Quantum Entanglement: Fundamentals, measures and application

Prof. Amarendra Kumar Sarma

Department of Physics

Indian Institute of Technology-Guwahati

Week-02

Lec 8: The EPR Paradox and Bell Inequalities

Hello, welcome to lecture 6 of this course. This is lecture number 3 of module 2. In this lecture, I'm going to discuss about the so-called EPR paradox, Einstein-Podolsky and Rosen paradox and also I will discuss about Bell's inequalities. Hopefully, this lecture would give you an idea how the concept of entanglement came into existence and also some philosophical aspects associated with it. Perhaps in lecture 1 of this course, I told you that Einstein never felt comfortable with the theory of quantum mechanics. He did not say that quantum mechanics is wrong, but he said that this theory of quantum mechanics is incomplete.

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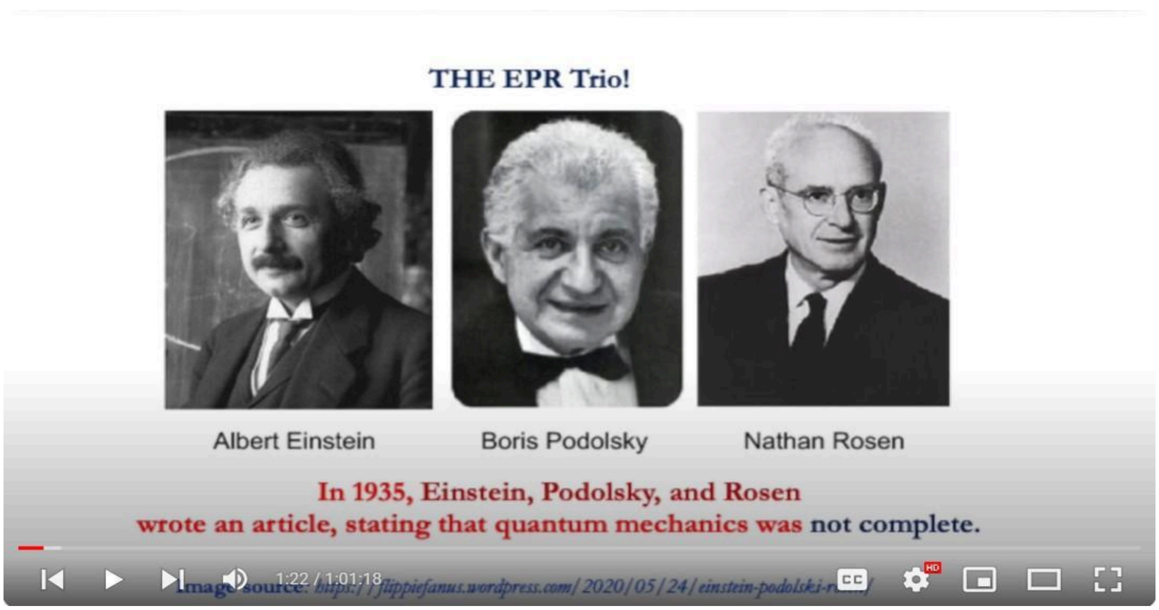

In order to prove his point, in collaboration with Podolsky and Rosen, he wrote a research article in 1935, giving birth to the so-called EPR paradox. I'm going to give you a very simple version of their argument that they provided in their article. However, before I discuss the EPR arguments, let me discuss a unique feature of quantum mechanics, that is, its non-local character.

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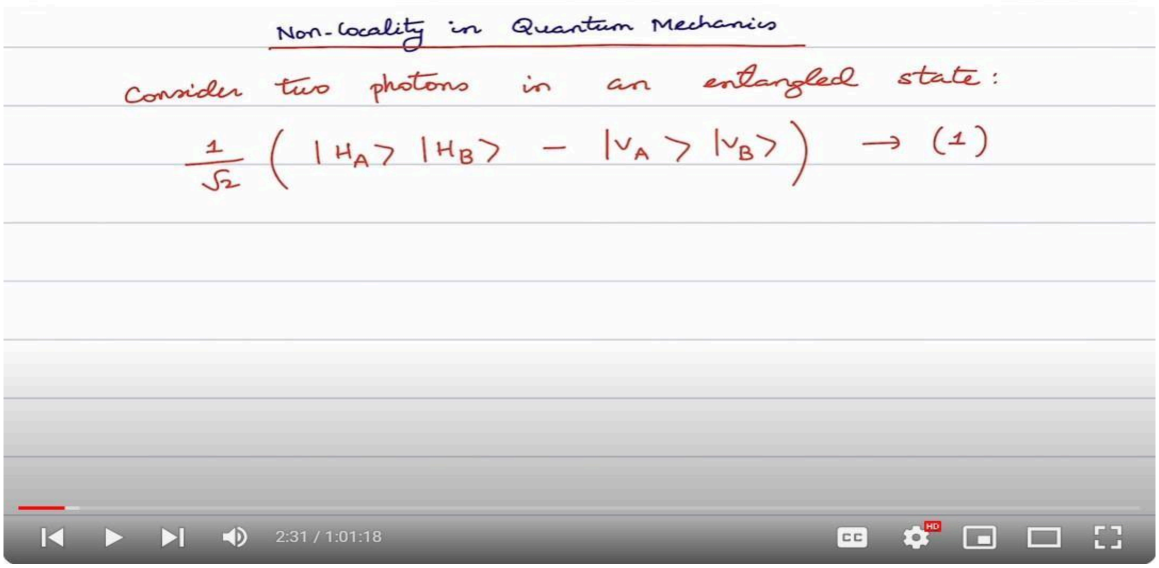

Let us consider two photons, consider two photons in an entangled state, in an entangled state given by this configuration. Let us say they form a singlet state and they are expressed by this configuration, ket HA, ket HB, minus ket VA, ket VB. Let me denote it as equation number 1.

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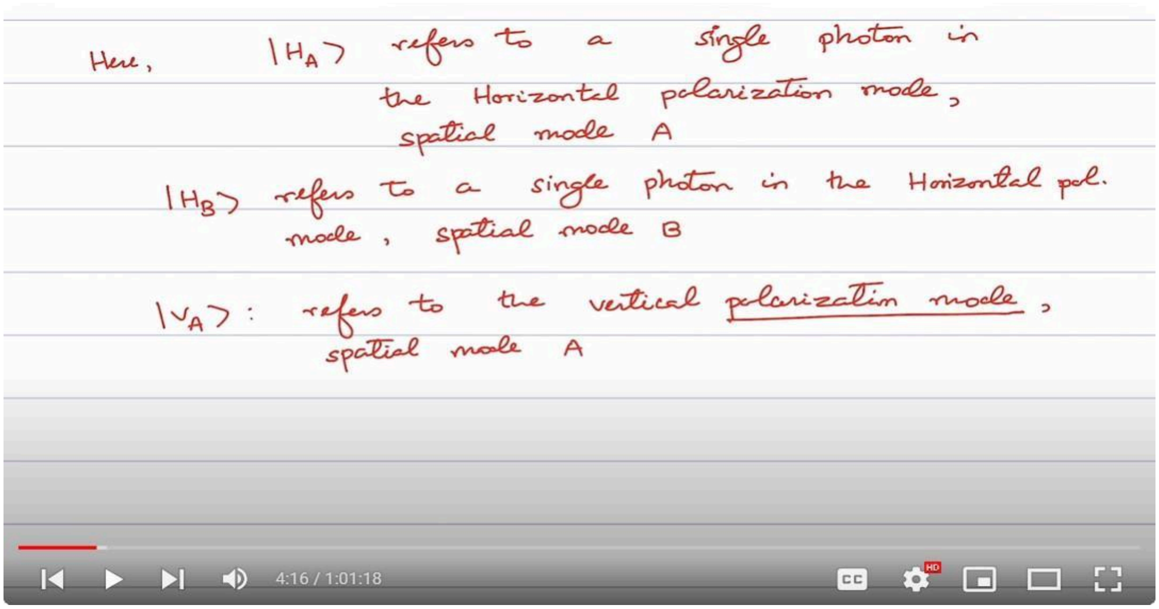

Here, here this ket HA refers to a single photon in the horizontal polarization mode, in the horizontal polarization mode and having the spatial mode A, having the spatial mode A, that means it is at the location A. On the other hand, HB, ket HB, refers to a single photon, a single photon in the horizontal, in the horizontal polarization mode, polarization mode, and it has the spatial mode B, that means it is at the location, say, B. Similarly, VA refers to the vertical polarization mode, vertical polarization mode and spatial mode here is A, spatial mode A for the single photon and similar way, VB refers to vertical polarization mode, okay, and spatial mode here is B, that means it is at the location B.

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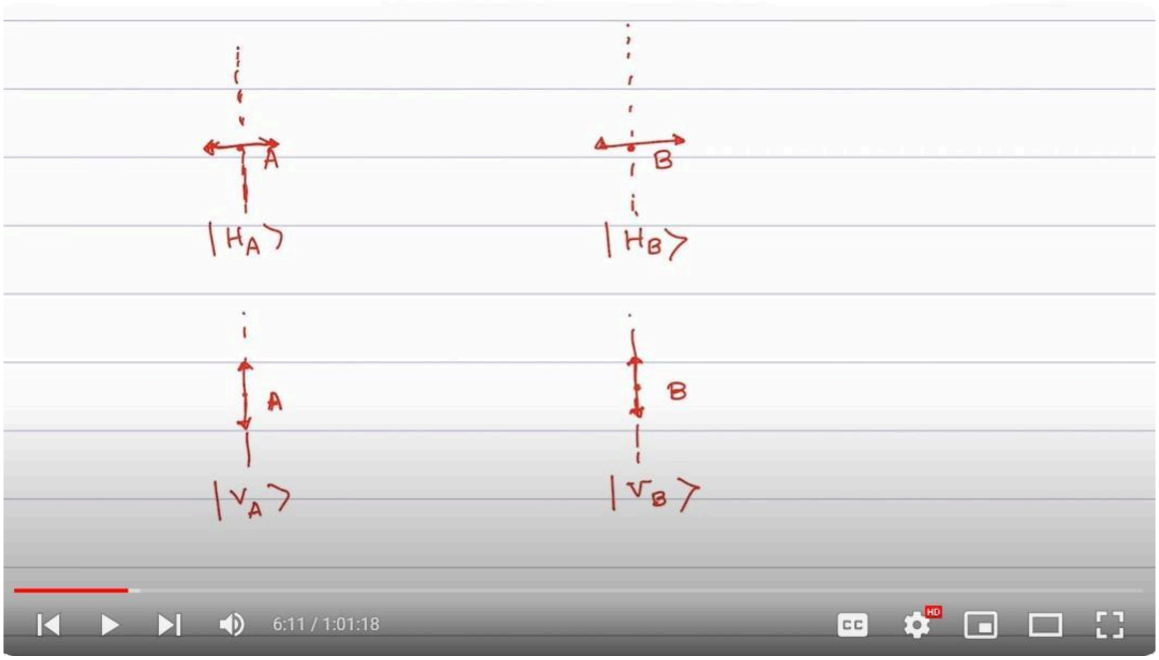

Diagrammatically speaking, what I mean to say is this, suppose, say, the first photon is at the location A or it has a spatial mode A and the other photon has spatial mode B and this first photon is horizontally polarized, if it is horizontally polarized, the other one is also horizontally polarized or if the first photon, that is photon at position A, position A is vertically polarized, the other one is also, other photon which is located at position B is also vertically polarized and in this case, this particular photon which is vertically polarized at the location B is represented by ket V B, this photon at A which is vertically polarized, it represented by ket VA, this photon, this particular photon which is horizontally polarized at location A is represented by ket HA and this photon at location B which is horizontally polarized, it represented by ket HB. So this is basically the scenario we are having.

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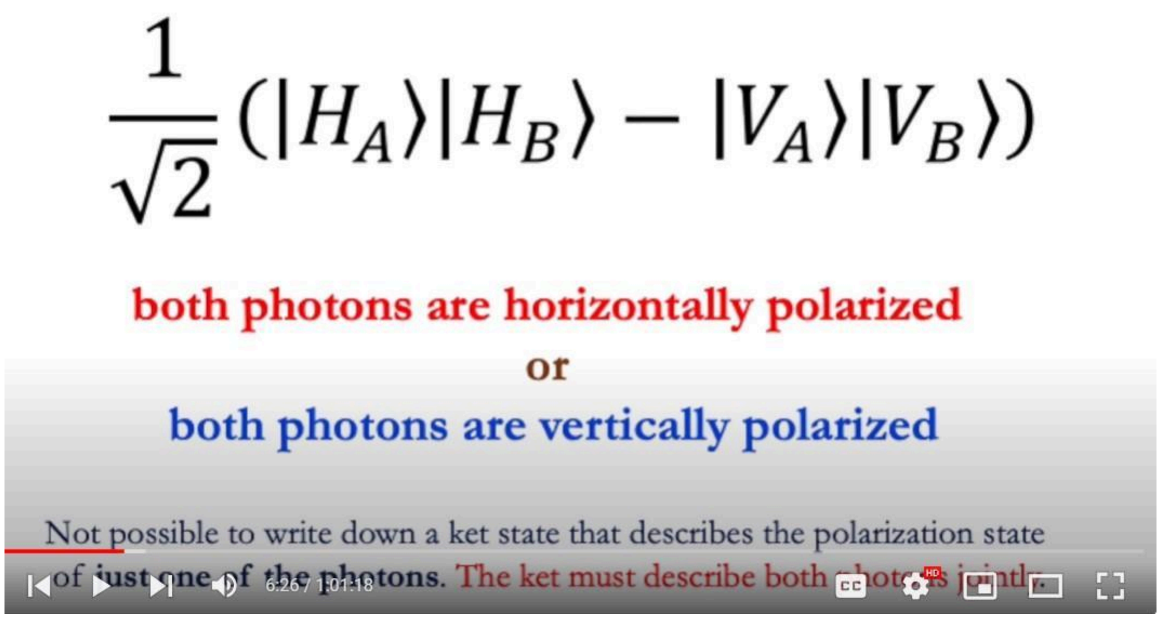

Now overall, the entangled state means that the photons are in a superposition of either both being horizontally polarized or both being vertically polarized. Now you have to know that it is not possible to write down a ket state that describe the polarization state of just one of the photons, the ket must describe both photons jointly and hence this term entanglement is used here.

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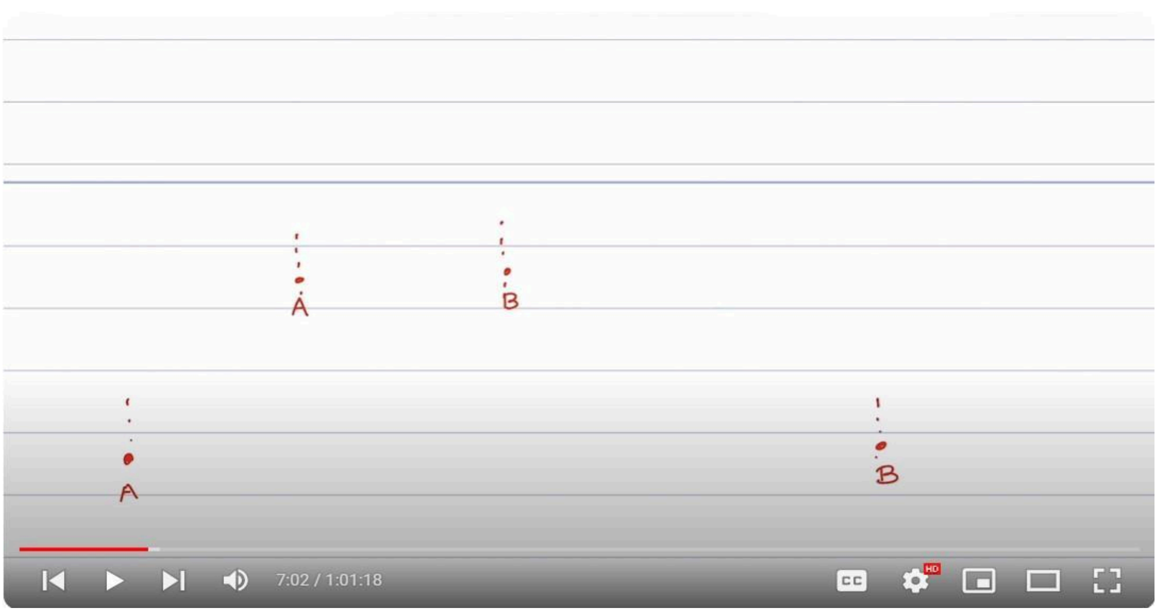

Now suppose the photons are initially at the location, say A and B nearby and I take them away to a faraway location. Suppose one photon is at here, the other photon is at B and this is at a very faraway distance, then if you do this carefully, then equation one that I have written will continue to describe this state but now we have a system described by a singlet ket whose individual parts may be space-like separated. If a measurement is made on one of the photons and it is found to be say horizontally polarized, the other one, the state of the other photon immediately becomes horizontal.

