Quantum Entanglement: Fundamentals, measures and application

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Week-02

Lec 6: Basic Technical Introduction to Quantum Entanglement

Hello, welcome to lecture 4 of this course. This is the first lecture of module 2. In this lecture I am going to give you a brief technical introduction to quantum entanglement also some necessary mathematical tools also I am going to discuss. So let us begin.

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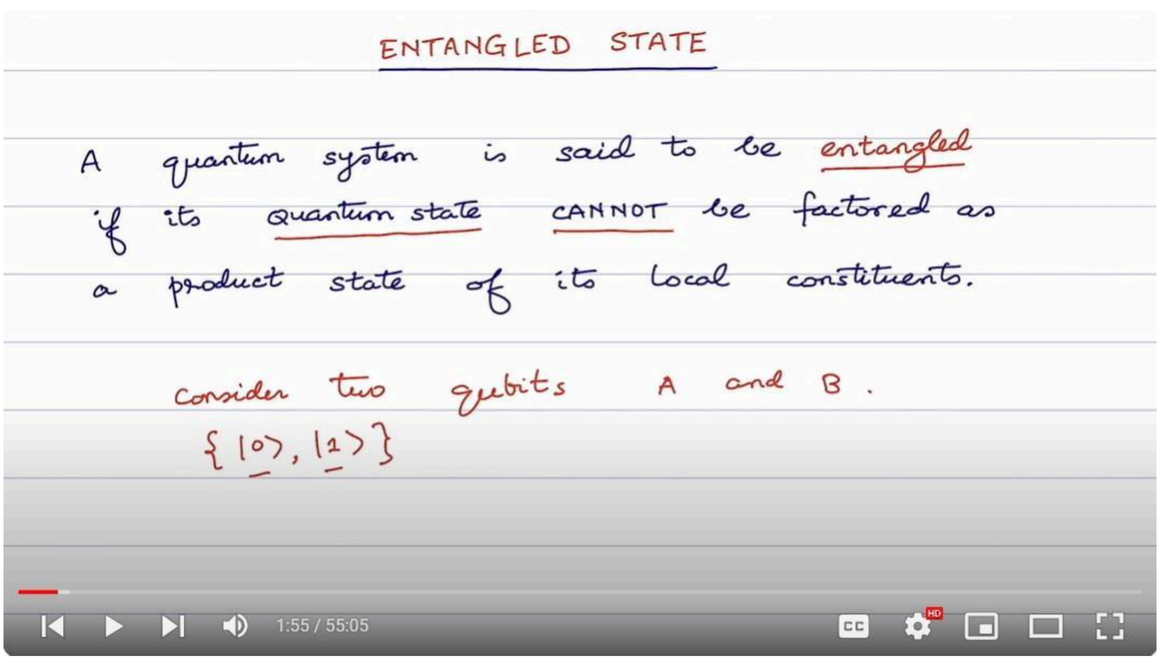

So what's an entangled state? To put very simply, a quantum system is said to be entangled if its quantum state cannot be factored as a product state of its local constituents. In other words, they are not individual system or particles, but are an inseparable whole. One constituent cannot be fully described without considering the other.

To understand it, let us consider two qubits. Let us consider two qubits, a composite system A and B. Okay, and the basis states we take to describe this qubit system, this two composite system is ket 0 and ket 1. Alright, where ket 0 may refer to a spin up and ket 1 may refer to spin down state of the qubit system if it's a two electron composite system.

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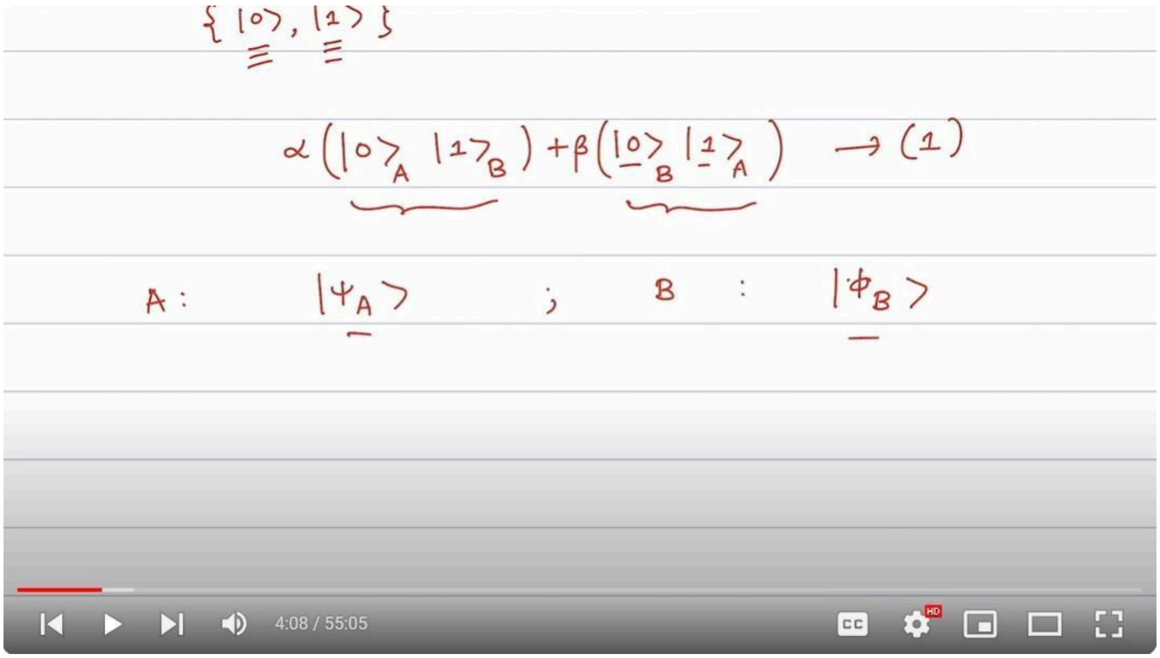

Now consider the situation where either qubit A is in ket 0 and qubit B is in ket 1 or qubit A is in ket 1 and qubit B is in the state ket 0. If a measurement is made, you get either of these two results only. Then the state of the composite system can be represented mathematically as follows. So say when you make a measurement, either you will get the system A to be in ket 0 and system B to be in ket 1 or you may get the system A to be in ket 1 and system B to be in ket 0.

So these are the two situations and basically you have a probability to get either this situation or that situation. So I can represent the state of this whole thing, this particular state would be represented by a state like this.

Now the question is suppose individually the system A is described by a state say ket psi A and system B is represented by say phi B. Whether I can express this particular state, let me say this is state 1, say this is equation 1 describe this composite system. That means either system A is in ket 0, system B is in ket 1 or if system B is in ket 0, system A is found to be in ket 1. Okay the question is whether I can express equation number 1 as a product state of psi A and psi B.

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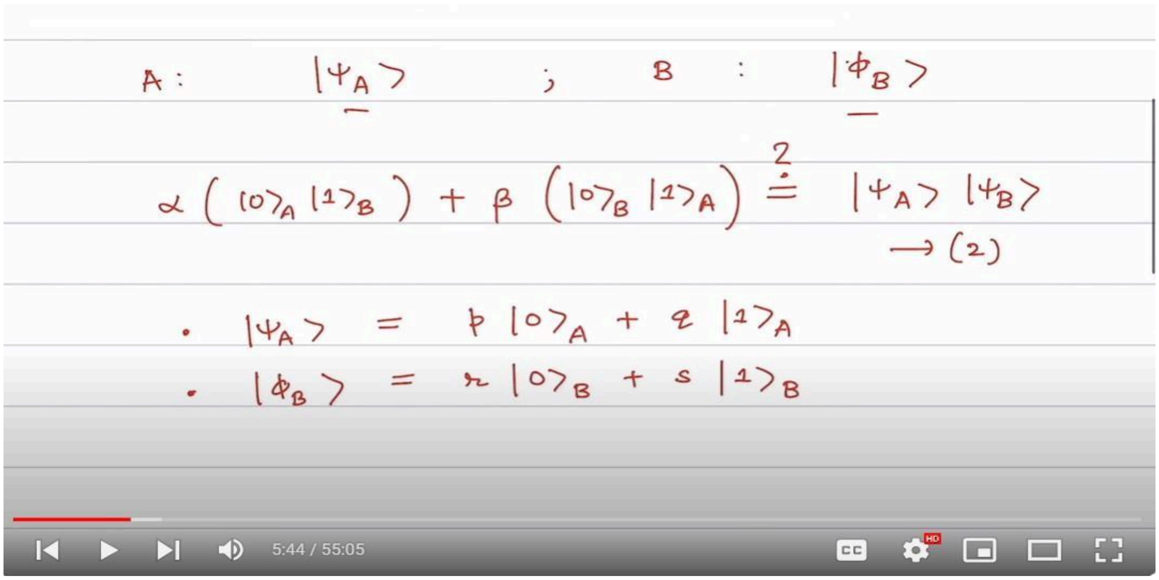

Alright so what I mean by this is that whether I can write say alpha ket 0 A plus not plus it's a two composite system ket 0 A ket 1 B plus beta ket 0 B ket 1 A. Whether I can write it as a product state like this or psi A psi B. So whether I can do that. Let me say this is my equation number 2. Okay let us see that whether we can do that or not. Suppose we assume that yes we can do that and then what is going to happen?

