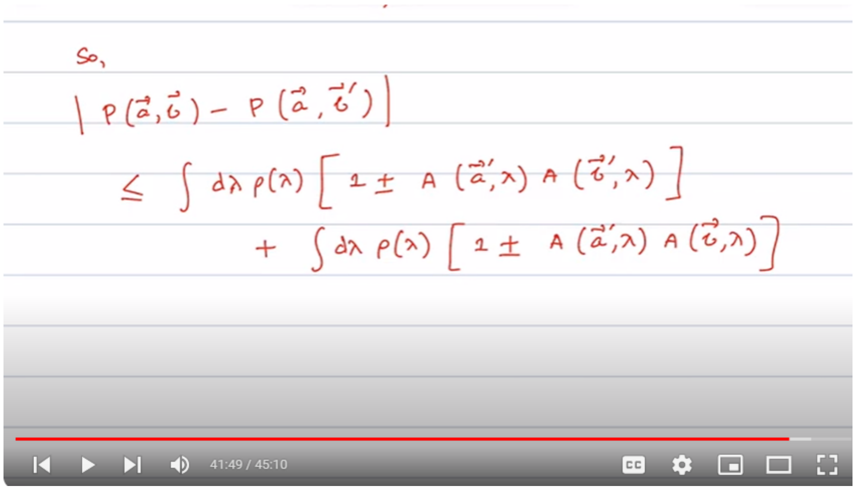

So utilizing this this information this particular information i can write modulus of p a b minus p a b dash that would be less than or equal to integration d lambda rho lambda one plus minus a a dash lambda a b dash lambda this is one expression one part and another part is integration d lambda rho lambda one plus minus a a dash lambda a b lambda okay.

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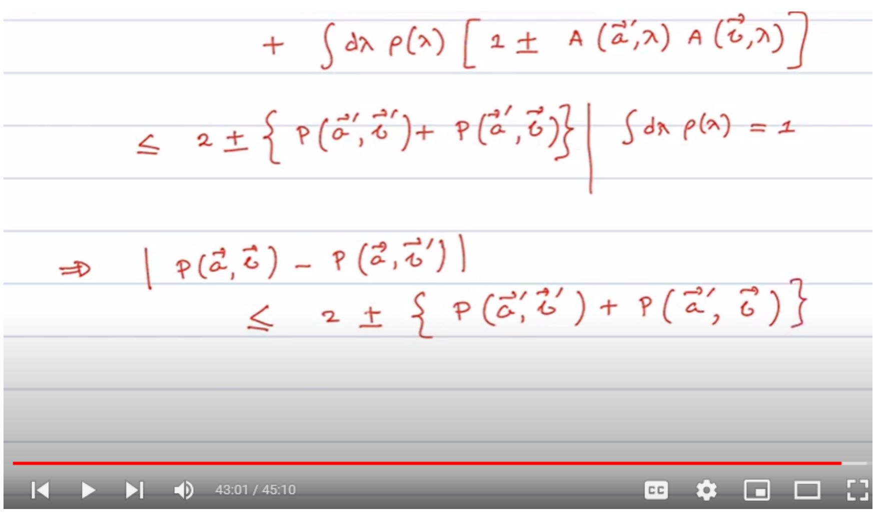

And this is actually equivalent to from here i can write this is equivalent to because you know that d lambda rho lambda is equal to one so utilizing this i can write it as two plus minus and this would be p a dash b dash plus p a dash b okay so this is the so-called Bell's inequality i have. So let me write again this implies that i am having modulus of p this is actually CHSH inequality that is what we wanted to prove this would be modulus of p a b minus p a b dash is less than or equal to two plus minus sum of p a dash b dash plus p a dash b okay.

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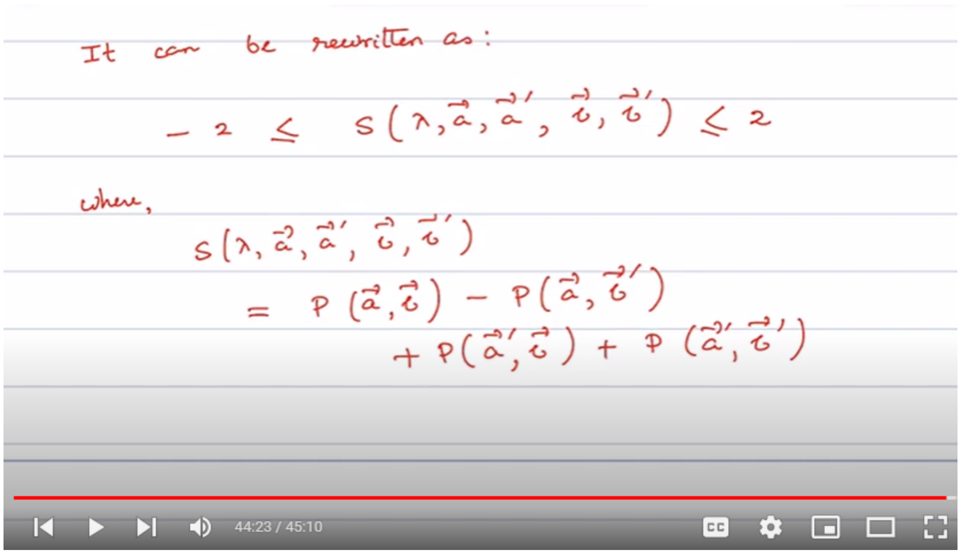

And this can be written in a different form l also it can be rewritten you can verify it it can be rewritten in an another form and that is this this minus two less than or equal to s a quantity we define as s lambda a a dash b b dash less than or equal to two and where this quantity called s lambda a a dash b b dash is defined as p a b it's very straightforward to very simple to verify it you please do it yourself it's simple a b dash plus it's algebra a dash b plus p a dash b dash okay now actually this is an important inequality CHSH inequality which is nothing but Bell's inequality it's a only a different form and based on this inequality a similar kind of CHSH inequality was proposed by clausar and his research group based on that they made an experimental proposal and later on elaine aspect did the experiment as you know Clause and Alaine aspect are two of the three guys who got the 2022 physics Nobel prize for their work on quantum entanglement.