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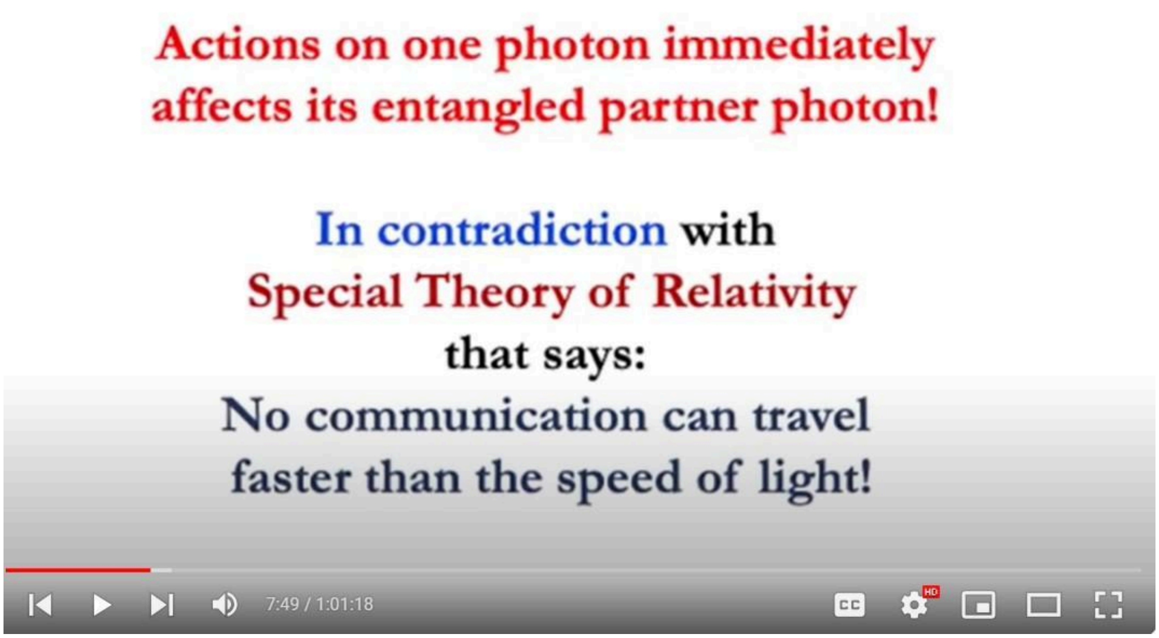

It appears that actions on one photon immediately affects its entangled partner seemingly in contradiction with spatial relativity and spatial relativity as you know requires that any communication is limited by the speed of light, that means no communication can travel faster than the speed of light. Because of this behavior of entangled systems, quantum mechanics is often referred to as a non-local theory. Now the orthodox view of quantum mechanics says that there is no contradiction at all and we are going to discuss this issue little bit later.

Now let us discuss EPR's thought experiment. The original EPR thought experiment used position and momentum as variables and you know that position and momentum are continuous variables.

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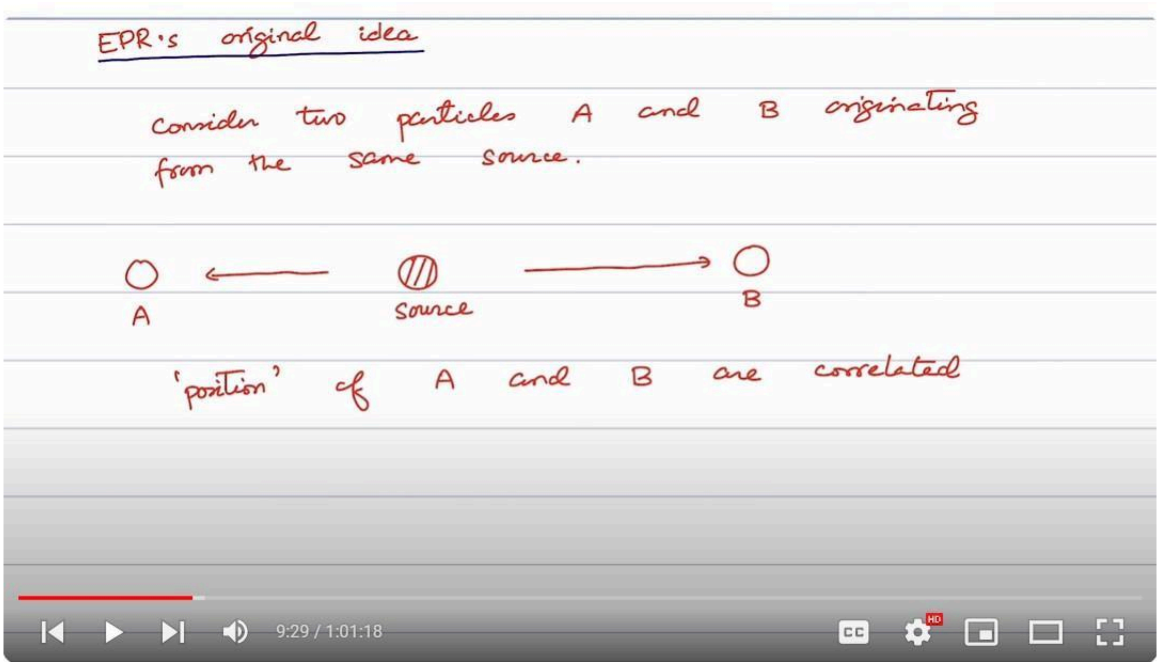

Consider two particles A and B originating from the same source. Consider two particles A and B originating from the same source. Okay, and they travel in the opposite direction. Suppose this is the source, this is the source and particle A travels in this direction, this is particle A and the particle B travels in the opposite direction and they travel in such a way that their position are always correlated. Position of A and B are correlated.

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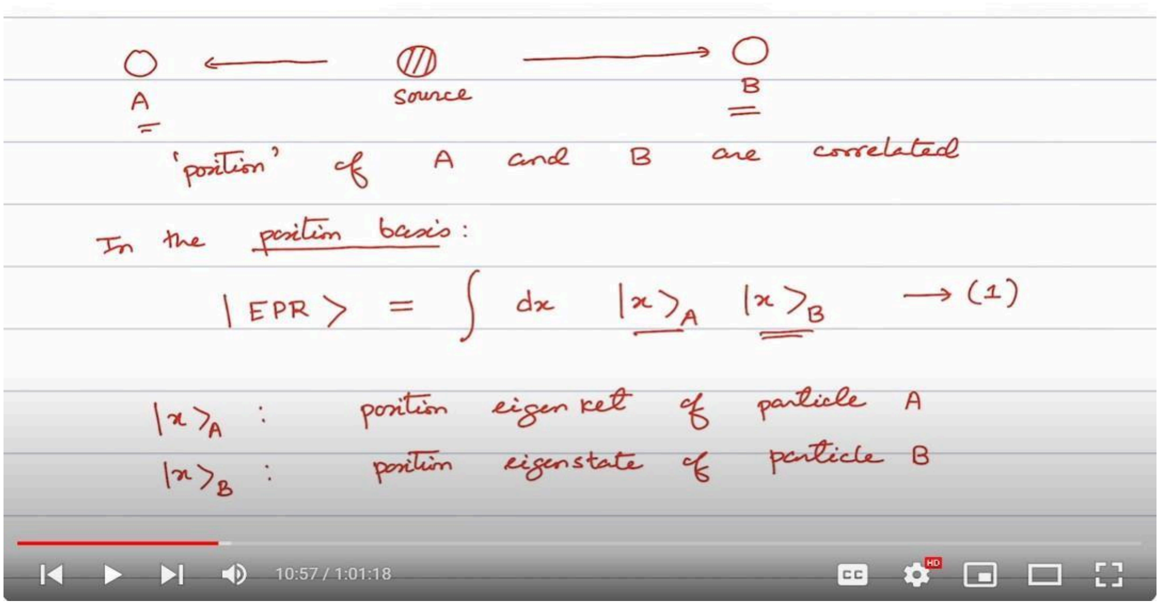

What I mean by this is as follows, in the position basis, in the position basis, it's a continuous basis, in the position basis, the state of the particles can be represented by, because I'm discussing EPR paradox, say the state is represented by a ket called EPR and the state is going to be written like this. Here X ket XA, ket XA refers to position eigen ket. This is position eigen ket or eigen state of particle A, particle A, while X ket XB is the position eigen state or eigen ket, it's a position eigen state of particle B.

So this correlation effectively means that if I measure position of particle A and I find it at XA, then immediately the position of the particle B will also be known and it would be at XB. And this is an important equation, let me denote it say equation number one.

By the way, some of you may have seen position basis, which is a continuous variable basis for the first time. So for them, let me digress a little bit and discuss very briefly about quantum mechanics of continuous variable. And I'm going to discuss very briefly about position basis as well as momentum basis, which is going to be very relevant for our discussion.

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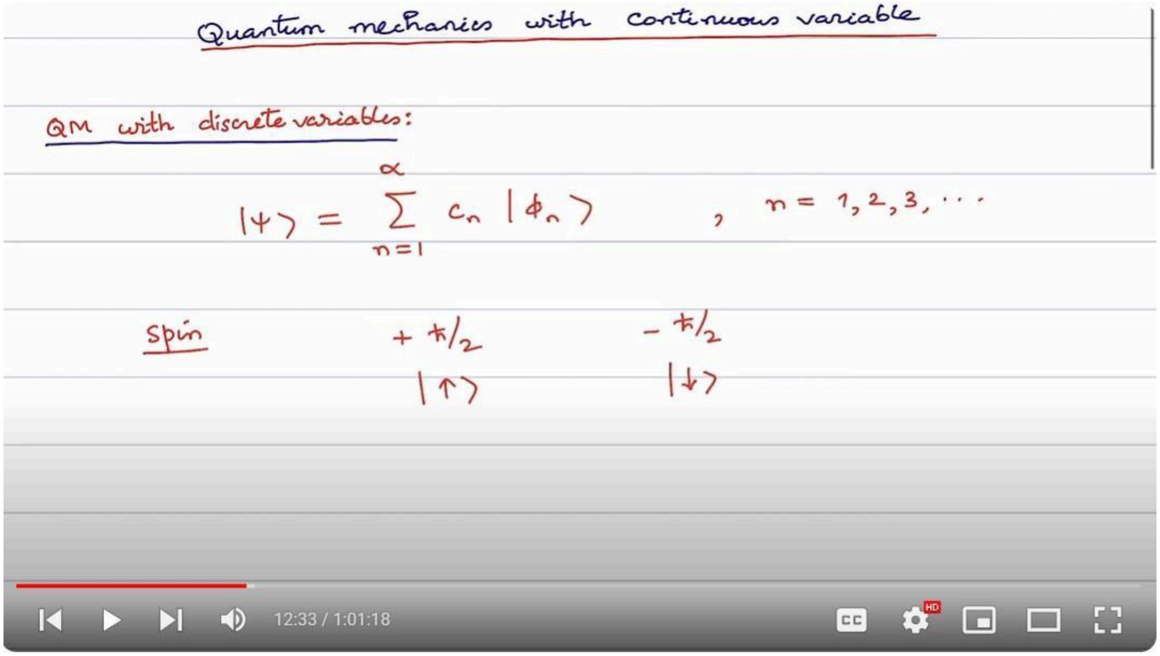

In general, the quantum mechanics principles, particularly postulate, except where we discuss are in the context of discrete variables. For example, when we write ket psi is equal to summation n is equal to one to infinity, cn phi n, here phi n are the eigen ket, cn are the complex coefficient. The use of the summation sign indicates that we are dealing with discrete variables in the sense that here n takes value one, two, three, and so on. It is discrete. The observables we consider in discrete variable quantum mechanics exhibit discrete eigenvalue spectra. For example, if I talk about an observable, say spin of an electron, it can take the value either plus h cross by two or minus h cross by two. As you know, plus h cross by two corresponds to the eigenstate, spin upstate, and minus h cross by correspond to the spin downstate.

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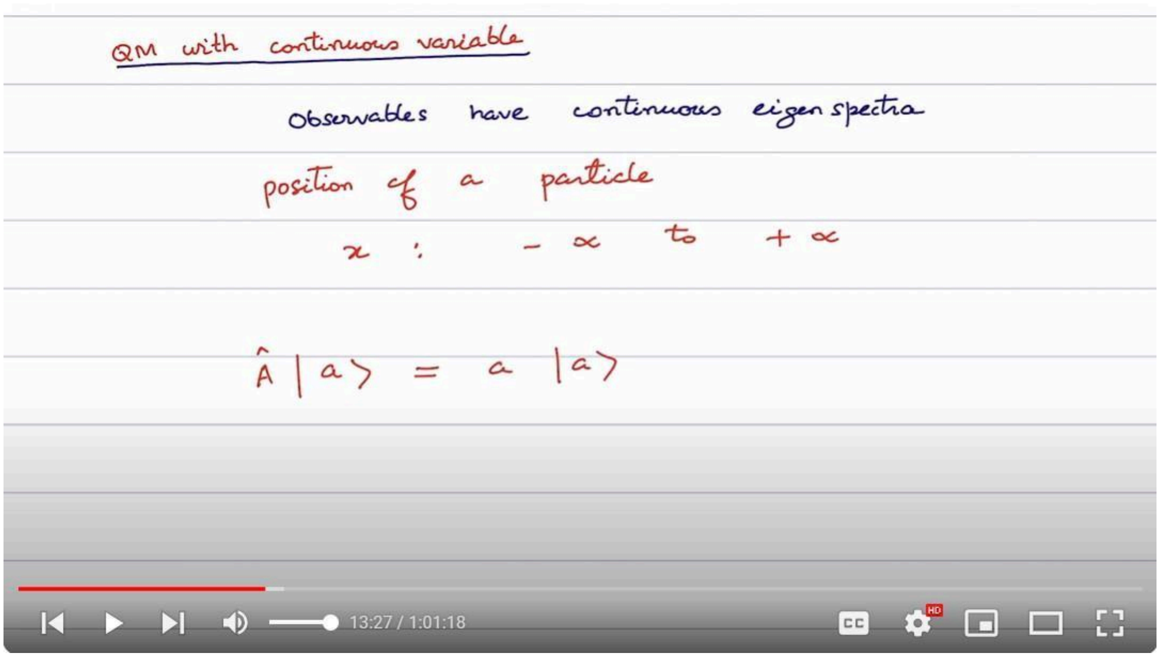

But if we go over to continuous variable quantum mechanics, there the observables have continuous eigen spectra. For example, say position of a particle. If I talk about position of a particle, let us say one dimension, if I talk in one dimension, the position x takes value from minus infinity to plus infinity. So it has continuous eigen spectra.

Now to discuss quantum mechanics with continuous variable, let me remind you about eigenvalue equation for discrete variable case, namely say when we have this eigenvalue equation, A applied on the eigen ket, say A is going to give me A A, and A is the eigenvalue here. This is eigenvalue, and this is my eigen ket. And we can extend this analogy to the continuous variable case.

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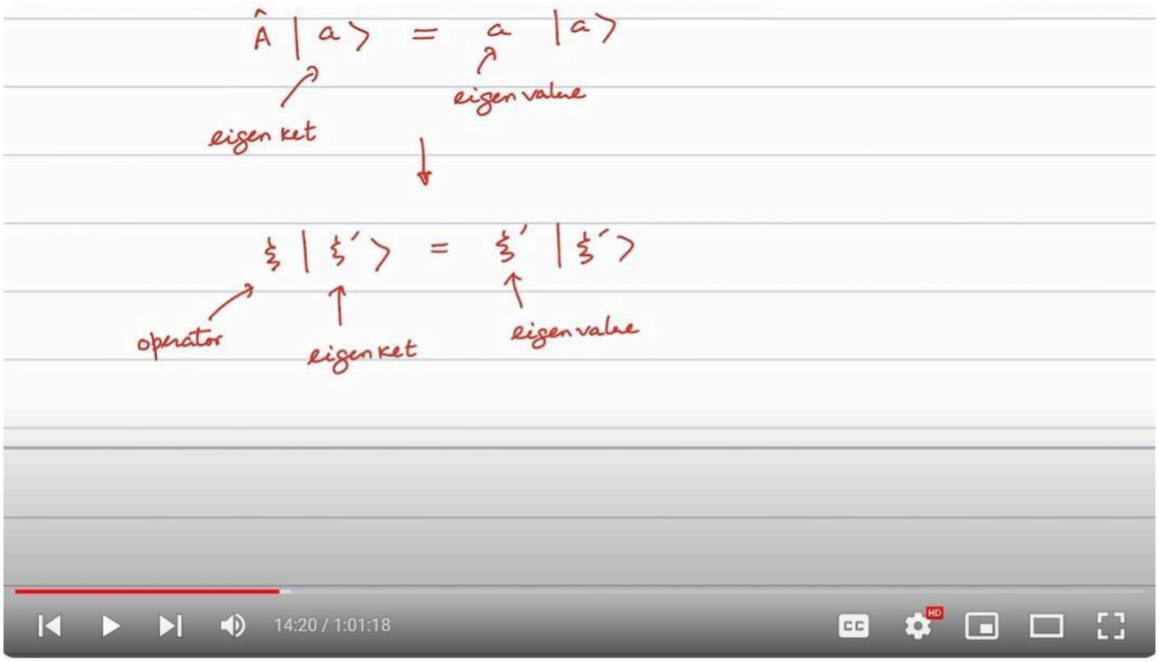

In analogy to this eigenvalue equation, I can write an eigenvalue equation for continuous variable also as follows. Xi operating on the eigen ket Xi dash is going to give me Xi dash, Xi dash is an eigenvalue, and I will get Xi dash, this is the eigen ket. So here this Xi is the operator, it's the operator, Xi dash is the eigen ket, and Xi dash is here eigenvalue. So this I'm talking in terms of continuous variable.