We have to analyze it. Let us analyze it. Now because both this psi A ket psi A and phi B can be expressed in terms of the basis states ket 0 and ket 1. So, psi A I can express as a superposition of ket 0 and ket 1. So let us say the superposition state is P0 A plus Q1 A. Superposition states P and Q are complex numbers. And so, system B for system B, I can write R say 0 B plus S1 B. right Where P, Q, R and S are complex numbers.

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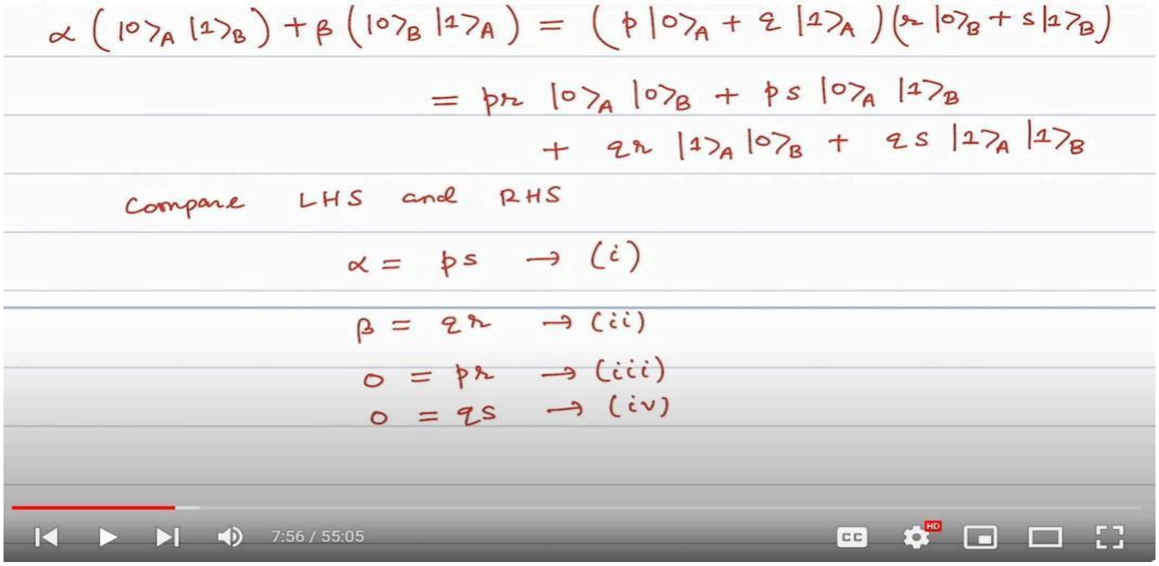

Now if I put these expressions in my equation 2 and expand it. So what I will get? So let me write equation 2 once again. Alpha 0 A 1 B plus beta 0 B 1 A. So then the right hand side if I expand. So I will get as okay let me first write the whole thing. P0 A plus Q1 A. This is psi A and for the other one I have R0 B plus S1 B. Okay if I expand it. If I then the right hand side I will get terms like this. say P R 0 A 0 B plus P S 0 A 1 B plus Q R 1 A 0 B plus Q S 1 A 1 B. Simple right?

Now if we compare both sides. Compare both left hand side, left hand side and right hand side of this equation. Right hand side. You can immediately see that I will get alpha is equal to Ps. Beta is equal to Qr. And you will have Pr is equal to 0 and Ps Qs. Qs would be equal to 0. Right? So say this is equation 1, 2, 3 and 4.

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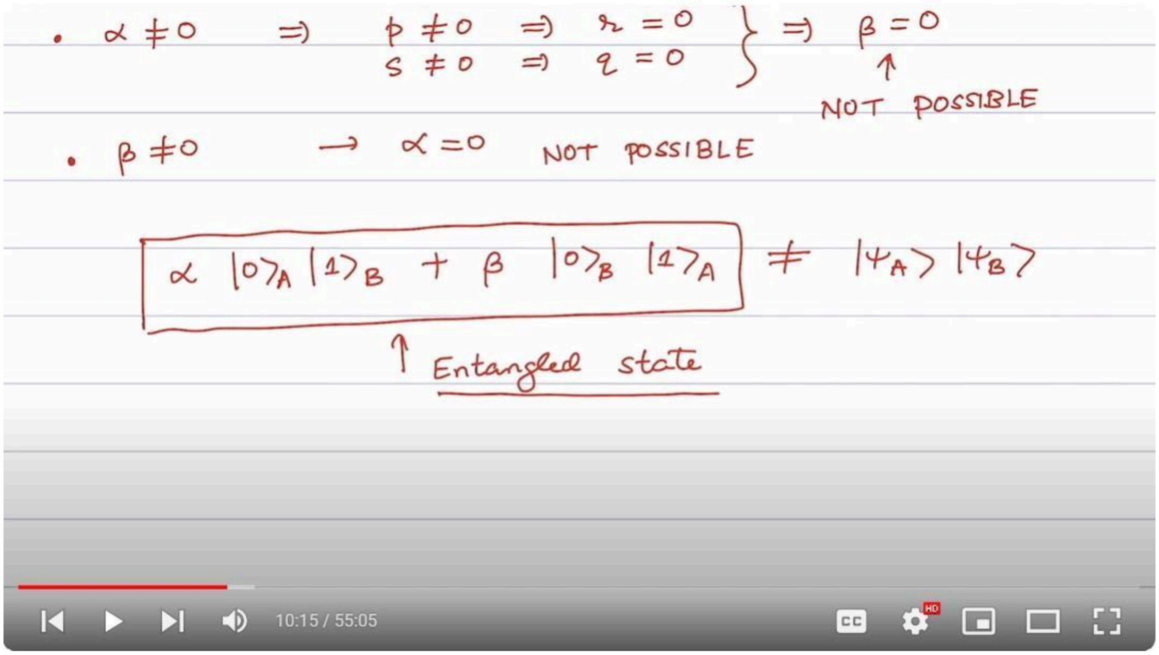

Now you see we cannot have alpha to be equal to 0. In fact beta can also be equal to 0. So first take this case. Say alpha is not equal to 0. If this is to be true and if all these equations has to be satisfied then we must have this implies that P cannot be equal to 0. Just look at this equation. Just look at this. If this is to be satisfied P into R. So R has to be equal to 0. Right? If equation 3 has to be satisfied, R has to be equal to 0. And also you see S cannot be equal to 0. This S cannot be equal to 0. So for equation 4 to be satisfied we must have S is equal to Q is equal to 0. Right? We must have to have Q equal to 0. But if that is so, this automatically implies that I must have beta is equal to 0. So this is not possible because my assumption is that alpha and beta both are non-zero. So this is not possible.

Similarly you can take the other case where if you impose this condition that beta cannot be equal to 0, it eventually lead you to the fact that alpha is equal to 0. Again this is also not possible. Therefore we can conclude that we cannot express this state alpha 0A 1B plus beta 0B This particular quantum state we cannot write it as a product state of the individual system psi A psi B. We cannot do that.

And therefore this particular state that we are considering here, this is an entangled state purely because of the fact that we cannot represent it as per the definition. We cannot write it as a product state of the individual constituents or subsystems. So, this is an entangled state.

Now let us consider this problem using the so called density matrix formula**.**

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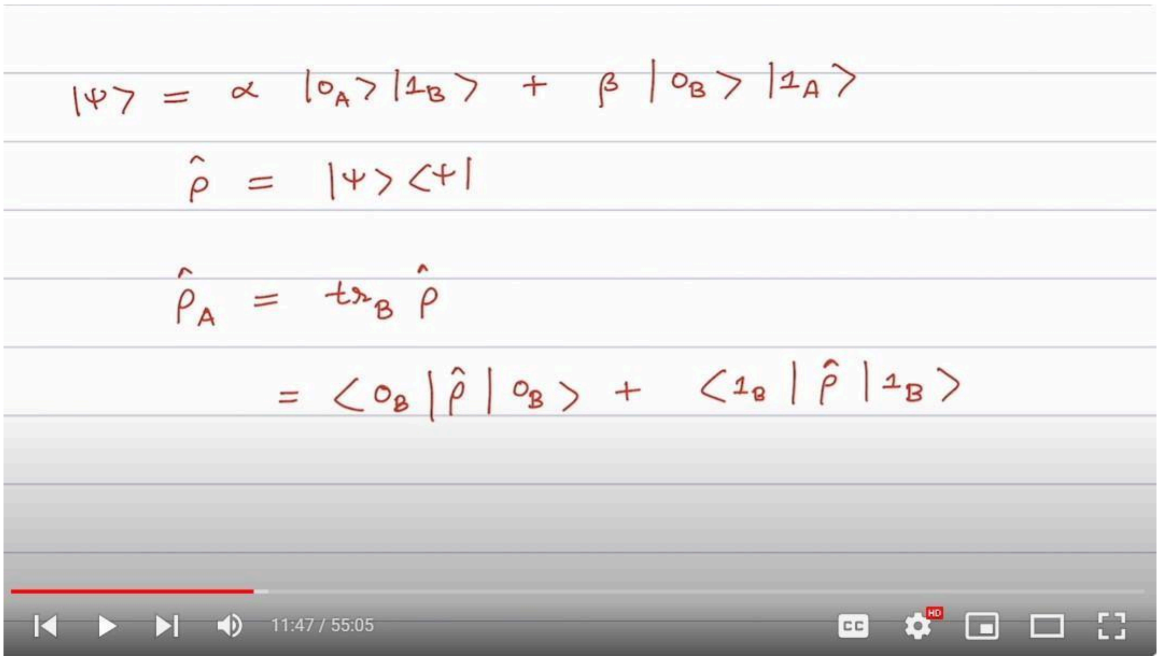

Let me write the entangled state in this particular form. So ket psi is equal to alpha 0A 1B plus beta 0B 1A. The corresponding density operator would be rho would be equal to ket psi bra psi. If we want to study system A specifically, we need to know the reduced density operator of A and that we can find out by tracing out B from the density operator rho.