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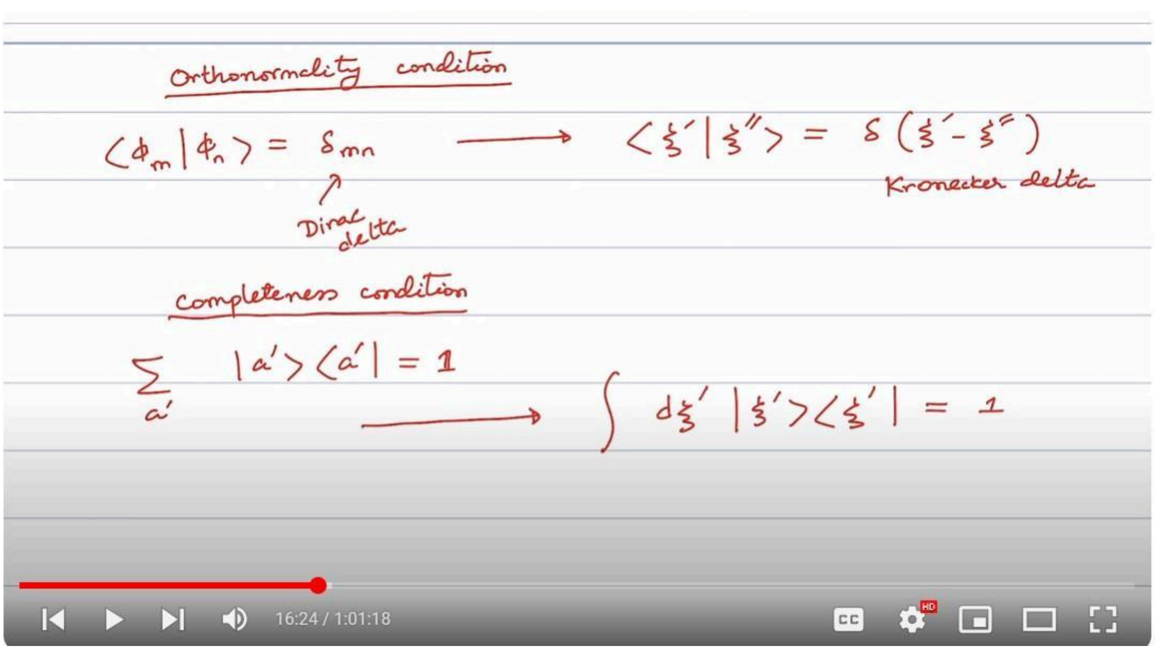

This analogy we can extend to a number of cases. For example, the so-called orthonormality condition, orthonormality condition that we encounter in discrete variable quantum mechanics is as follows.

It says the scalar product of this eigen basis, phi m phi n is equal to delta m n, where delta m n is the Dirac delta function. This means that the state phi m and phi n are orthonormal. In this we can extend to continuous variable as follows. We'll write in continuous variable, the scalar product of say Xi dash and Xi double dash would be equal to, now instead of this Dirac delta here, here we are having Dirac delta. In continuous variable case, we are going to write a Kronecker delta. That would be delta Xi dash minus Xi double dash. That means if Xi dash is equal to Xi double dash, then this is going to give me one. If Xi dash is not equal to Xi double dash, this is going to give me zero. So this is Kronecker delta.

The same thing we can do for the completeness condition as well. So let me write the very important completeness condition. In discrete variable, we write the completeness condition as follows. We write summation say A dash, A dash, A dash is equal to identity operator. It is identity. We can extend these two continuous variables. Summation is now going to be replaced by a integral and we'll have d Xi dash, ket Xi dash, bra Xi dash would be equal to one.

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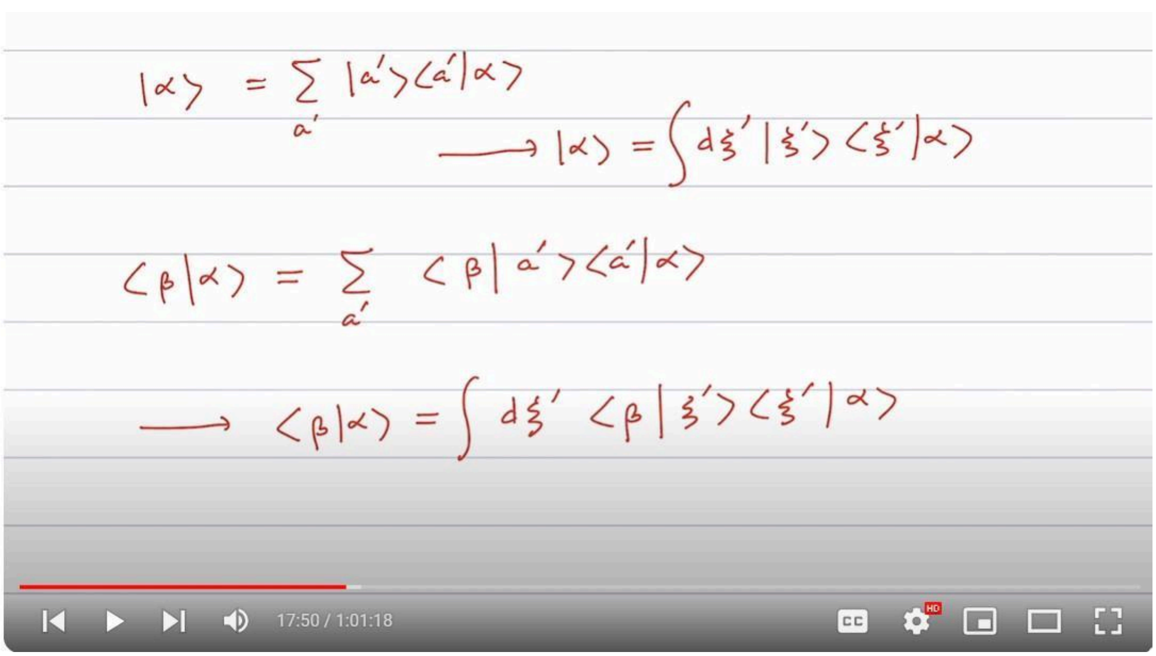

Now any arbitrary state, ket alpha, we can write in discrete variables using this completeness condition as follows. We can write A dash, A dash, A dash, alpha. The same thing we can do in continuous variable case also. ket alpha is equal to integration, d Xi dash, Xi dash, Xi dash, alpha.

All right? Now what about the scalar product of two quantities? To catch say ket beta and ket alpha in discrete variable, we write it as this. Say summation A dash, beta, and we are going to just use the completeness condition. We are going to sandwich this thing inside and then this is what we'll get. And the analogy for continuous variable is again simple and this would be simply replaced by this integration, d Xi dash, beta, Xi dash, Xi dash, alpha. Okay, as you can see, going from discrete variable to continuous variable appears to be very straightforward.

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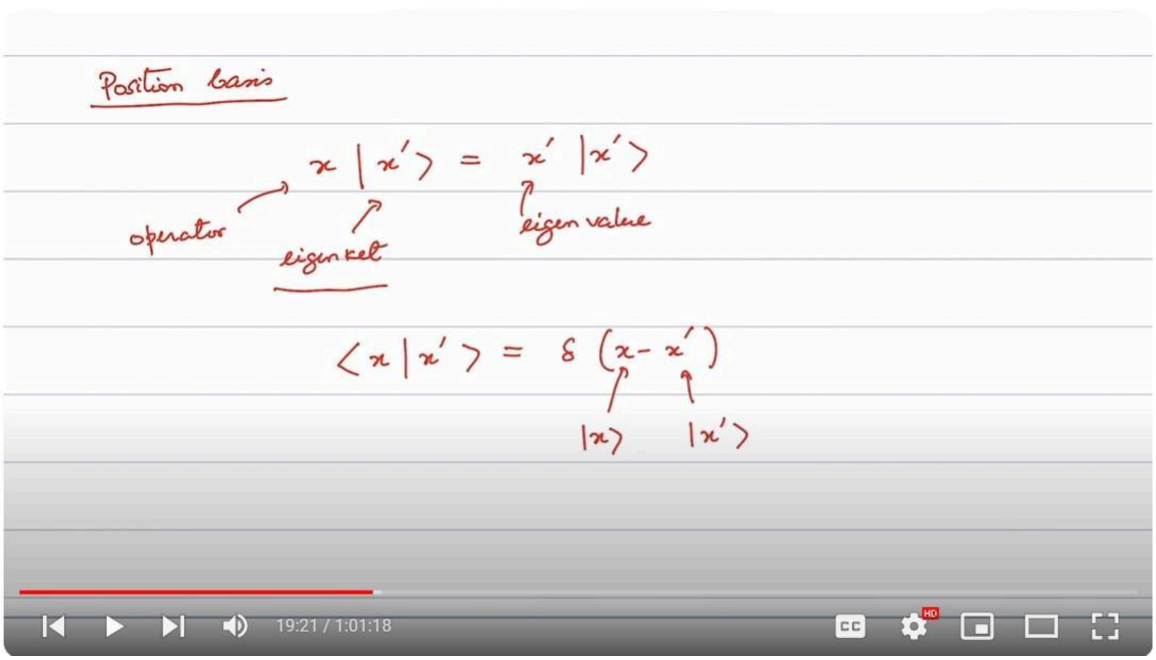

Now we'll discuss about the position and momentum basis, which is extremely useful in continuous variable quantum mechanics. So let us discuss about position basis first, then we'll go to momentum basis. Say the position operator X satisfy the eigenvalue equation, X X dash is equal to X dash X dash, where X dash is the eigenvalue, position eigenvalue. This is eigenvalue.

And ket X dash is the eigenket. And X is the operator. I'm not using the cap sign here. I'm avoiding it, but you should understand that I'm talking about operator here. Now this eigenkets, ket X dash form a complete set called the position and they are called position basis and satisfy the orthonormality condition. The orthonormality condition is scalar product of X X dash is equal to Kronecker delta X minus X dash. Here, this X corresponds to the eigenvalue corresponding to the eigenstate ket X and X dash is the eigenvalue corresponding to the eigenstate X dash. Okay.

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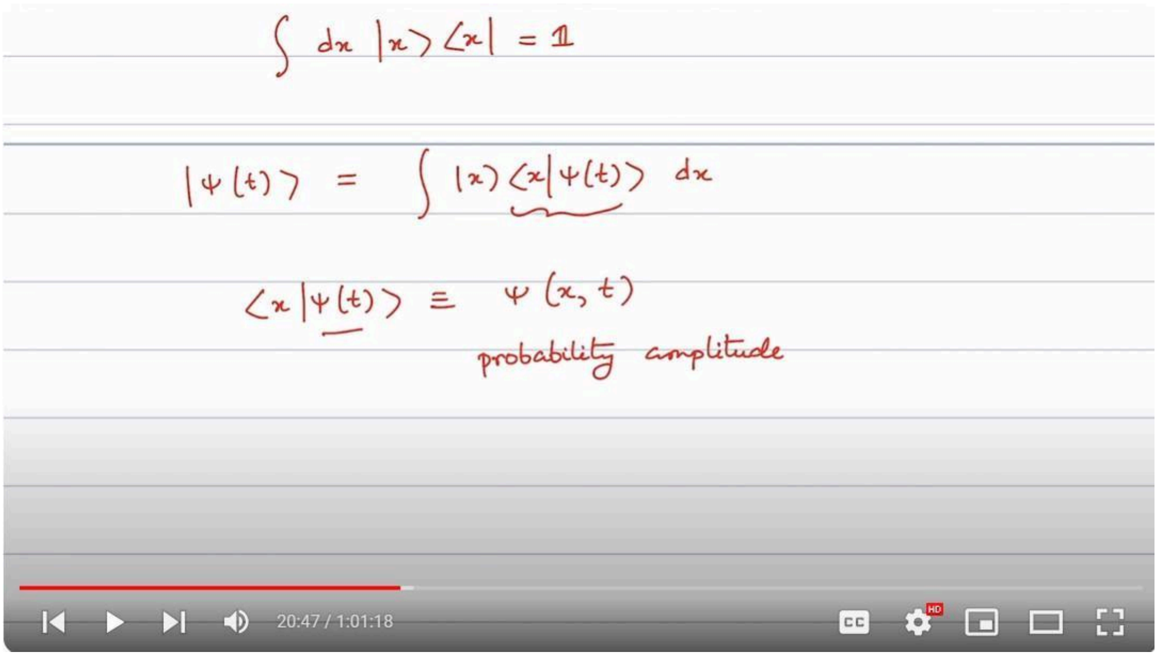

And also the completeness condition satisfy by this position, eigenkets are as follows, d X ket X bra X is equal to one and we can expand any arbitrary ket in the position basis. For example, suppose we have this eigenket psi of T, ket psi of T, I can use position basis. Basically, I have to use the completeness condition.

Then I have to, I can write, then I can write it as follows. So simply this is what I will have, right? Now here, this quantity, you see this is a scalar quantity and it's just a number and it is generally denoted by, we can denote this quantity as psi X T. Now you may recognize that this is a complex number, and this is called a probability amplitude. This is the probability amplitude and also known as the wave function. Effectively it means that when the system is in the state psi of T, it is in a position eigenk level by X.

The quantity psi of X T is termed as the wave function in quantum mechanics as I have said, and it's basically a state vector in the position basis. It is also called position space wave function.

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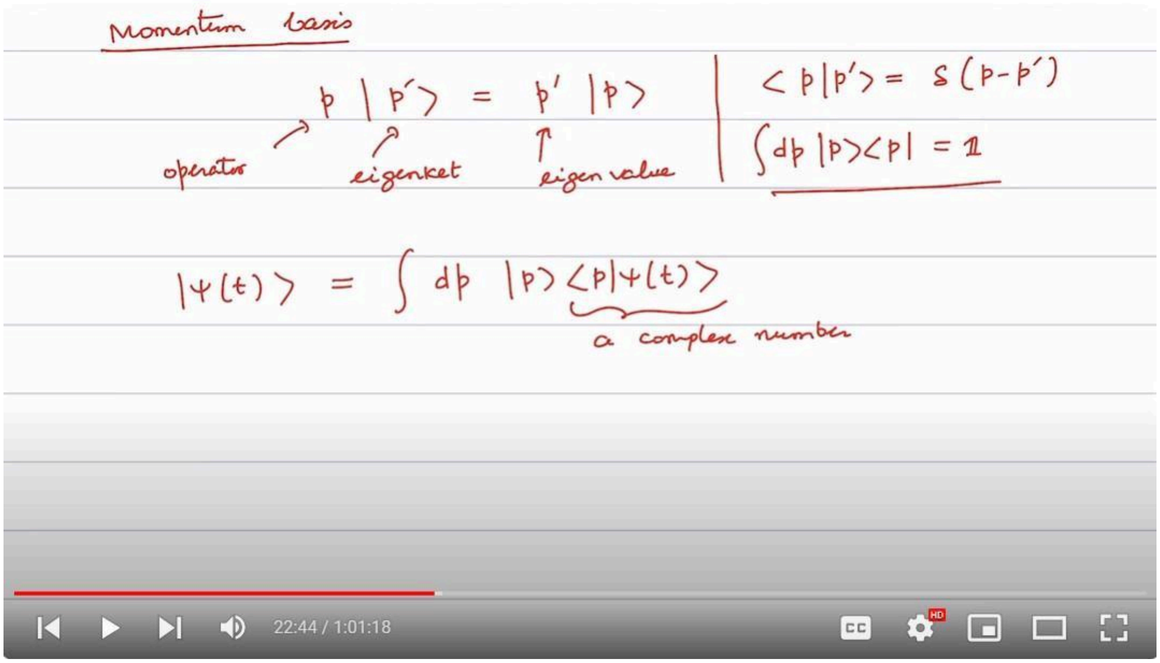

Now let me discuss about momentum basis. So momentum basis is actually straightforward. It is exactly similar to what we have for position basis. In the momentum basis, we can again write a eigenvalue equation of this type, say P ket P dash is equal to P dash ket P, where P is the momentum ket, P dash is the momentum eigen ket. This is the eigen ket and P dash is the eigen value and P is the operator. This is the operator and it also has the similar kind of properties.

For example, the orthonormality condition would be P P dash would be equal to delta P minus P dash and the completeness condition would be dP ket P bra P is equal to identity operator. And here also I can write an arbitrary state, psi of T in terms of momentum basis as follows.

I can write simply, I have to use this completeness condition and I will write ket P bra P psi of T. As you see, this guy is again a complex number. It's a complex number.