So we have to do the trace operation over the density operator. So this is operator. So let me put the operator sign here. So because the basis states are ket 0 and ket 1, but we have to, when we will take the trace, we have to take the basis state in the system B. We are now tracing out B. So therefore I have to do the operation this way. So I will take the trace first over 0 ket 0 and then using the other basis state. So this is what I'll have.

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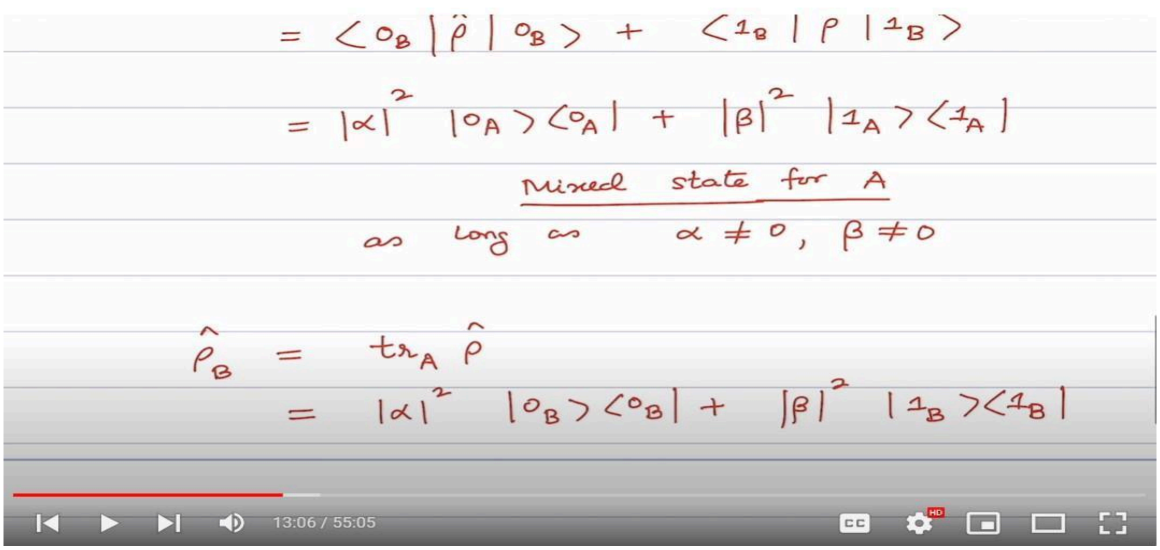

And in fact, if you do it, we have actually done this kind of problem in lecture 3. And in the first problem solving session. So I encourage you to work it out. If you do it, then this is what you should obtain. You will get mod alpha square 0A 0A plus mod beta square 1A 1A in this form. And as you can see, this is the form of a mixed state for system A. It's a mixed state for system A as long as both alpha and beta are non-zero. Okay.

Similarly, for the system B, we can write rho B, we have to then trace out A. If you trace out A, then you can show that you are going to get mod alpha square 0B 0B plus mod beta square 1B 1B.Okay. Clearly, when one of the systems is considered with regard to the other, actually without regard to the other, it is generally a mixed state. And what does it mean? It means that you don't have the full information about the system. In fact, this gives us a hint about how to characterize entanglement according to the degree of purity of either of the systems.

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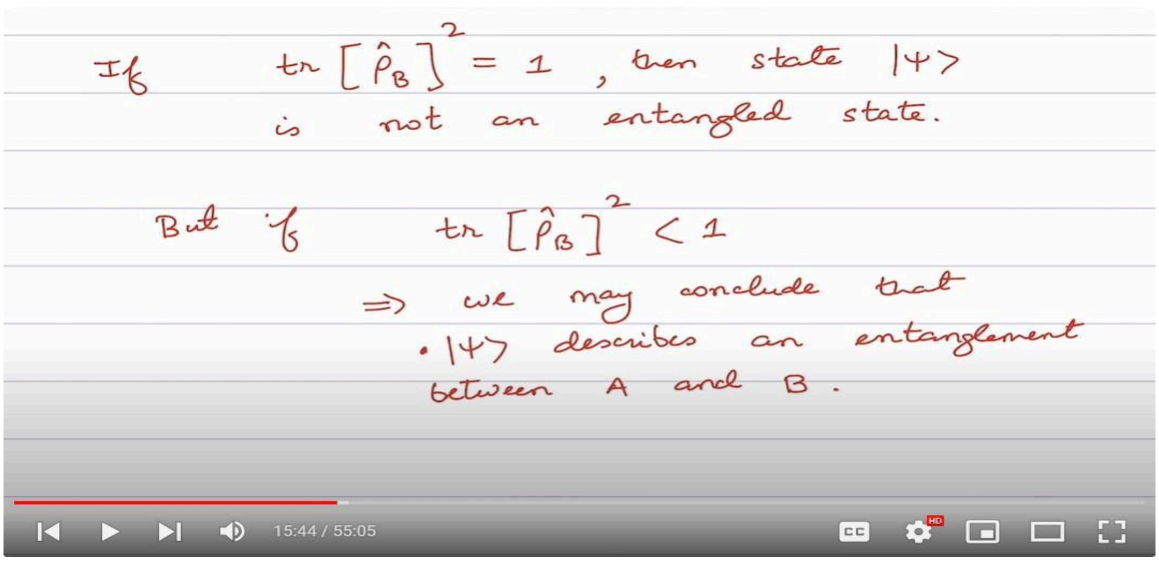

So, for example, if say, trace of rho B square is equal to 1, then the state ket psi is not an entangled state. Because it is going to be a completely pure state, right? The system B will be a pure state. That means you have the complete information about the system B and you need not have any information, need not have to know any information about system B. System B is in the mixed state, this is you see, the system B is in mixed state. That means, again, I am reiterating that you don't have the complete information of the system B. So, overall you have information about system A plus B only, but individual information is not there. Now, if trace rho B square is equal to 1, that means you have complete information about the system B.

That means it is in a pure state. So, if you find, that means, but if trace rho B square is less than 1, it may say, it implies that we may conclude, we may conclude that the state psi describes an entanglement, entanglement between A and B, between A and B. So, you can either find out trace of rho B square or trace of rho A square. If you find that it is less than 1 in both cases, then we can say that this state psi represents an entanglement between the system A and B.

When we talk about quantum entanglement, we deal with composite systems. A composite system is a system which contains two or more parts in it. In this course, mostly we are going to talk about composite system which contains two parts in it. And such systems are called bipartite system. The example just I have considered is an example of quantum entanglement in pure states. Later on, I am going to talk about entanglement of mixed state as well. But for now, let us discuss quantum entanglement of pure state a little bit more technically.