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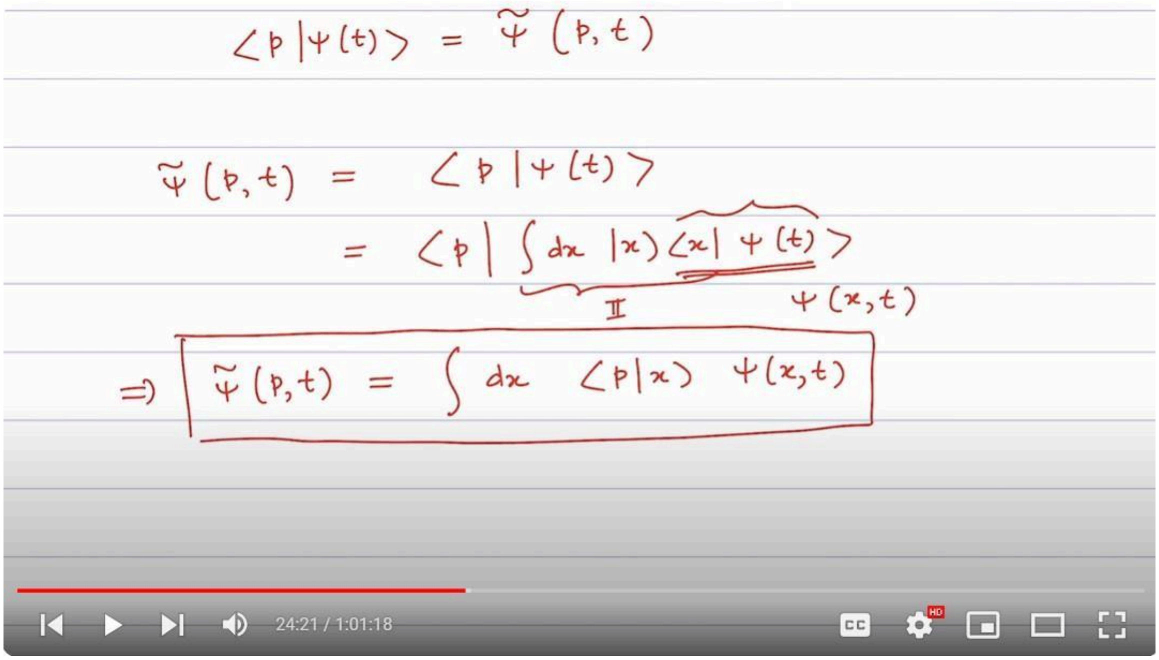

And this has also a name and in short notation, we can write P psi of T is equal to psi tilde of PT. P psi tilde of PT. And this is the so-called momentum space wave function corresponding to the state vector psi of T. Now there is a connection between the two basis, position basis and the momentum basis.

To show the connection, let me write again, psi of PT, that's the momentum space wave function is equal to P psi of T. Now let me apply the completeness condition from the position basis. And let me sandwich this here, dX, XX, which is the identity. And I will have psi of T. And this is identity. I can always put it, sandwich it between two scalar products. Using this, I can express psi tilde of PT, the momentum space wave function is equal to, as you can see, this guy here, you see this guy here is nothing but psi of XT.

So utilizing that, I can write integration dX, PX, scalar product of PX. This is a number, psi of XT. So this is the connection between the position space wave function psi XT and the momentum space wave function, psi tilde PT. And we have to work out this quantity. It can be worked out. I think I will do that in the problem-solving session number two.

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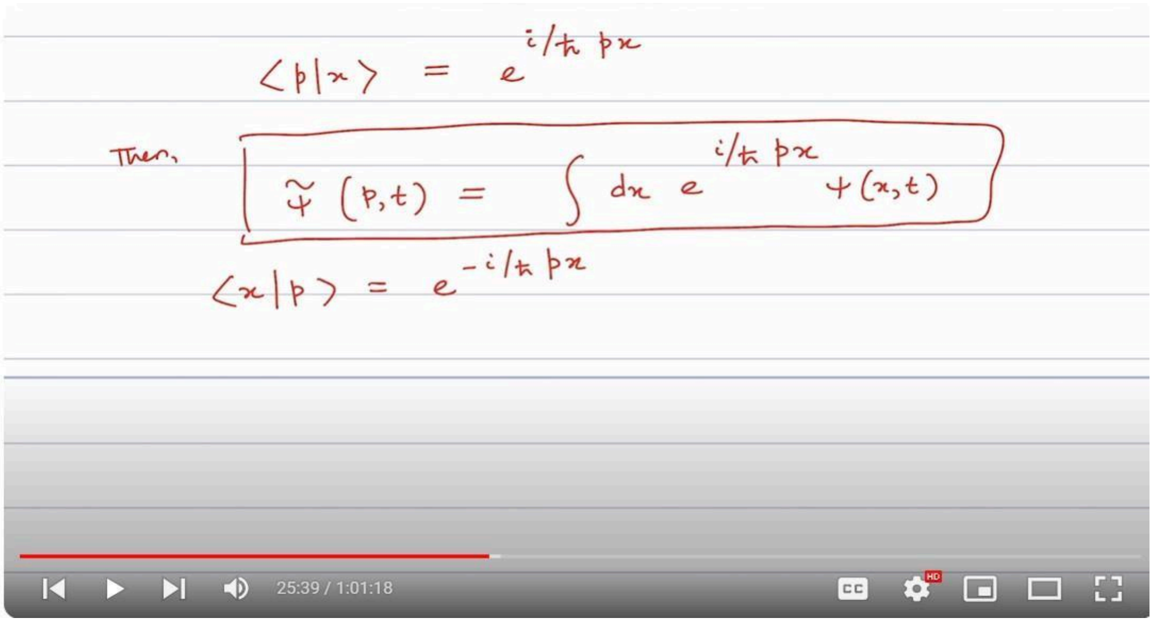

It can be shown that this scalar product of P and X is equal to e to the power i by h cross PX. Then we have psi tilde of PT is equal to integration dX e to the power i by h cross PX psi of XT. In the similar way, we can show that this quantity XP is equal to e to the power minus i by h cross PX. So as you may recognize that from this expression, psi of XT is the inverse Fourier transformation of psi tilde of PT or psi tilde of PT is the Fourier transform of the position wave function psi of XT.

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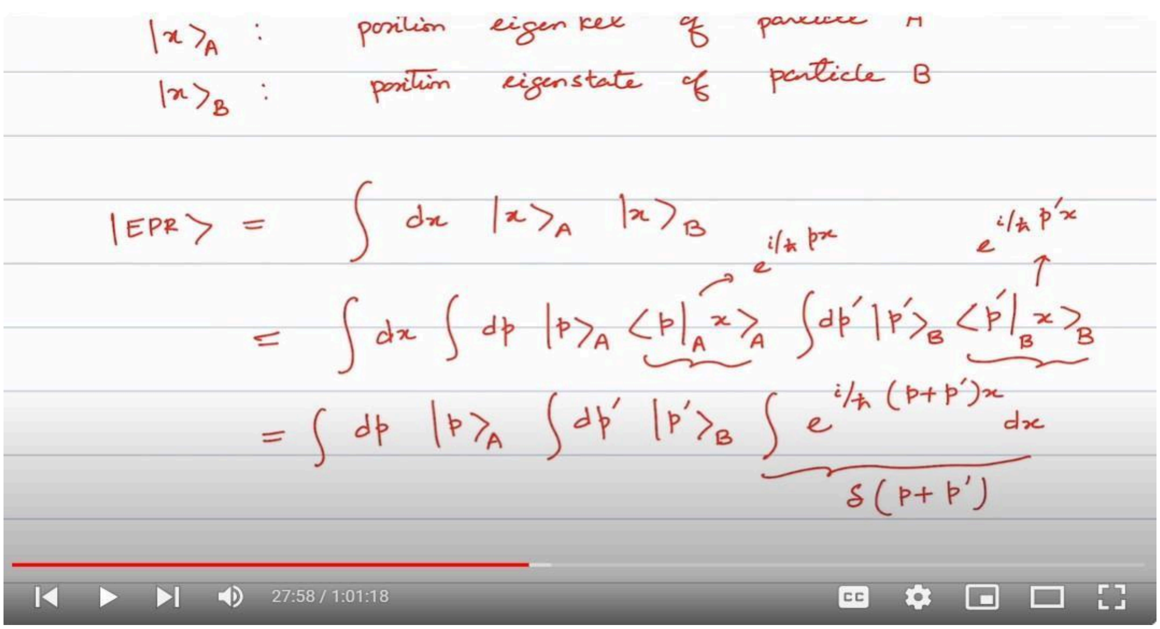

Let us now come back to this equation number one. Now in the momentum basis, we can write this EPR state as follows. ket EPR is equal to, so we have dX XA XA XB and if I use say the completeness condition in the momentum basis, let me write DP PA, PA, right? This is identity.

Then I have here XA and for the other, let me sandwich another completeness condition in momentum basis here. Let us say it is dP dash, P dash B, P dash B and I have here XB. I think it is easy to follow. If you look at it carefully, we'll be able to follow it easily.

And this I can now write as dP PA, dP dash, P dash B and integration because this quantity and this quantity, I know this quantity is E to the power I by h cross PX. And this quantity here is E to the power I by h cross P dash X. So therefore I can write here integration E to the power I by h cross P plus P dash X dX. And this guy is nothing but the Dirac delta function delta P plus P dash.

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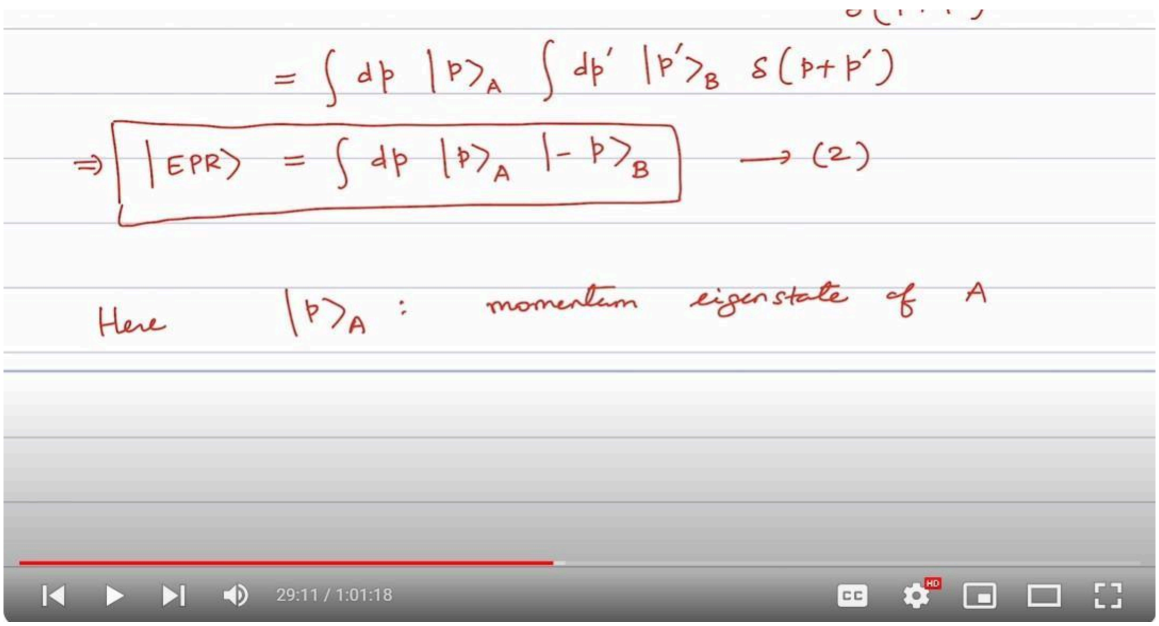

So therefore I can write, let me write it clearly, DP PA, DP dash, P dash B, delta P plus P dash. Now applying the properties of the Dirac delta function, I can write dP PA minus PB. So if your state in the momentum basis, I get it as this one. Here this ket PA, this is the momentum eigenstate, momentum eigenstate of particle A. Now from this equation, let's say this equation number two, this is the EPR state in the momentum basis. From here you can see that the momentum is exactly anti-correlated for this state.

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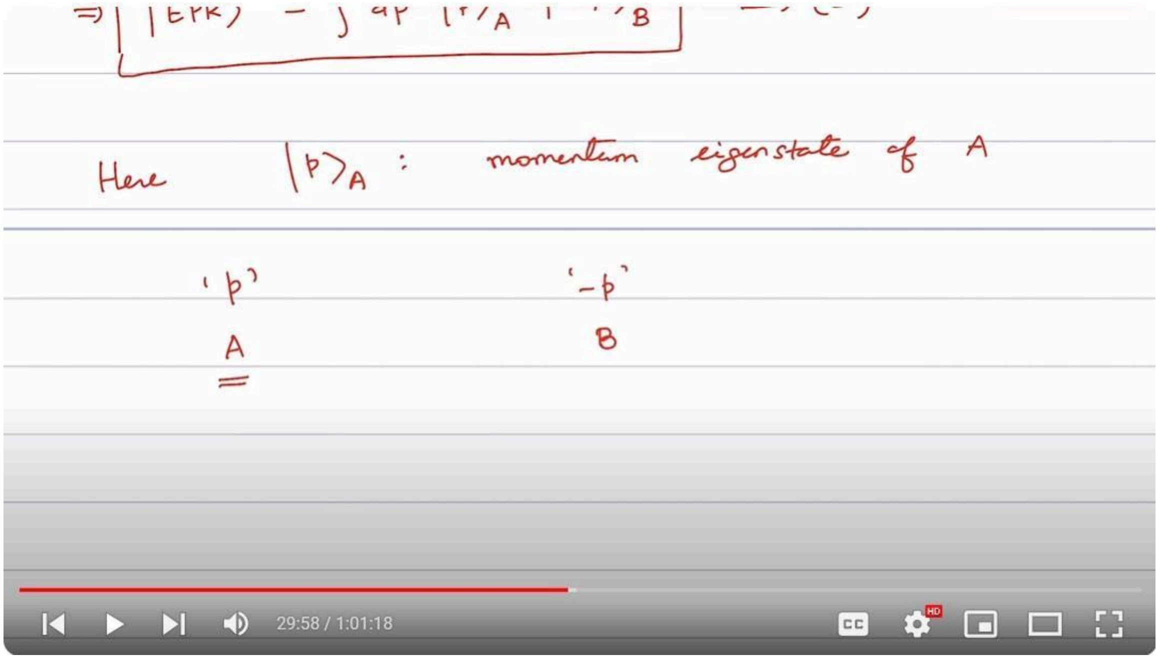

If a measurement of particle A is made, if momentum measurement is done, and if we find that the momentum of particle A is P, then immediately the momentum of the particle B would be minus P, it would be opposite to that of particle A.

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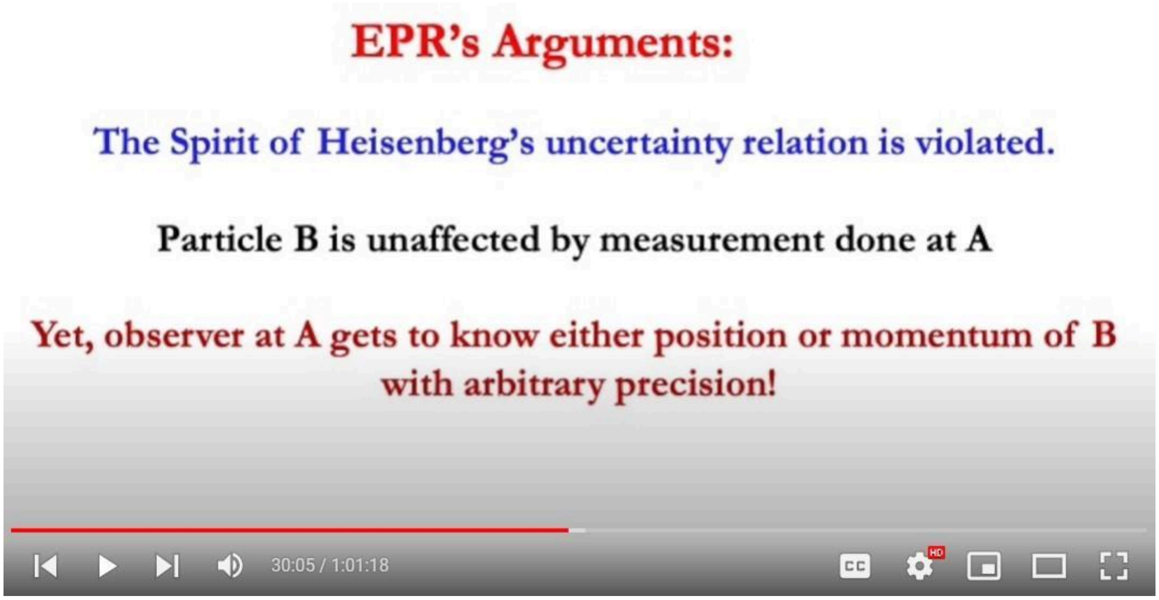

And EPR argued that this situation violated the spirit of the Heisenberg uncertainty relation. Because the argument is that that measurement is done on particle A only, and the particle B is separated from A in a space-like way. That means it is separated from A by a far away distance. So measurement that is done on particle A by an observer at A is not going to affect anything on the particle B, right? So it was, according to them, it was very, it's a kind of very bizarre situation here that without making any kind of measurement on B, we can know the, or the observer at A can know the position or the momentum of B with arbitrary precision.

And because the choice of the measurement was arbitrary and apparently it did not affect the real situation of the other particle, it seemed that the particle B must have some well-defined value of this observable in the very first place, contrary to Heisenberg uncertainty relation. And the quantum description implies that a connection between the particles, even when far apart, which Einstein termed as spooky action at a distance.

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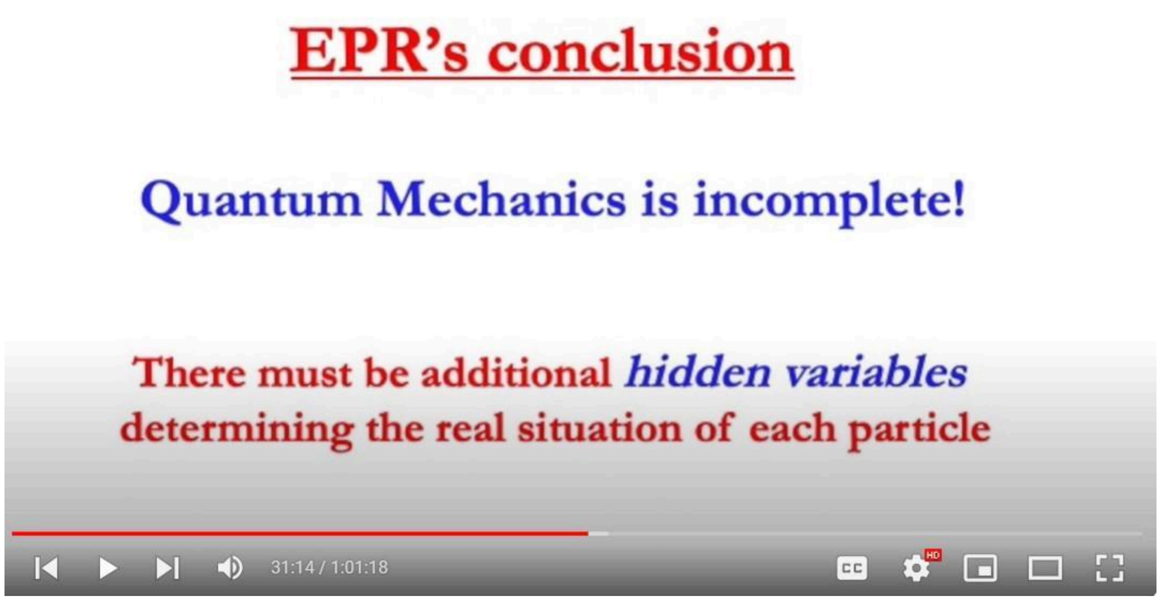

So EPR's conclusion was that quantum mechanics is incomplete and there must be some additional hidden variables determining the real situation of each particle. Now there's another version of EPR, which was put forward by British physicist David Bohm involving discrete variables. Now let me give you a simplified version of Bohm's EPR version of thought experiment.

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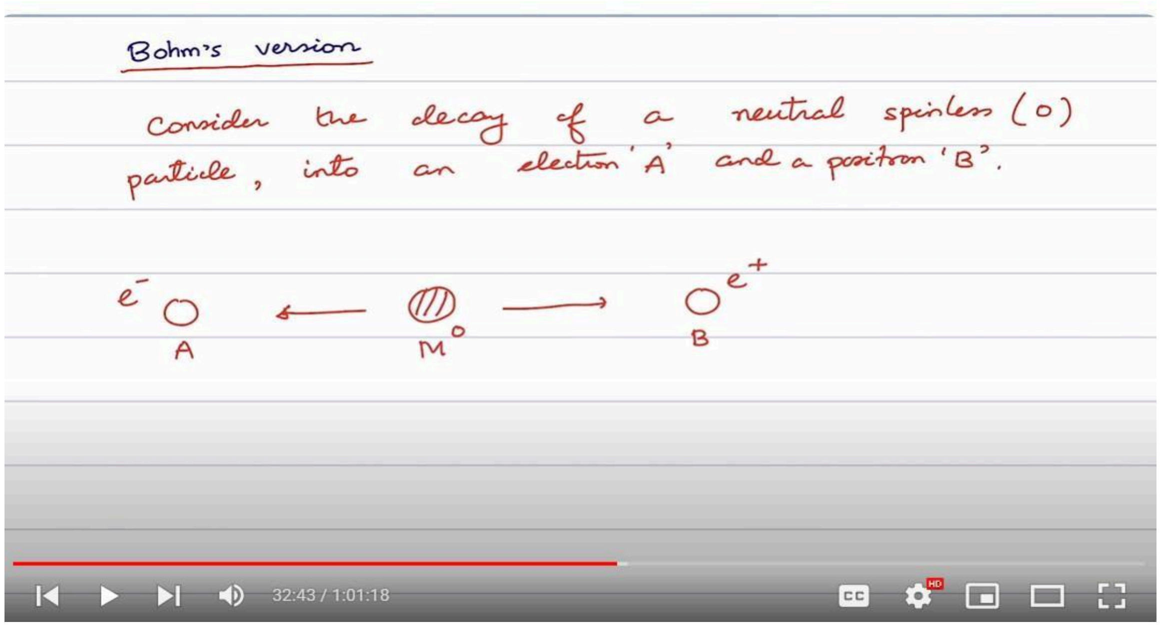

Consider the decay, consider the decay of a neutral, that means a charge-less particle, a neutral and also spin-less, that means spin zero, a spin-less particle into, suppose it is not spin-less, it is decaying into an electron, let me say electron A and a positron and a positron, let's say B, okay. That means if you have a neutral particle, say M, and its charge is zero, it is decaying into an electron, electron is going this way. Suppose this is electron A, it charges E minus and the positron is B, it charges E plus. Now because it has started with a zero spin, the source has spin zero, the total angular momentum, spin angular momentum has to be conserved,

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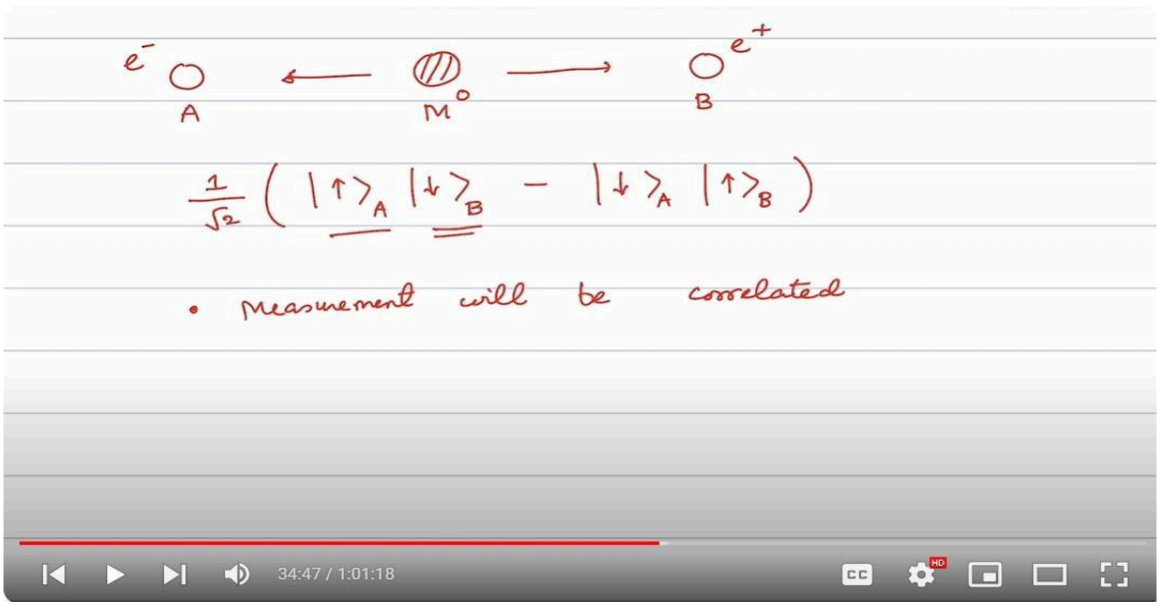

So this A and B, this electron and the positron are described by a singlet configuration, say this one given by this configuration, one by root two. If the electron is found to be in the upstate, the positron has to be in the downstate so that the spin angular momentum is conserved or if the electron is found in the downstate, the positron has to be in the upstate.

Now quantum mechanics is completely clueless which combination you will get. However, it does say that the measurement will be correlated, so it does say that the measurement will be correlated, it would be correlated. Now you will get each combination half the time on the average, that means either you will get electron to be in the upstate, positron to be in the downstate or the electron in the downstate and the positron in the upstate, you are going to get half of the time on the average if several measurements are carried out.

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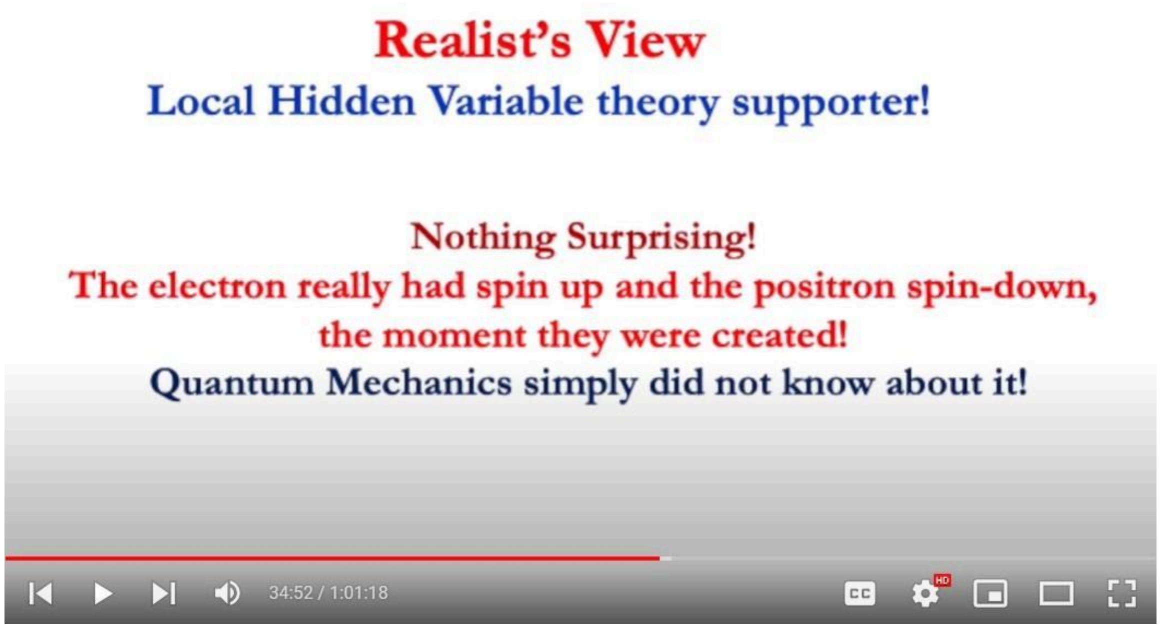

The apparent trouble is that if we know that A is in spin upstate, right, if A is in spin upstate, then B is in the spin downstate and for sure. However, A and B may be separated from each other even say by 100 light years. Now if you consider this as spooky action at a distance, in fact, in essence, there are two views about this paradox. One is the realist view. Realist says that there is nothing surprising, the electron really had spin up and the positron spin down from the moment they were created. It is just that quantum mechanics did not know about it.

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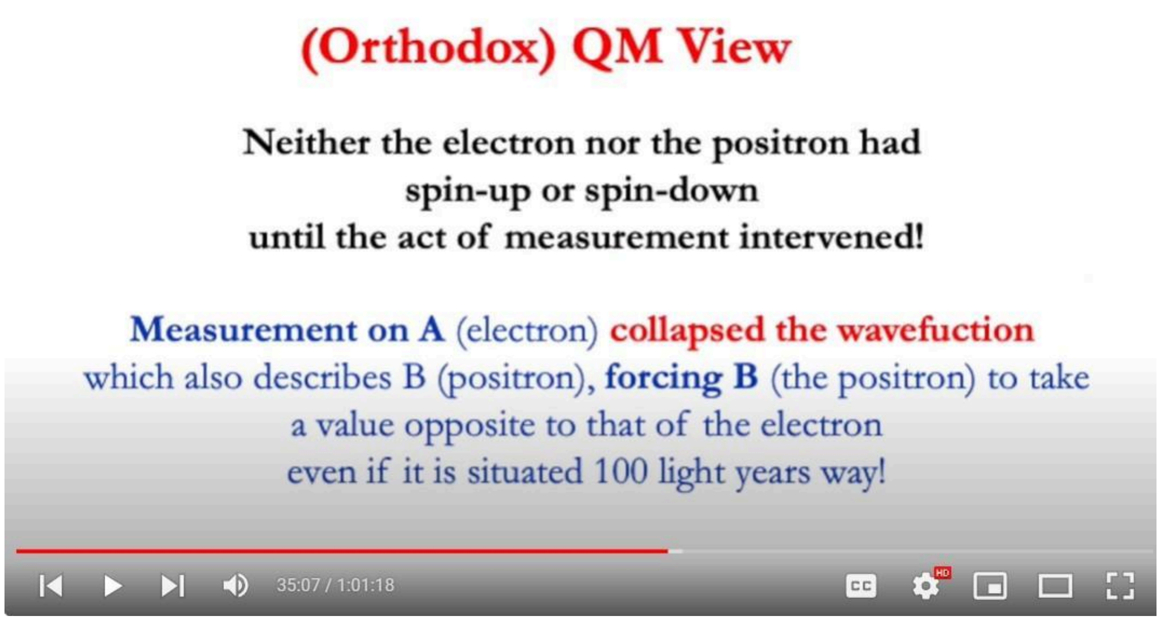

On the other hand, the quantum mechanical view, which is the orthodox view, is that neither particle had either spin up or spin down until the act of measurement is intervened. Our measurement of the electron A, collapse the wave function and immediately produce the spin of the positron B even if they are at 100 light years away or whatever distance they are away.

The fundamental assumption on which EPR's argument raised is that no influence can travel greater than the speed of light. This is the principle of locality. The EPR paradox was considered quite philosophical till quite some time because no experiment could be designed to test EPR's thought experiment. However, in 1964, John Stewart Bell came up with a excellent research article and there he gave a theoretical basis by which experiment could be designed and quantum mechanics could be tested.

John Stewart Bell came up with a theorem which is now called Bell's theorem. It consists of a probe that shows that the predictions of quantum mechanics differ from the predictions of the so-called local hidden variable theory. And he came up with what is now very popularly known as Bell's inequalities. We are going to discuss about Bell's inequalities as simplified version of that now.

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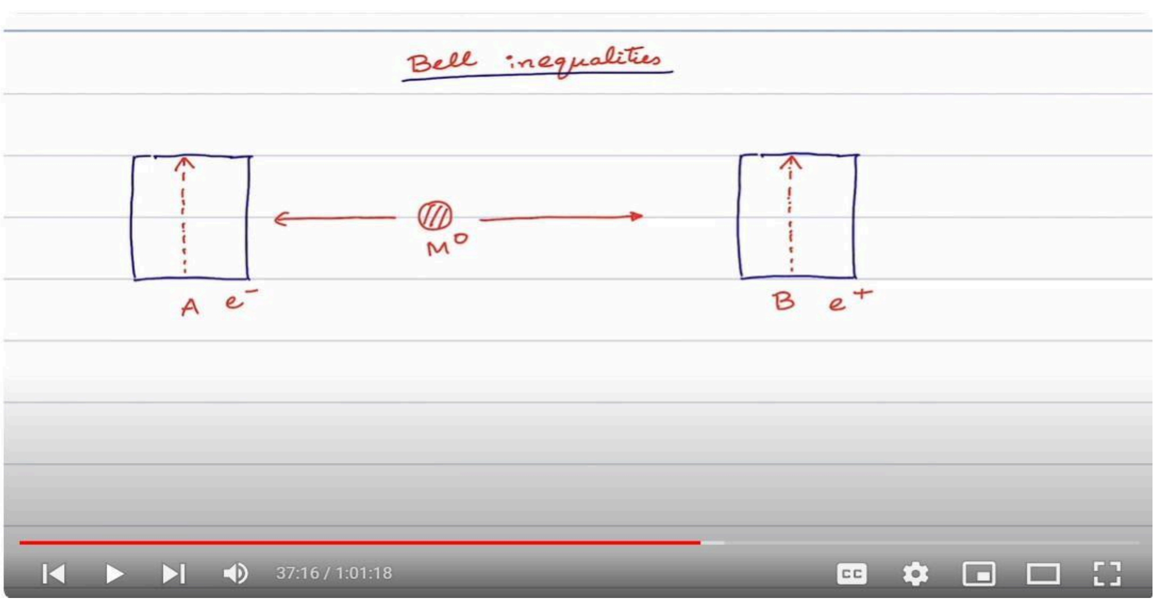

Let us go back to Bohm's EPR experiment. In that experiment, we had a neutral particle, say M0, and this particle decays into two other particles, one particle being the electron, say A, and it had charge E minus and the other one is a positron going in the opposite direction, denoted by B, and it has charge E plus. In the experiment, the orientation of the detector, detecting the electron and positron spin were parallel to each other as shown in this figure here, and they are in the same direction.

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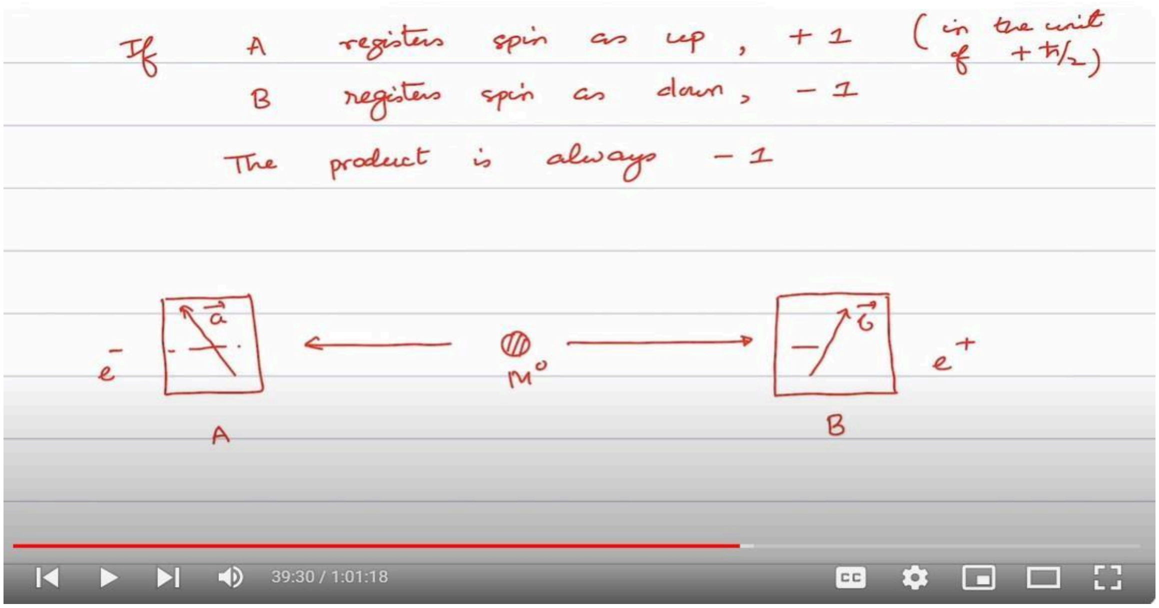

Now if, say, the detector at A registers, registers spin as up, that is, say plus one in the unit of h cross by two, in the unit of plus h cross by two, so rather than writing plus h cross by two, I am writing plus one, so it is in the unit of h cross by two. If A registers spin as up, B registers, as we have already discussed earlier, spin as down, that is minus one. And the product is always, of these two results, the product is always minus one.