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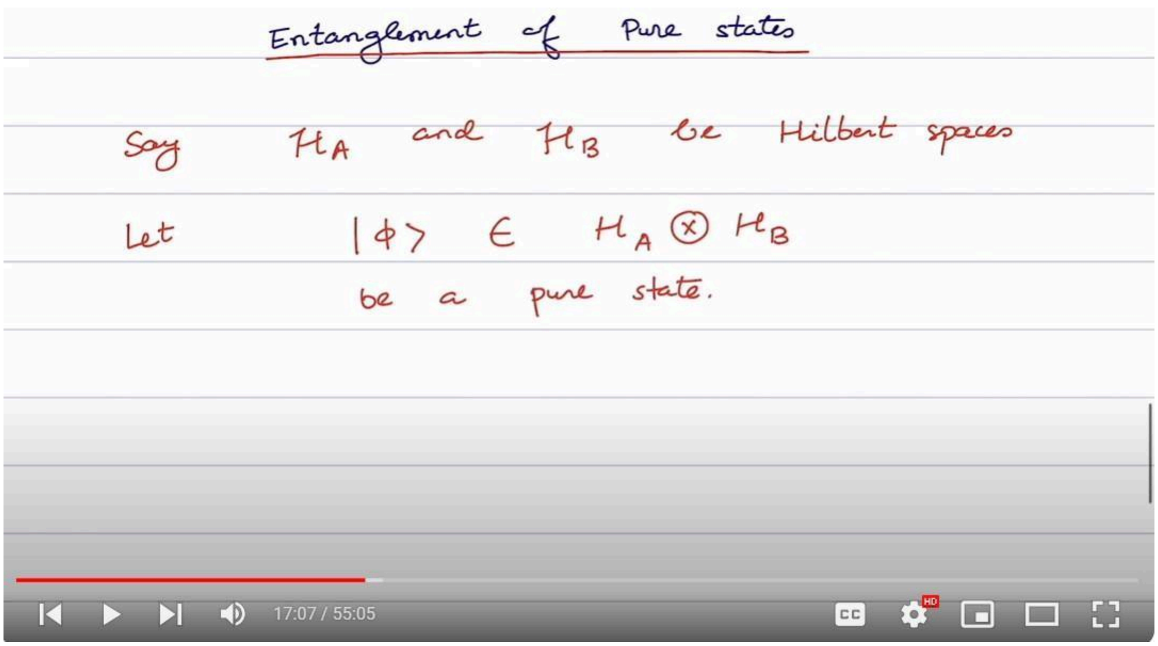

Let us say, HA and HB are Hilbert spaces be Hilbert spaces. And let us say, there is a state phi which belongs to these two Hilbert spaces, that belongs to the direct product of these two Hilbert spaces of system A and B. And it is a pure state. Let phi be a pure state.

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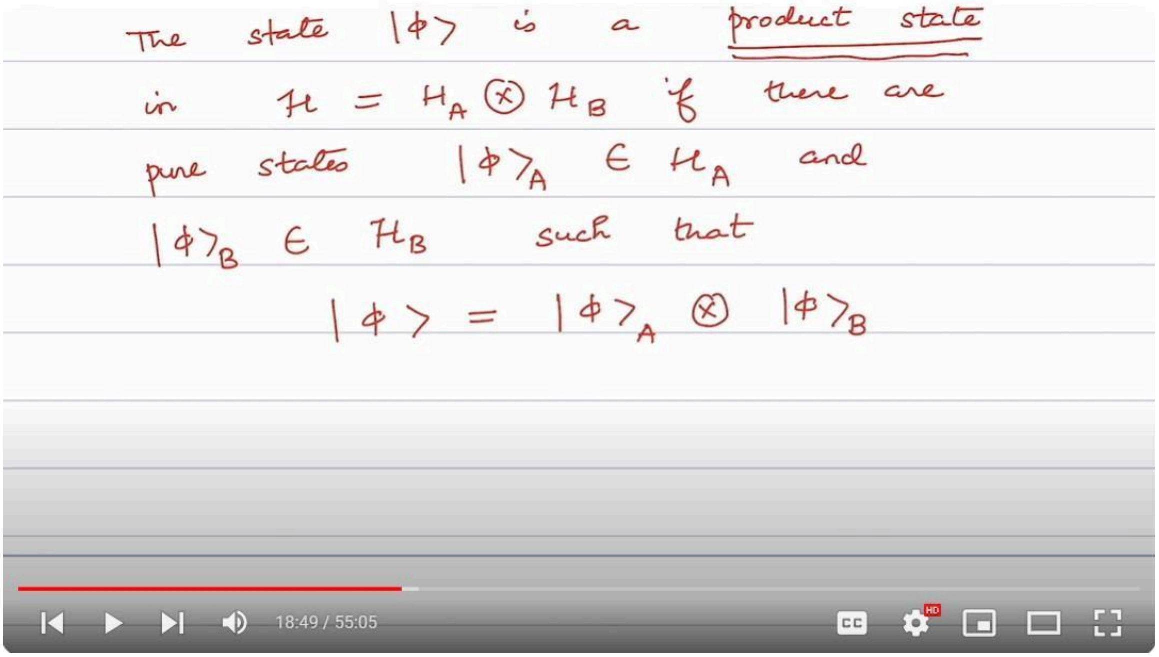

Now this state, phi A, this state phi is a product state, is a product state in the Hilbert space H is equal to, which is the direct product or the tensor product of the Hilbert spaces HA and HB.

If, basically I am giving you the definition of product state. So, the state phi is a product state in the Hilbert space H, which is equal to the direct product of Hilbert space of A and Hilbert space of B. If there are, there are pure states, say phi A belonging to the Hilbert space A and another state, say phi B belonging to the Hilbert space B. This is basically a technical definition. It is such that, this phi, I can write is a tensor product of these two pure states phi A and phi B. Okay, you can consider it as I said as a definition of product state. A pure state phi is separable if it is a product state in Hilbert space H, otherwise the state phi is entanglement.

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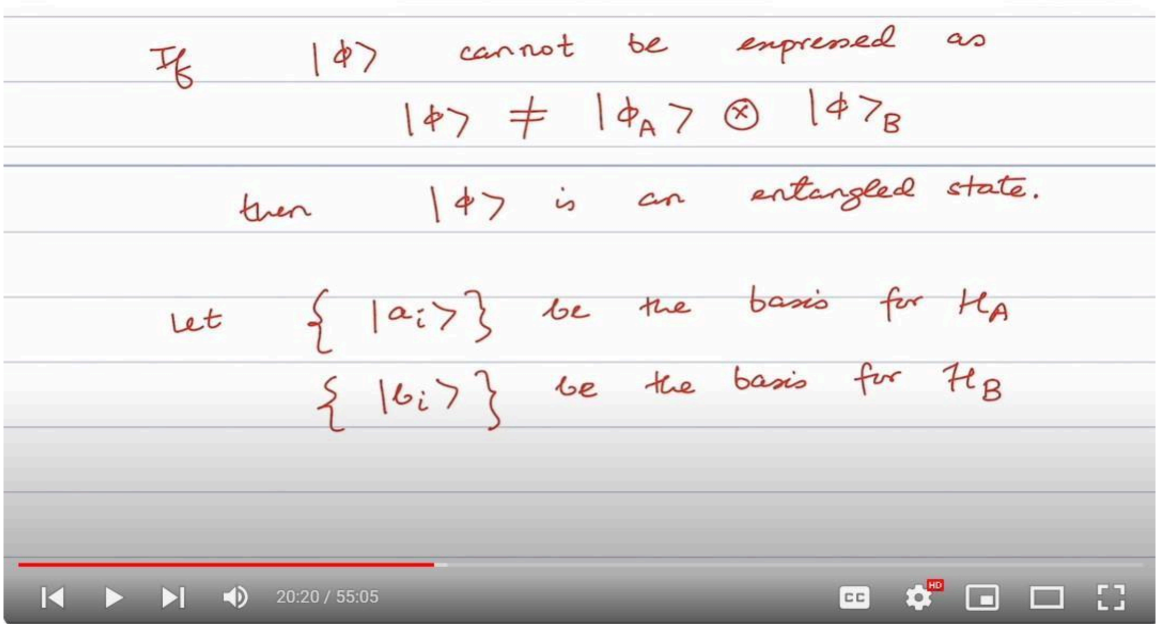

If phi cannot be expressed as product state, that means we already know if this is not equal to the tensor product of these two states, then phi is an entangled state. So, this you can consider is as definition. However, the fact that the pure states fulfilling the condition of entanglement is not immediately obvious. So, let us say to understand it further, so let I have Ai be the basis, be the basis for Hilbert space HA and Bi corresponds to the basis, be the basis for Hilbert space corresponding to system B.

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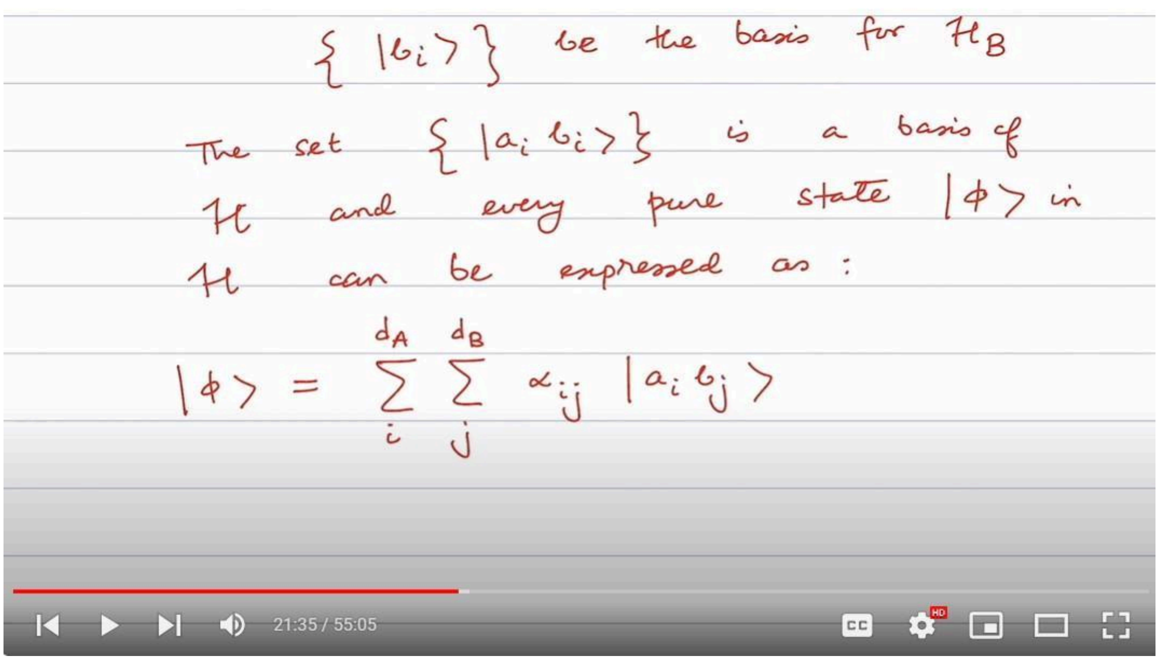