Bell suggested a generalization of the above experiment, and he allowed that the detector exist, exist to rotate independently. As I said that in this Bohm's experiment, the detector axis are parallel to each other, but Bell suggested that, okay, let's say the detector axis can rotate, suppose this is the detector axis at A, it is along the direction, say, A, not necessarily that the detector at B has its axis along A, rather, say, it may be along some other direction, say, directed along B, okay? And this particle, again, it's getting decayed into two particles, A and B, but the detector orientation may not be parallel to each other, here at A, you are having electron, and here at B, you are having positron.

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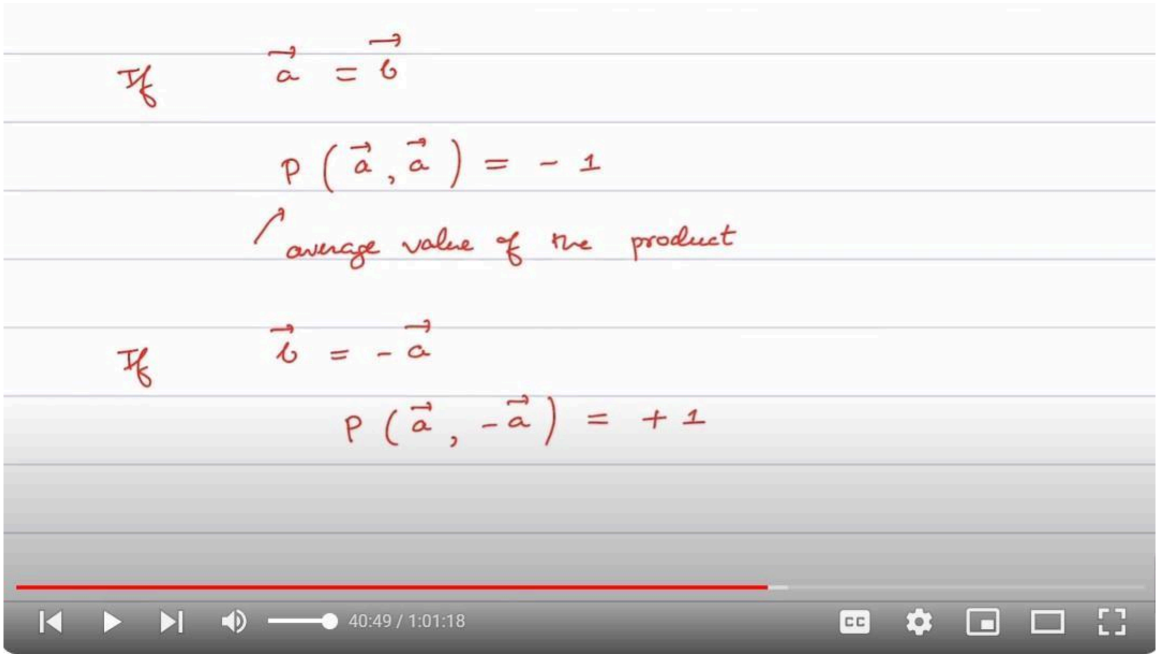

If A is equal to B, that means the detector axis are parallel, you recover the original EPR experiments of David Bohm. Bell proposed to calculate average value of the product of the spins for two given set of orientations. So what he says, that the product of these two measurements, if both the axis are parallel to each other, at detector A and B, you are going to get the product as minus one, so here P refers to average value of the product, so this refers to average value of the product, by product, I means the result that you obtain at detector A and B as regards the orientation of the spin, okay? Now, on the other hand, if, say, B is minus A, that means the detector axis at B is anti-parallel to A, then obviously the product of these two measurements you are going to get is to be plus one, right?

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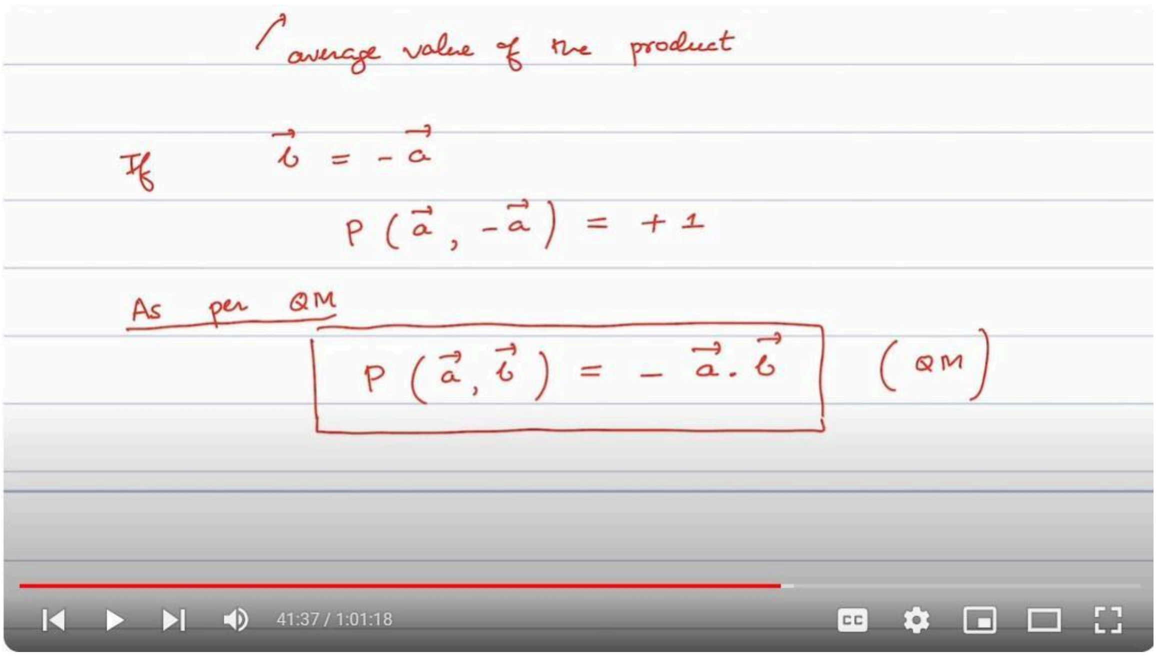

From this, we can write for arbitrary orientation as per quantum mechanics, these results that I am discussing is using quantum mechanics, so as per quantum mechanics, so as per quantum mechanics, we have this result, general result that product of the two measurements at the detector A and detector B is going to be minus one minus A dot B, so this is quantum mechanical result.

This relation, this particular relation, we are going to prove in the problem solving session by giving a proper example of two spin-half particles.

Now, what the so-called local hidden variable theory says about this average or expectation value?

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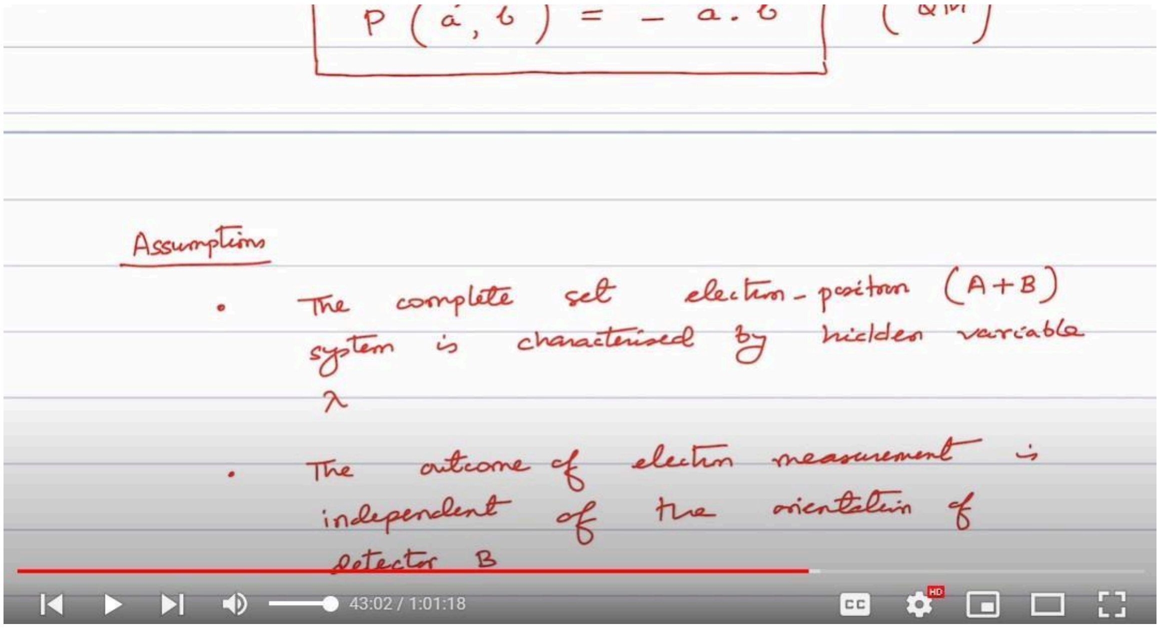

Let us find out, we are going to make some assumption first. Assumptions, first say the complete set of, the complete set of electron-positron system, electron-positron system, which is A plus B system, is characterized by, is A plus B, characterized by hidden variable, hidden variable lambda, we don't know what is this hidden variable and we have no control over it.

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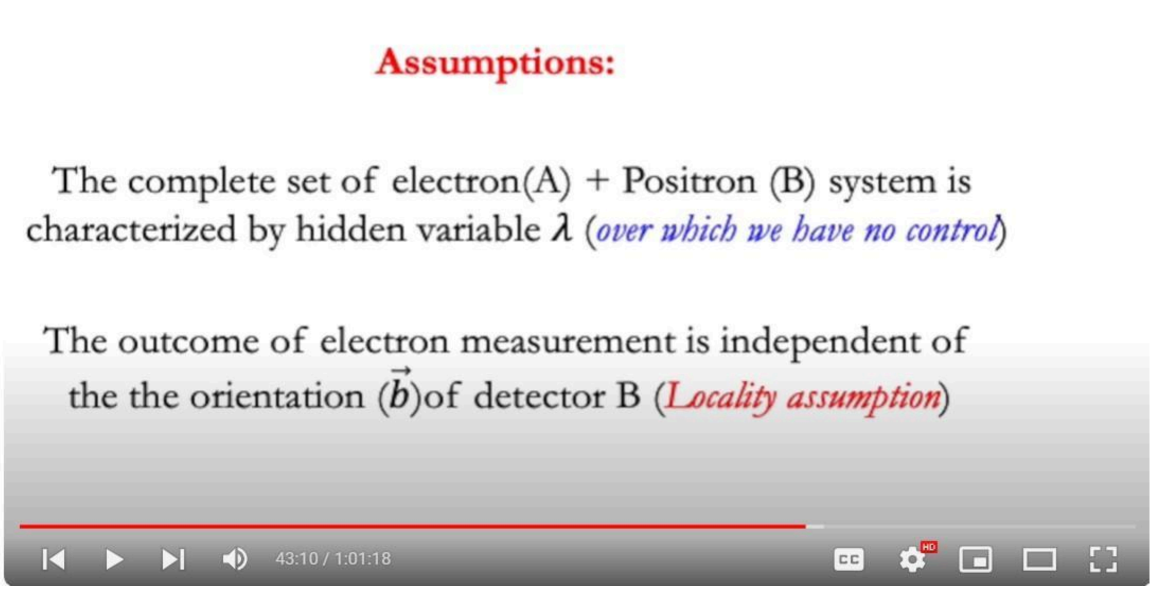

Also assume that the outcome of electron measurement, electron measurement is independent of the outcome independent of the orientation of the detector at B, detector B, so orientation means we don't know whatever be the direction of B, the outcome of electron measurement is not going to depend on that.

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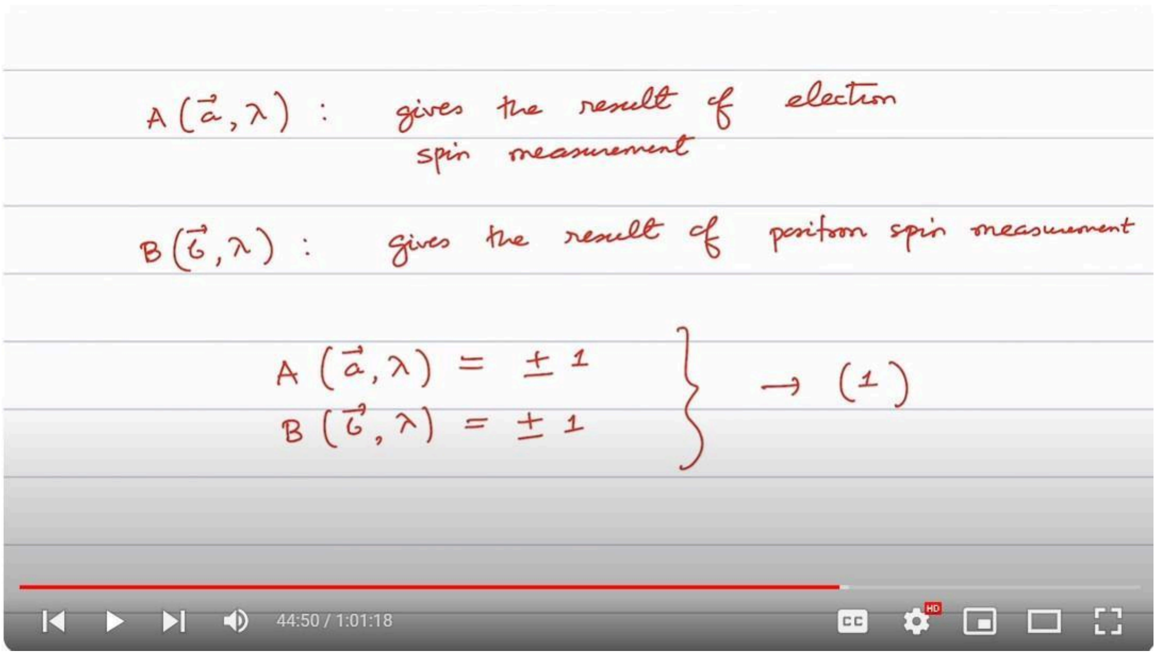

Suppose there exists some function, say A, a function of the orientation axis of the detector at A and hidden variable lambda, and this function gives the result of, gives the result of electron measurement, electron spin measurement, spin measurement, and we also have a function B, which is a function of the detector orientation at B and hidden variable lambda, and this gives the result of positron spin measurement, result of positron positron spin measurement, okay.

Now this function can take only values plus one or and minus one, so A, this function A, A lambda, it takes value either plus one or minus one, and B takes value, because the spin can be either plus half or minus half, that's why both A and K can take value plus one or minus one, let me say this is an important proposition, so let me say this is equation number one.