The set of basis, this set Ai Bi, this is basically the direct product of these two basis Ai and Bi is a basis of the Hilbert space H and every pure state phi in the Hilbert space H can be expressed as, we are now discussing technical things, so therefore you have to listen carefully. I can express this pure state phi as summation over the basis, i goes from say da, da refers to the dimension of the Hilbert space A and j goes from to db, that is the dimension of the Hilbert space B and I have this complex coefficient alpha ij Ai bj.

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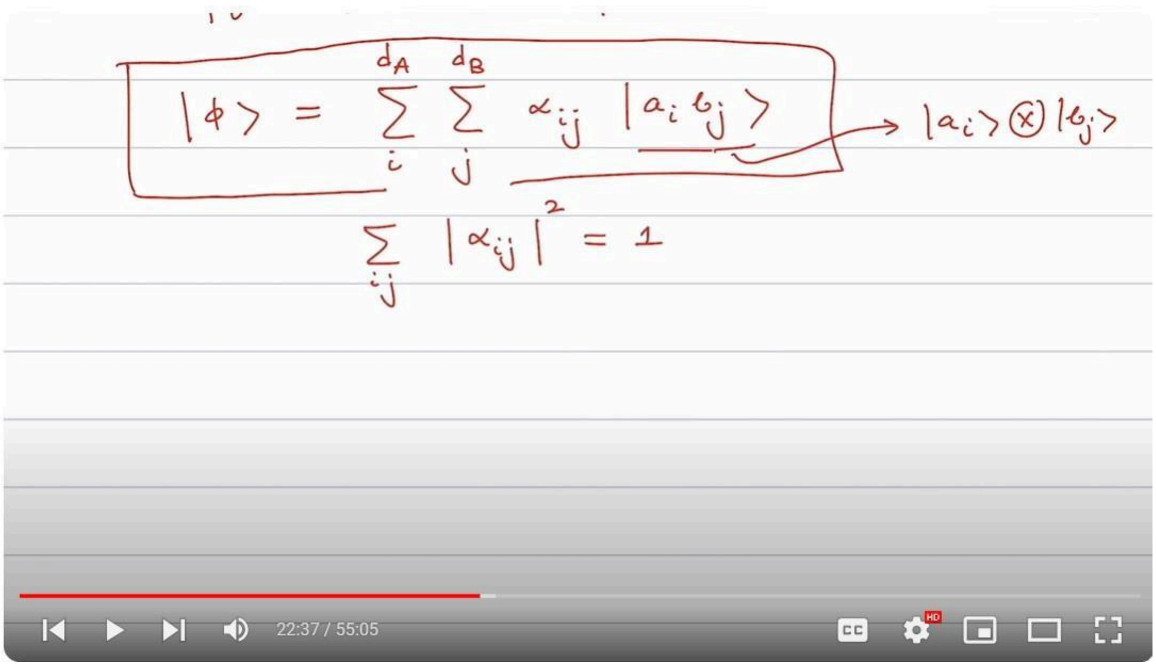

And also obviously the condition this complex coefficient alpha ij has to satisfy is alpha ij mod square is equal to 1. I think similar stuff we have discussed when I first mentioned about bipartite system in an earlier class, so here I am giving a general definition.

Therefore it is necessary to find a state such that it can be written as a linear combination, so if you have a pure state like this it is necessary to find a state such that it can be written as a linear combination of tensor product, this is tensor product, this is the short hand notation for the tensor product Ai cross bj of the basis states of the parts but cannot be written as a tensor product of two states from the respective parts directly.

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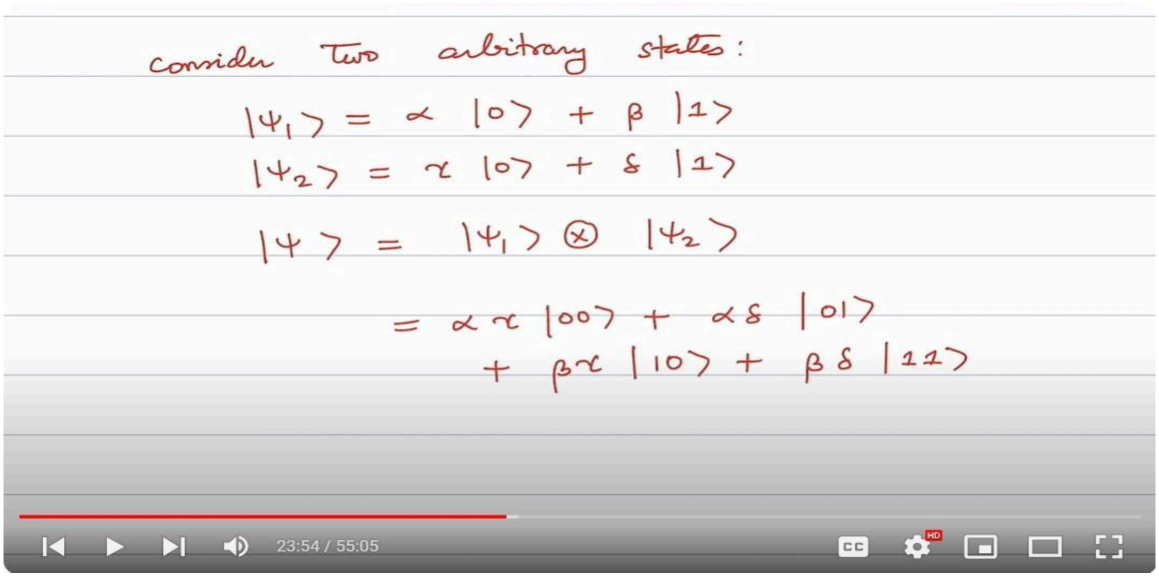

To illustrate it let me give you an example, let us consider these two states, consider two arbitrary states, two arbitrary states, let us say I have a state say ket psi 1 in the basis state k0 and ket 1. So alpha 0 plus beta 1 and another state I have is psi 2 that is say gamma 0 plus delta 1. Alright, now an arbitrary product state psi, ket psi can be written as the tensor product of psi 1 and psi 2. Okay, and if I open it up then what I am going to get, I will get alpha gamma 00 plus alpha delta 01 plus beta gamma 10 plus beta delta 11, check carefully this is what you should get.

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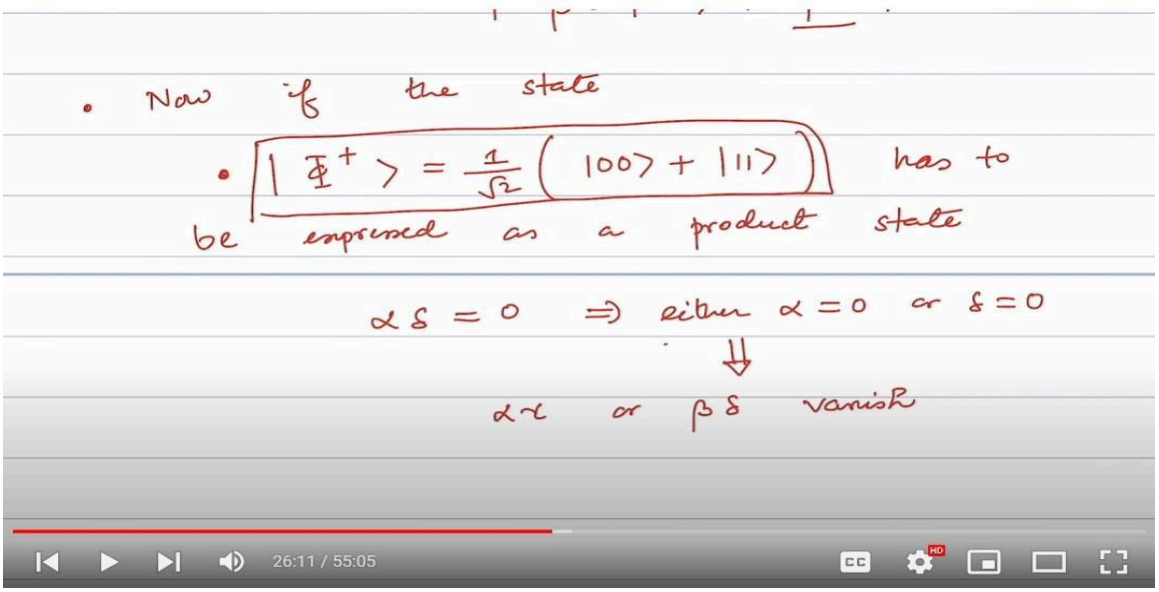

Now say, now say if we have this state, now if the state phi plus is equal to 1 by root 2, 00 plus 11 were to be a product state. Suppose I have this state and I have to write it as a product state then I think we have done a similar type of example at the beginning of this lecture. So if you look at it, if I have to write this as a product state, phi has to be expressed as a product state, what I should have?

If you look at this expression here, from here you see that means product state of this type, so if this is to be written as a product state, this phi plus has to be written as a product state. If you analyze it, you will find that you should have alpha delta is equal to 0, right? And this implies, this is going to imply that either alpha is equal to 0 or delta is equal to 0. But this means again, this in turn is going to imply that alpha gamma and or alpha gamma or beta delta will vanish, okay? This alpha gamma or beta delta is going to vanish and therefore this going to lead us to a contradiction with the assumption that this guy is a product state.

So clearly this is not a product state, that is what it means.

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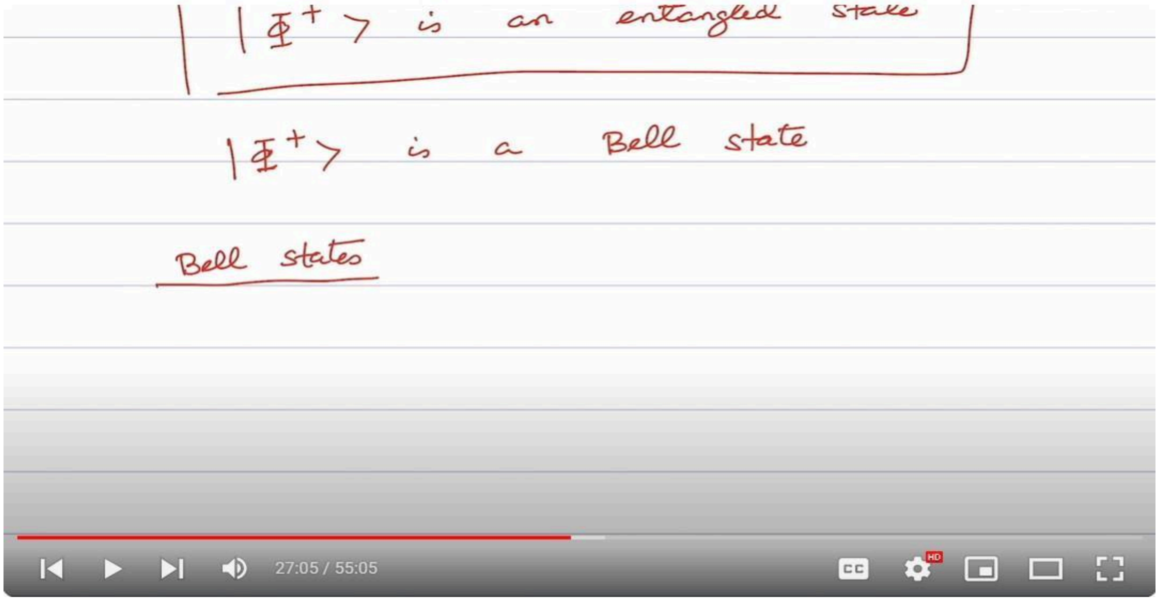

What it is then as per the definition, phi plus is an entangled state. I hope you are getting it. In fact, phi plus is one of the Bell state, is a Bell state. We are going to discuss Bell's inequality and so on in the next lecture.

And this kind of state does exist, experiment shows that such kind of states does exist and there are four Bell states and all of them serve as an example of entangled state.

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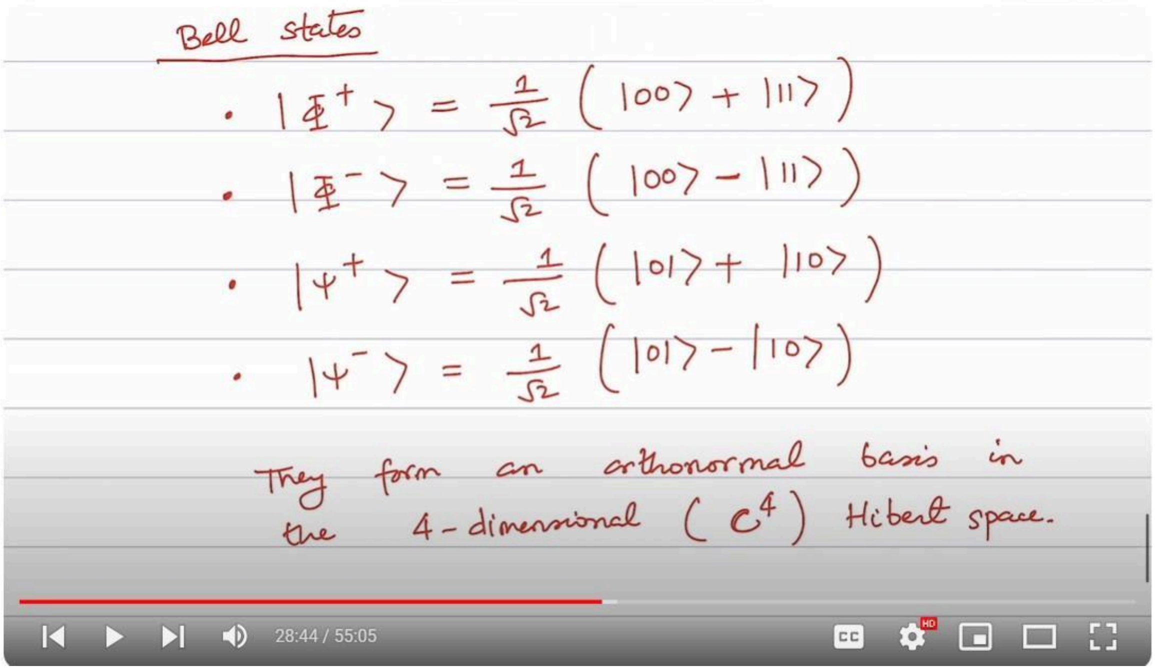

These four Bell states are, so let me write down all the Bell states here. You can remember them. So these Bell states are, all of them are entangled states.

Phi plus already we know it is 1 by root 2, 0 0 plus 1 1. Phi minus is 1 by root 2, 0 0 minus 1 1. Okay? You see the difference in the sign here. And then we have state, phi plus 1 by root 2, 0 1 plus 1 0. And we have, phi minus is equal to 1 by root 2, 0 1 minus 1 0. Alright? So it is interesting to note that these four Bell states are mutually orthogonal to each other.

These four Bell states are mutually orthogonal to each other and because of them, they form an orthonormal basis. They form an orthonormal basis, orthonormal basis in the four dimensional complex Hilbert space. Four dimensional, which is in short notation I can write C4 Hilbert space. This is important, it is used in quantum information and quantum computation a lot.

We will now discuss a very convenient method to characterize quantum entanglement.
This method is particularly useful for bipartite system in the context of bipartite quantum entanglement. This method is called Schmidt Decomposition Method. I am going to discuss it, but before that I have to remind you about some other decomposition method from linear algebra so that you can understand this method quickly.