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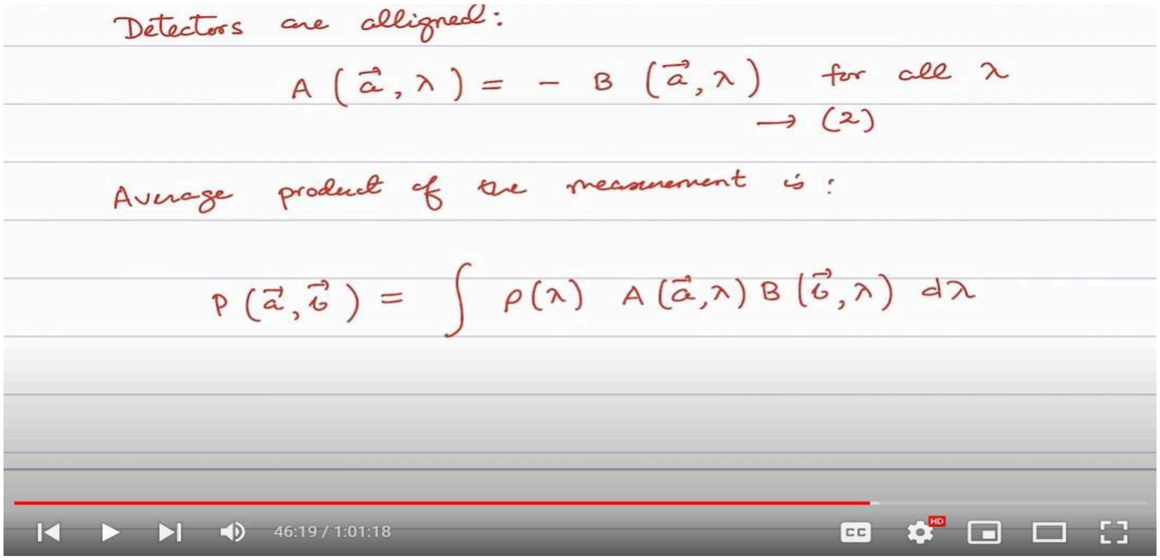

Now when the detectors are aligned, suppose the detectors are aligned, that means the orientation of the detector at A and B are parallel, when the detectors are aligned, then we'll get A of A lambda is equal to minus B of A lambda, because as you know, if the detector A detects that the spin is plus one, then obviously the detector B should give the opposite result of A, and that is going to happen for all hidden variable lambda, and this is our proposition number two, now the average product of measurement or expectation value, average expectation value or the average product of the measurement, product of the measurement is, it is given by PAB is equal to rho lambda, I'll explain what it is, product is A A lambda B B lambda d lambda,

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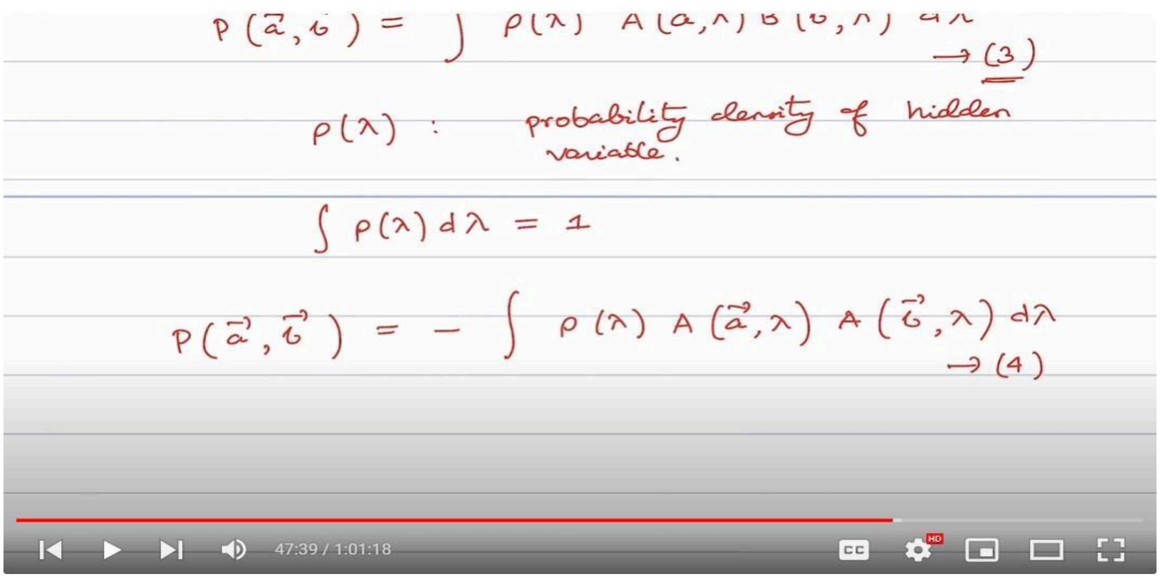

where rho lambda is the probability density of hidden variable, this is probability density of hidden variable, hidden variable, and just like any probability distribution, it should satisfy this condition, rho lambda d lambda should be equal to one, because it has to take some hidden variable out of all the distribution, and now using this equation, say this is my equation number three, using equation two, equation two in equation three, I can rewrite equation three as follows, I can rewrite equation three as this, PAB is equal to minus rho lambda A A lambda A B lambda, D lambda, so here I have used this equation number two in equation number three, and that's how I get this equation, so let me name it as equation number four.

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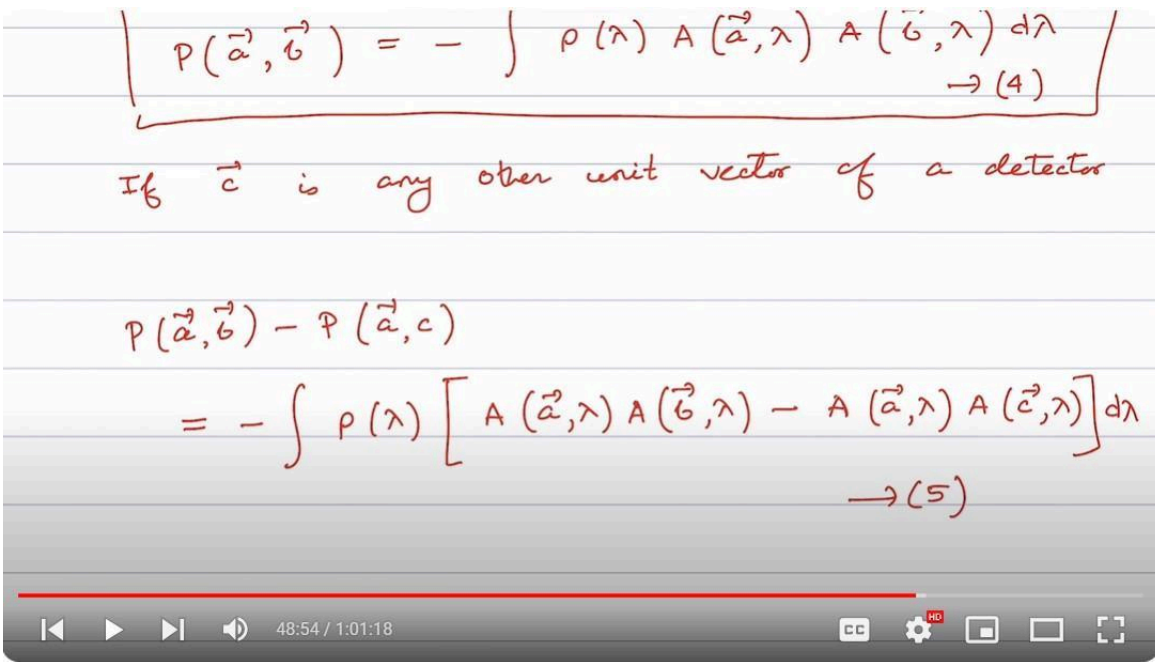

Now if I take another orientation of the detector, it may be a detector A or at B, say if C is any other orientation, or any other unit vector, or orientation of a detector, then I can write, because now I have this equation number four I have, using this equation, I can write PAB minus PAC, I just have to use equation four and then I will get very easily this expression, rho lambda A A lambda B, now I'm replacing B by A, so I will have A B lambda minus A B lambda, A A lambda, A C lambda, D lambda, okay, let me say this is my equation number five.

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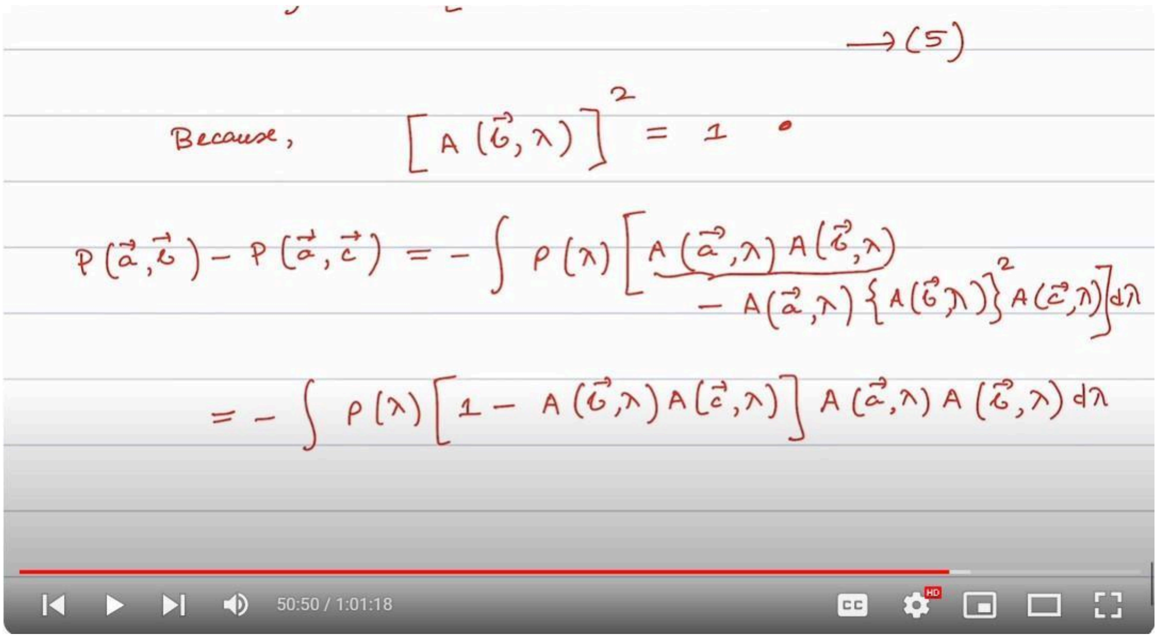

Now because of the fact that A B lambda whole square is equal to one, because A B lambda can take value either plus one or minus one, so utilizing this now I can rewrite further this equation number five, so I urge you to actually if you take pen and paper along with me you will be able to follow the steps very easily, so this equation number five now I can write as this, this one I will write minus rho lambda,

let me write down all the steps carefully, A lambda A B lambda minus A A lambda and in between this let me sandwich this guy, this expression let me sandwich that is A, because this is equal to one B lambda whole square, then I have A C lambda d lambda, okay,

so using this it is straightforward to see that I can write this equation as this rho lambda, I will take A A lambda A B lambda common, I will take it out, I will have one minus A B lambda B A C lambda, I will have A, I am taking it common now, A A lambda A B lambda d lambda, I hope all of you can follow it,

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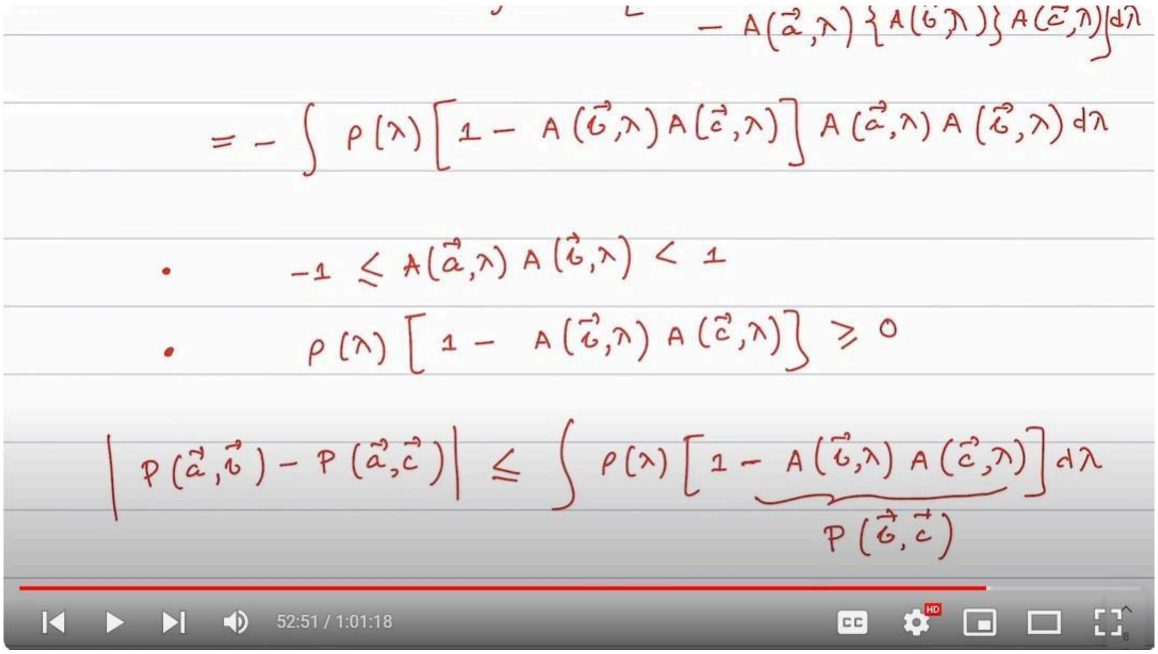

now we are going to use the fact that this product A A lambda A B lambda, it can take value either one or minus one or plus one, so it basically lies between minus one and plus one, right, it can maximum, it can take plus one and minimum, it can take minus one, so it lies between then, between these two values and moreover the fact is that rho lambda one minus A B lambda A C lambda, this is also going to be greater than or equal to zero, because rho lambda has to be a positive quantity, this probability, so this quantity has to be greater than or equal to zero,

so utilizing these two properties to information, I can therefore write P A B minus P A C is equal to, if I take the modulus, then I am going to get this inequality very in a straightforward way, that this would be less than or equal to rho lambda one minus A B lambda A C lambda d lambda,

now you see that this guy, this quantity is nothing but with this minus sign, this quantity is nothing but P B C, right, so along with of course rho lambda, if you see the definition of, let me again remind you, this expression if you see, so using that definition, along with this integral, this integral is also there, so this expression, let me say this is my equation number six, I think I have written this is equation number five,

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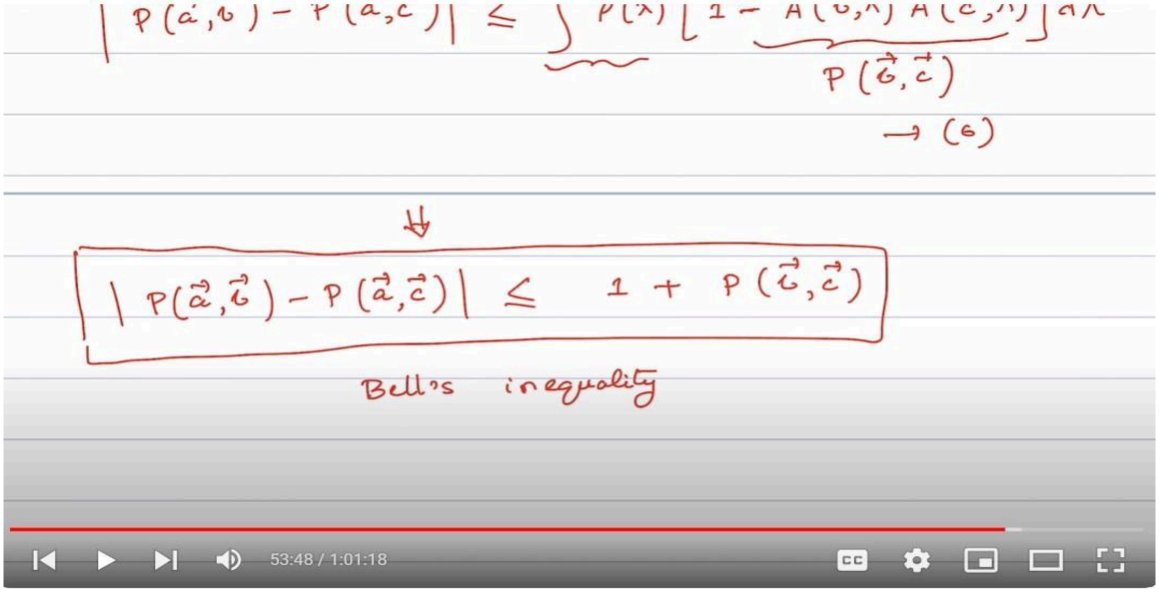

so this equation number six, I can write in a compact form as follows, that is P A B minus P A C is less than or equal to one plus P B C, and this is the very famous Bell's inequality, this is the so called Bell's inequality, it is one of the form, there are many many forms in literatures, many other forms are there, but primarily this is one of the most popular form of Bell's inequality, and these holds for any local hidden variable theory,

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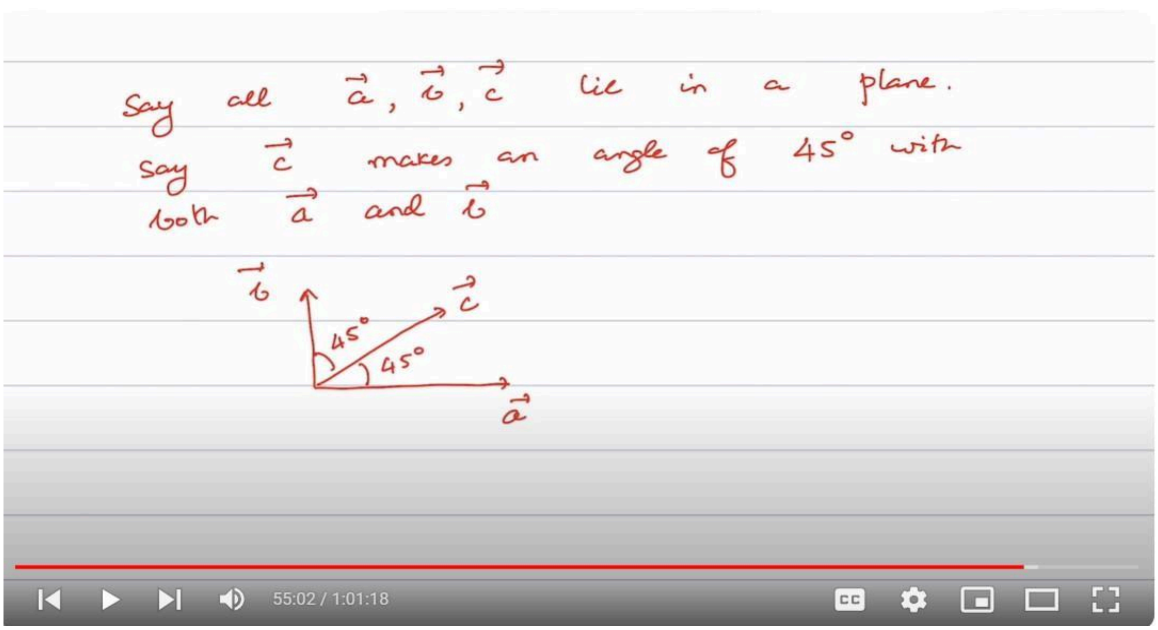