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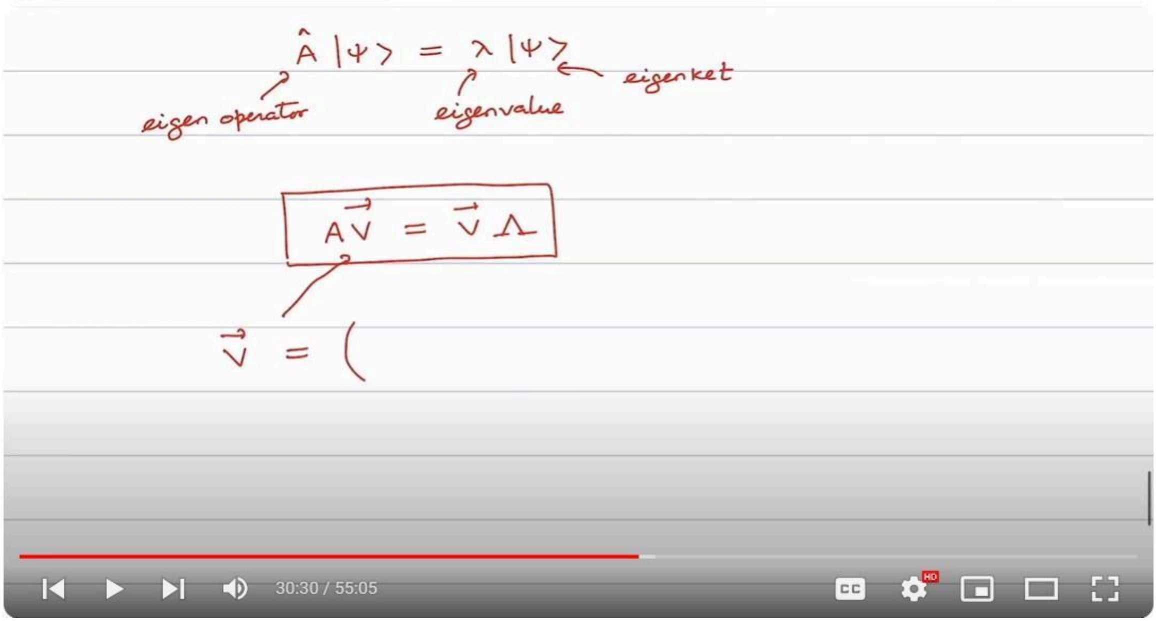

Say A is a linear operator in the Hilbert space and it satisfy the eigenvalue equation A ket psi is equal to lambda ket psi. Where lambda is the eigenvalue and ket psi is the eigen ket and A is an eigen operator. You know that the operator A can be represented by a matrix and all of you also know that how to find eigen values and eigen vectors of a matrix. Now let me invoke a result from linear algebra which you may have seen that relation which goes like this. Av where v is a matrix and Av is equal to v capital lambda.

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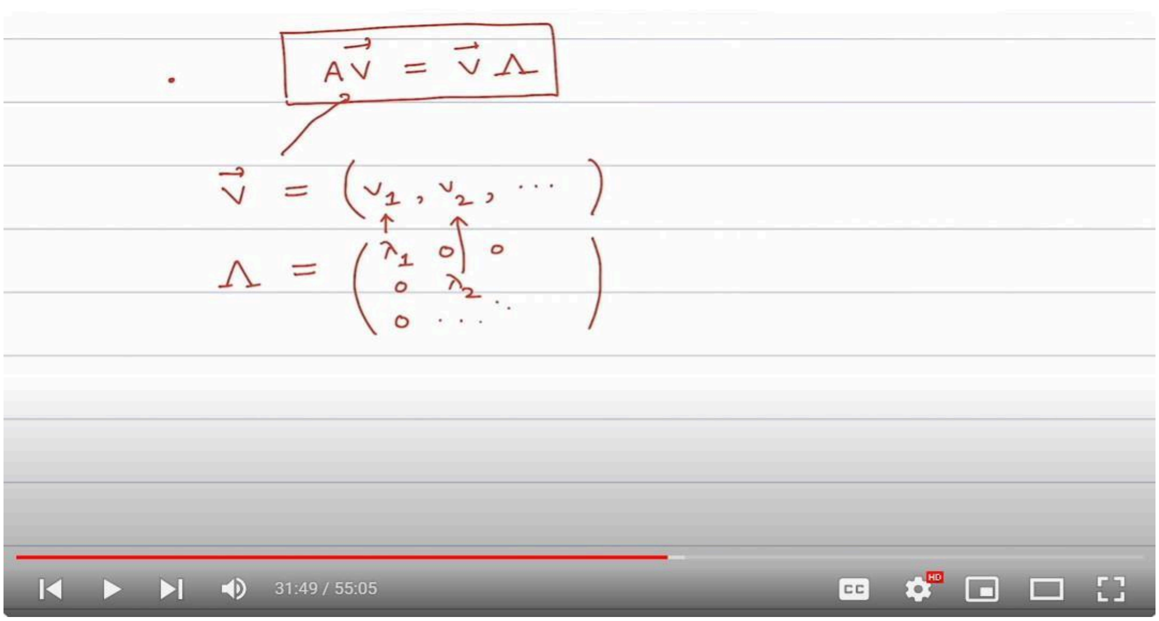

I am going to explain what are these. So this is the relation where v is a matrix formed by the eigen vectors of matrix A formed by eigen vectors v1, v2 and so on of the matrix A. And this capital lambda it is a diagonal matrix formed by the eigen values of the matrix A and it eigen values are say lambda 1, lambda 2 and so on. Which are its diagonal elements.

The convention is that you arrange this capital lambda matrix in such a way that the eigen value lambda 1 is the one which has the highest value. And lambda 2 is less than lambda 1, lambda 3 is less than lambda 2 and so on. And the vector v or which is the matrix v is formed in such a way that v1 is the eigen vector corresponding to the eigen value lambda 1, v2 is the eigen vector corresponding to the eigen value lambda 2 and so on. Those of you who have seen this particular relation for the first time I can illustrate it by an example.

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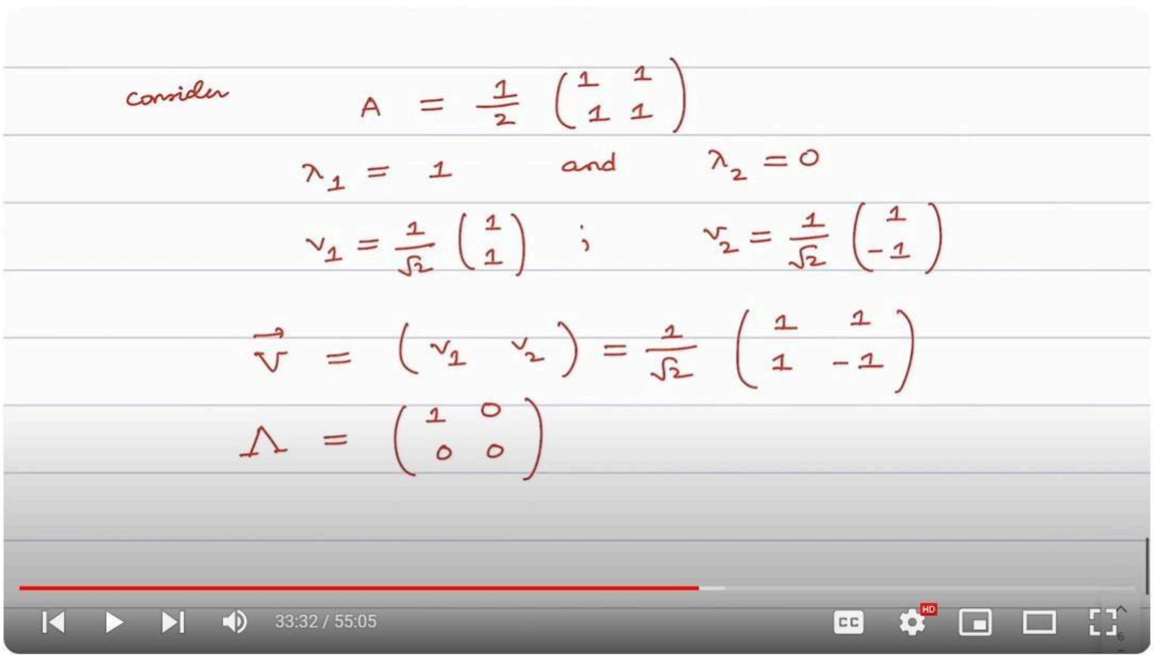

Let us consider a 2 by 2 matrix for illustration purpose. Say consider a matrix A, a 2 by 2 matrix. Let us say it is half 1, 1, 1, 1.And you can easily find out the eigen values of this matrix. You just have to set up the characteristic equation and solve it. And if you do that you will get the eigen values as lambda 1 is equal to 1 and the other eigen value would be lambda 2 is equal to 0. You can find the eigen vector corresponding to lambda 1 and you can show that it would be 1 by root 2, 1, 1. That's the eigen vector corresponding to lambda 1.

And v2 which is the eigen vector corresponding to lambda 2 is equal to 0. That would be 1 by root 2, 1, minus 1. Now using this v1 and v2 eigen vectors you can now write down the matrix v and that would be constituted by v1, v2. Explicitly speaking, writing you can write it as 1 by root 2, 1, 1, 1, minus 1. Ok. And this capital lambda, that's the diagonal matrix would be 1, 0, 0, 0.