Now the question is, does the quantum mechanics predictions compatible with Bell's inequality or not, to do that, let us say all three vectors, say all three vectors, A detector orientation, A B C lie in a plane, lie in a plane, and say C makes an angle, say the orientation C makes an angle 45 degree, angle of 45 degree, with both, we can always have this kind of situation in experiment, with both A and B, that means the A and B are orthogonal to each other, the detector positions of detector A and detector B are perpendicular to each other, and C is making an angle of 45 degree with both of them, so this is 45 degree, this is 45 degree,

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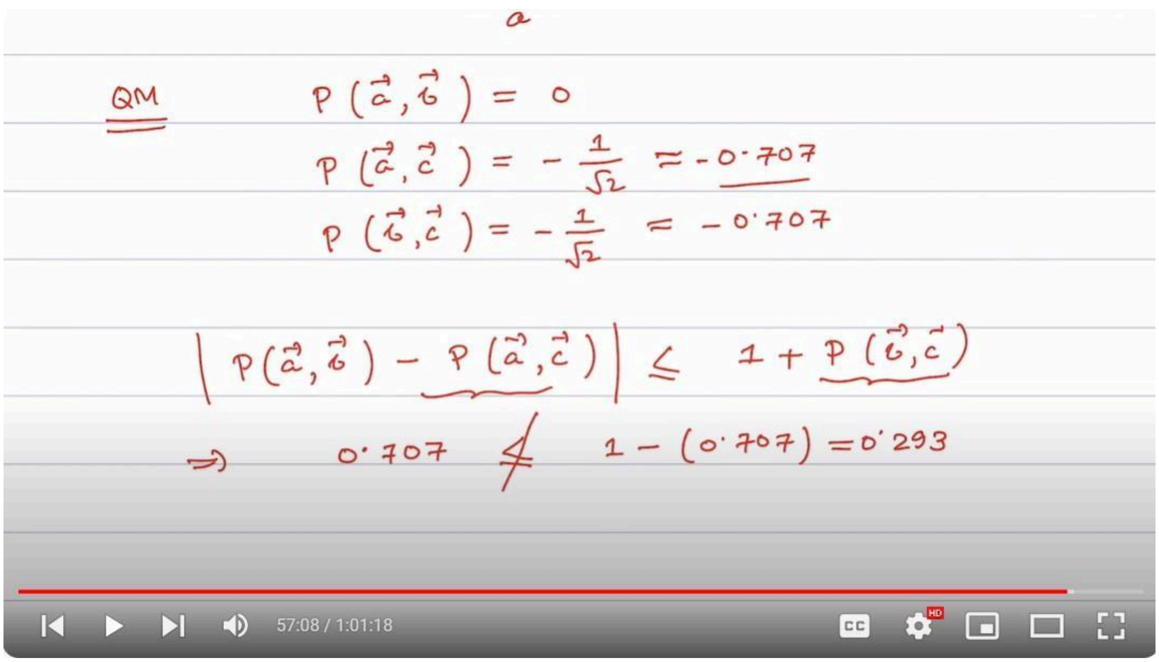

Therefore quantum mechanics, what quantum mechanics says, if this is the orientation, as per quantum mechanics, we know the result, P A B is equal to minus A dot B, which is equal to zero, A and B are by the way unit vectors, so because the angle is 90 degree between them, so cos 90 is zero, so P A B is equal to zero, on the other hand, P A C is, it is cos 45 degree, with a minus sign, with minus A dot C, so therefore it is minus one by root two and P, we are left with another quantity, that is P B C, P B C is also minus B dot C, and because the angle is 45 degree, this is one by root two, and one by root two is approximately 0.707, okay, this is minus 0.707.

Now let us see whether Bell's inequality, if we put whether that inequality is satisfied by this quantum mechanical results, if I put this here, this is my Baye inequality, one plus P B C, now P A B is equal to zero, and this equal to this quantity, one by root two, so therefore if I am taking the modulus, so this would be 0.707, and this would be less than or equal to one, and P B C is minus one by root two, so it is one minus approximately 0.707, so this is equal to 0.293.

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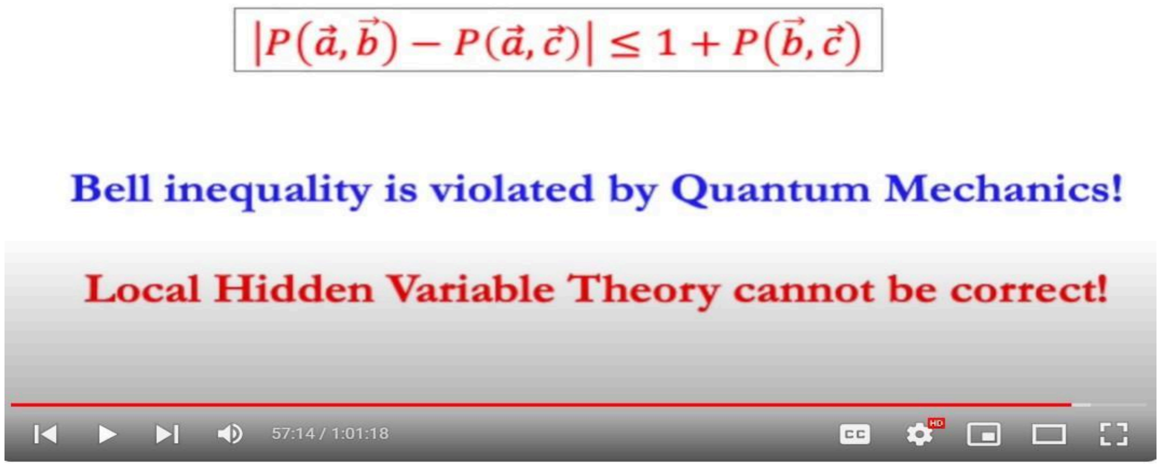

Now it is very clear that 0.707 is definitely greater than 0.293, so therefore this Bell's inequality is not obeyed by quantum mechanical results, so quantum mechanical is not consistent with Bell's inequality, and therefore we can discard the so-called local hidden variable theory.

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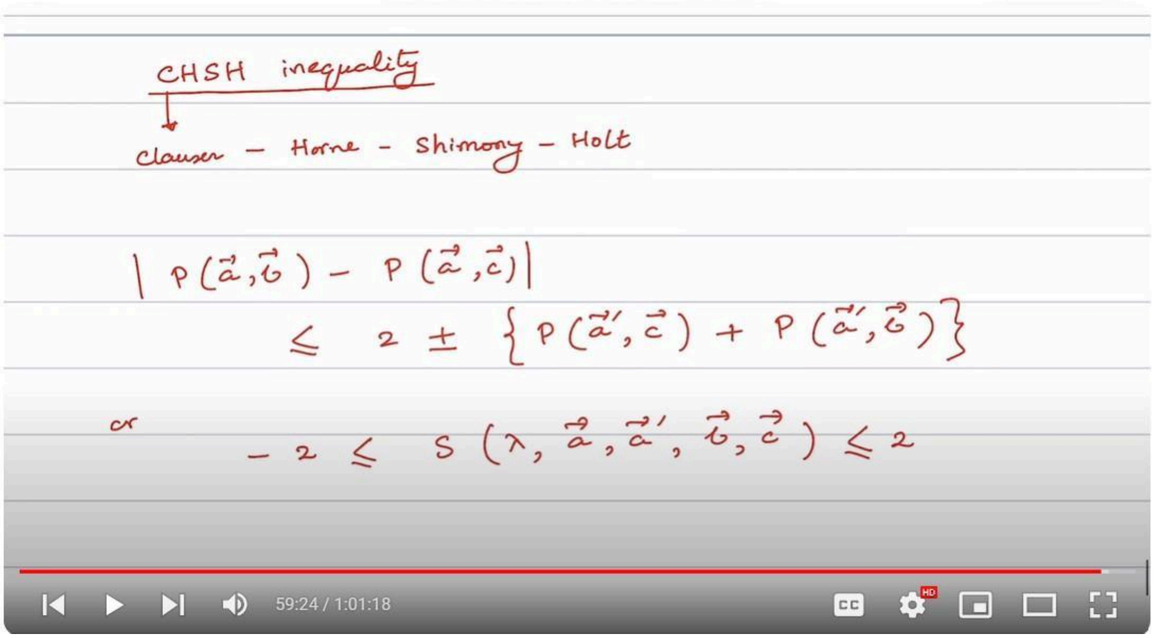

One important alternative form of Bell's inequality is the so-called CHSH inequality, it is similar to Bell's inequality that we have discussed, and it can be worked out using the form that we have discussed, and this inequality is named under four physicists, they are C stands for closure, H stands for horne, S stands for simony, and the other H stands for holt, and H stands for halt, so this is a very relevant inequality in terms of experiments are concerned,

and the form of this inequality is as follows, it is the modulus of P A B, A B are the, as I said, the detector orientation, P A C, this is less than or equal to 2 plus minus P A C, this is another orientation of the detector at say A, and this is another is P A B. Okay, and or we can write in a more convenient form this inequality is also written as minus two less than or equal to S lambda A A dash B C, it is less than or equal to plus two,

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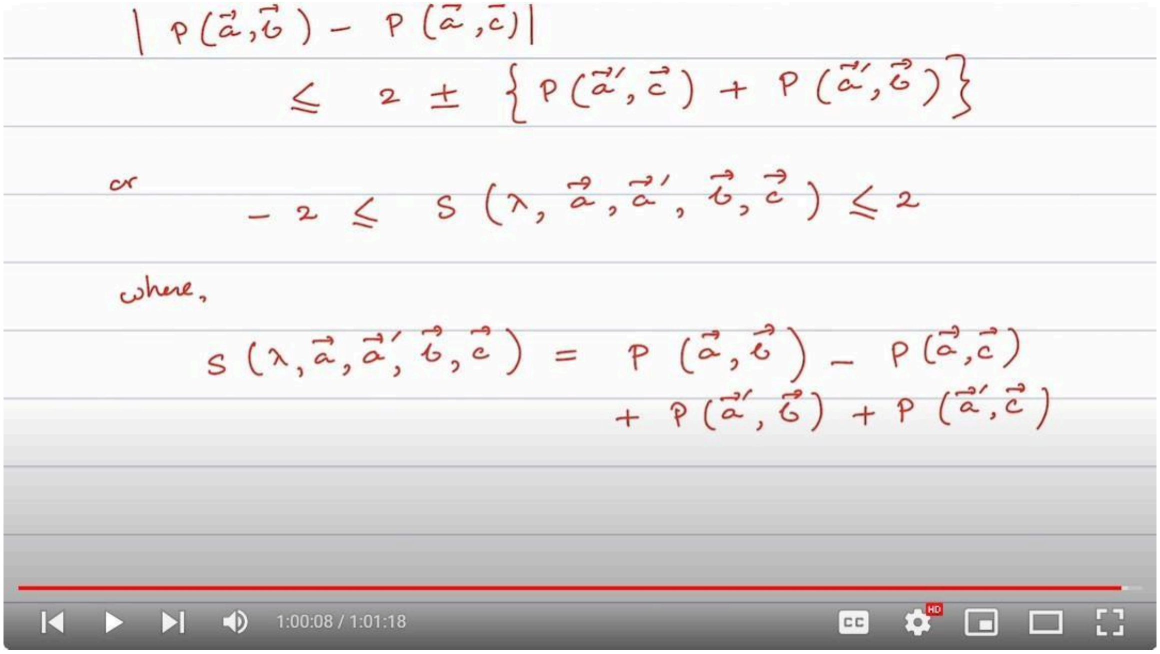

where this quantity S, which is a function of lambda and the orientations A A dash B C is equal to P A B minus P A C plus P A dash B plus P A B A dash C, the proof of this inequality is straightforward and we'll do that in problem solving session two.

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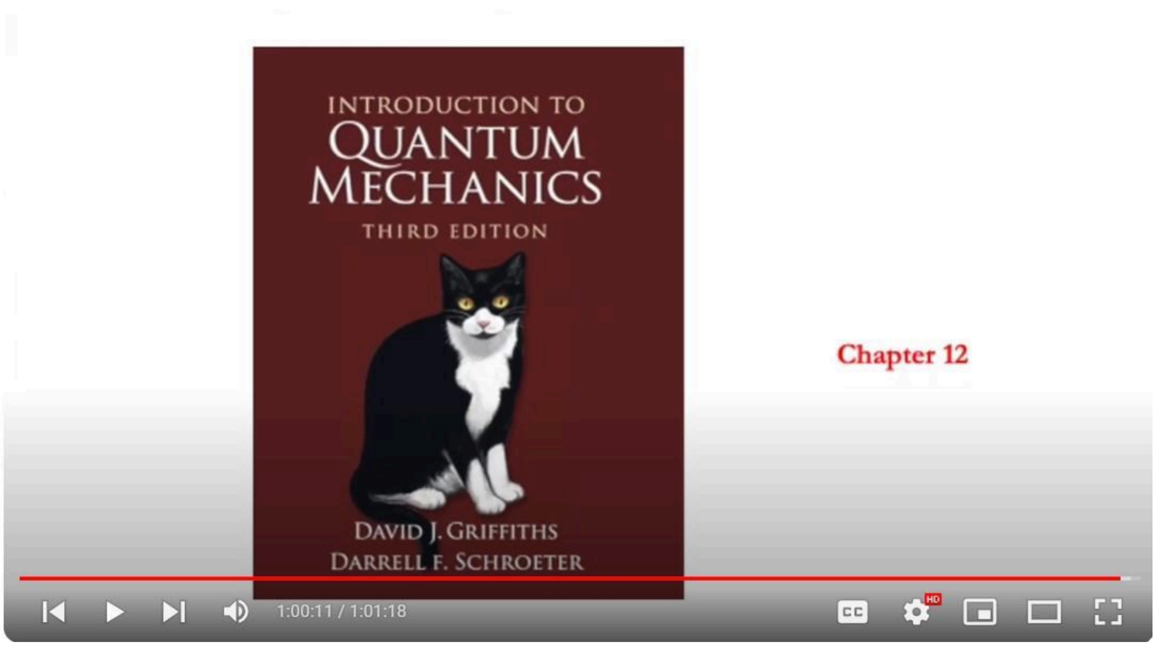

By the way, the Bell's inequality that I have proved earlier is inspired by Griffith's quantum mechanics book, chapter 12,

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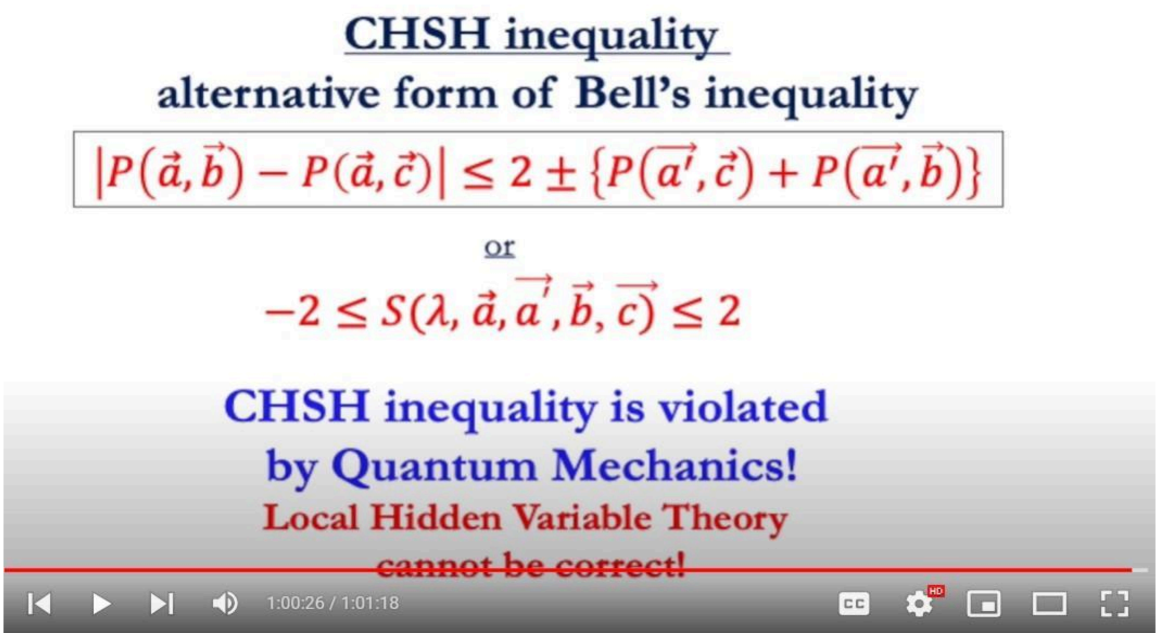

now as regards this CHSS inequality is concerned, Closer and his group experimentally demonstrated that quantum mechanics violate this inequality, thereby discarding local hidden variable theory.

Let me stop for now, with this lecture, we have completed the second module of the course, I encourage you to go to the problem solving session number two, in the next lecture, we'll start discussing quantification of quantum entanglement, so see you in the next lecture, thank you so much.