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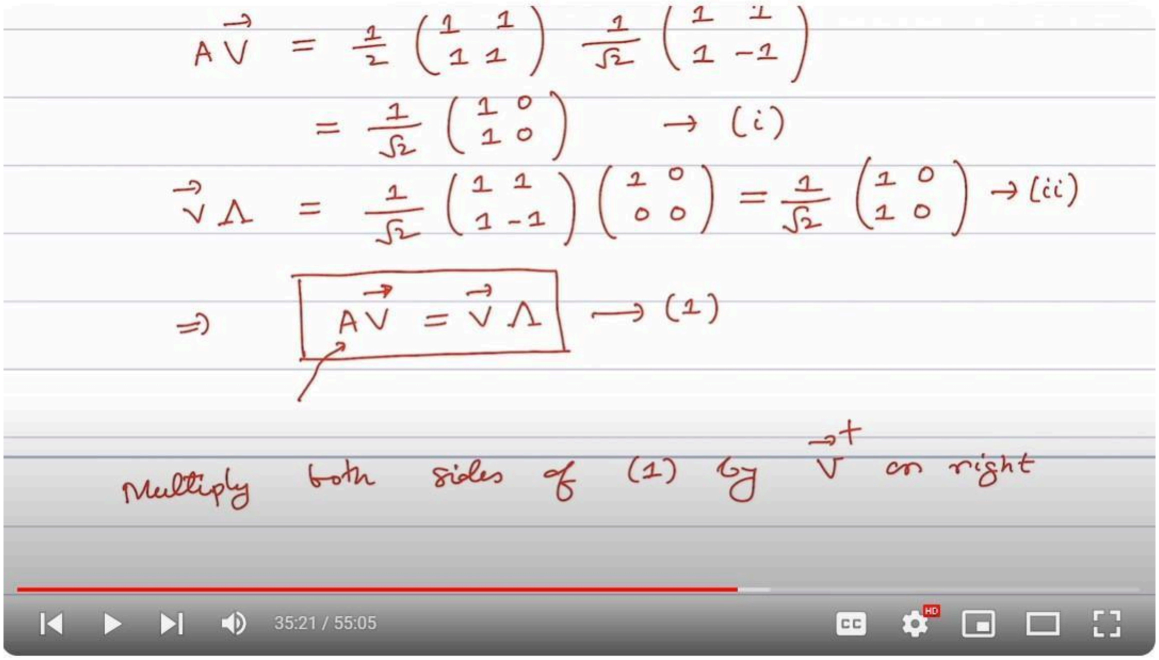

Now let me quickly verify whether the relation that I have written here is obeyed by the matrix A or not. To do that let me first multiply A and V and A matrix is half 1, 1, 1, 1 and V matrix is 1 by root 2. It is 1, 1, 1, minus 1. If you do the multiplication you will get 1 by root 1, 0, 1, 0.Let us say this is my equation number 1.

And now if I multiply V and capital lambda then I will get V is equal to 1 by root 2, 1, 1, 1, minus 1. And capital lambda is 1, 0, 0, 0. And you will get as a result of the matrix multiplication you will get 1 by root 2, 1, 0, 1, 0. And let me say this is equation number 2.

As you can see equation 1 and 2 matches. So they are equal. So this implies with this example I have just proved that Av is equal to V capital lambda. This is a very important relation and very useful. Let us now separate this matrix A from this relation. To do that let me multiply both sides of this equation. Say this is equation number 1.

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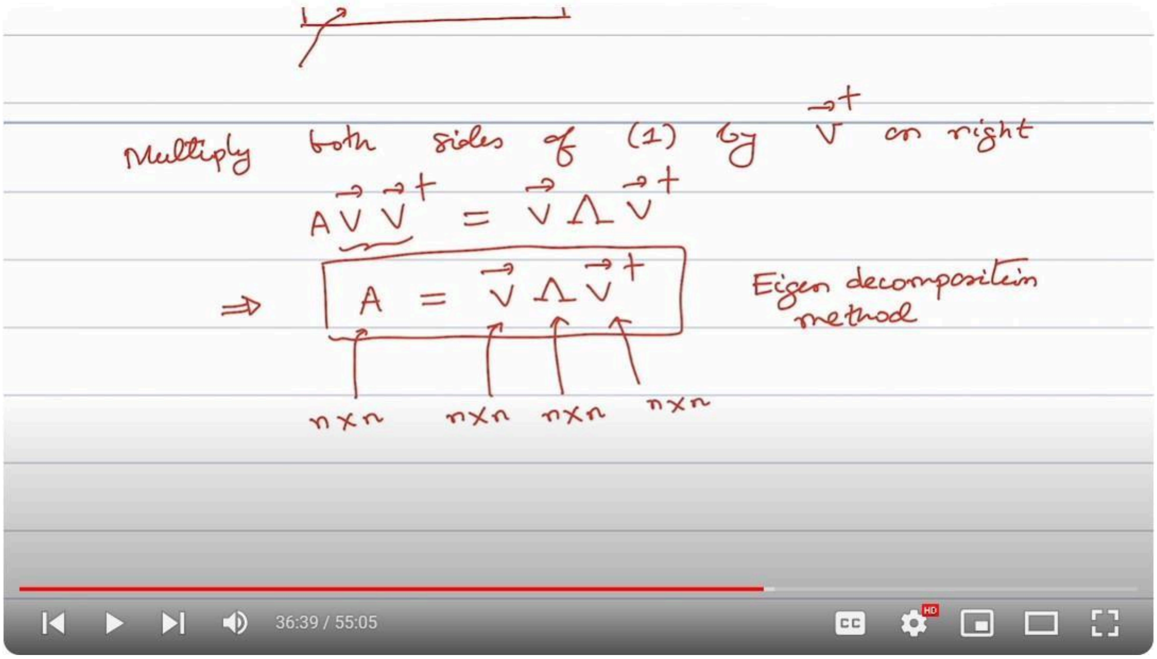

Multiply both sides of equation 1 by V dagger which is the Hermitian conjugate of the matrix V on right. Okay. So that means what I am doing is Av V dagger is equal to V capital lambda V dagger. Now you see V V dagger is an identity matrix.

So I will get on the left hand side A is equal to V capital lambda V dagger. This way I am able to decompose the matrix into three parts and this method is called Eigen decomposition method. This method is called Eigen decomposition method. Why Eigen decomposition? Because A is an Eigen matrix. So effectively we have decomposed a square matrix into another three square matrix.

Here A is a square matrix of dimension say n by n. That means n rows n columns. V is also a square matrix of dimension n by n. Capital lambda which is a diagonal matrix. It has also the dimension of n by n. And V dagger which is the Hermitian conjugate of the matrix V. It also has the dimension n by n.

Many times the Eigen matrix given to us may not be a square matrix. It may be a non-square matrix. That means it may have say m number of rows and n number of columns. In that case is it possible to decompose that matrix into three parts the way we have done it for the Eigen decomposition method? Yes it is possible provided the three parts obey certain peculiar kind of forms. And this method is known as the singular value decomposition method where one of the parts is going to be a diagonal matrix and diagonal matrix we are always interested because it has it is easy to handle and it contains lot of important information.

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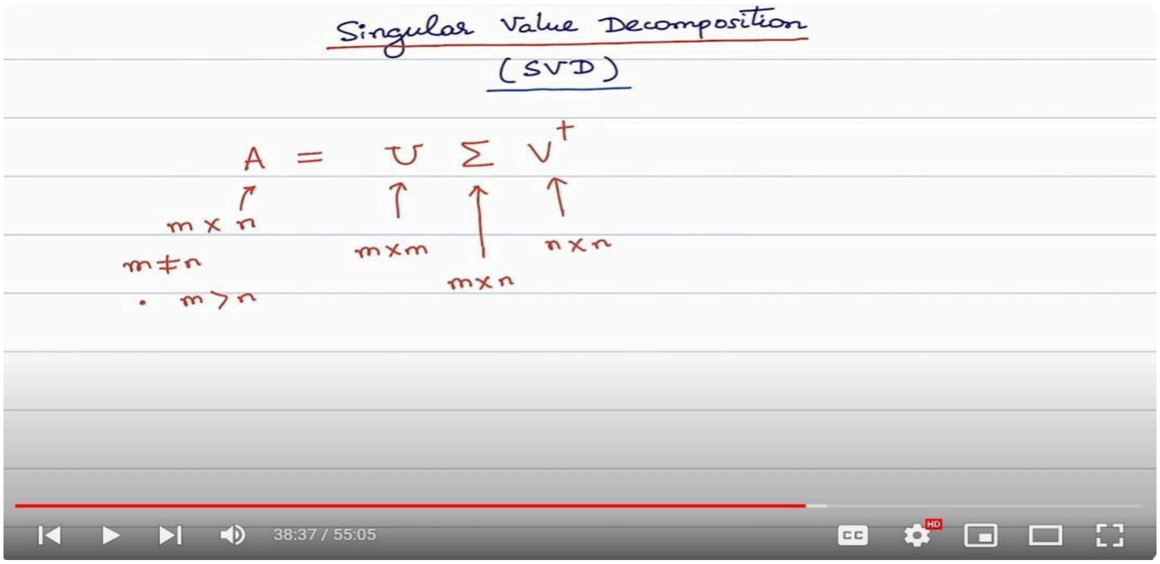

So our goal is to decompose the matrix A as follows. A would be equal to capital U, capital sigma and capital V dagger. I am going to explain all these components one by one. A is a non-square matrix. That means it has say m rows and n columns where m is not equal to n. And also I am going to assume that m is greater than n. That means the number of rows is more than the number of columns.

Capital U is a square matrix having the dimension of m by m. That means m rows m columns. And V dagger which is the Hermitian matrix of corresponding to the matrix V. It is a square matrix having the dimension of n by n. And this capital sigma is a diagonal matrix having the dimension of m by n. m rows n columns. Okay.

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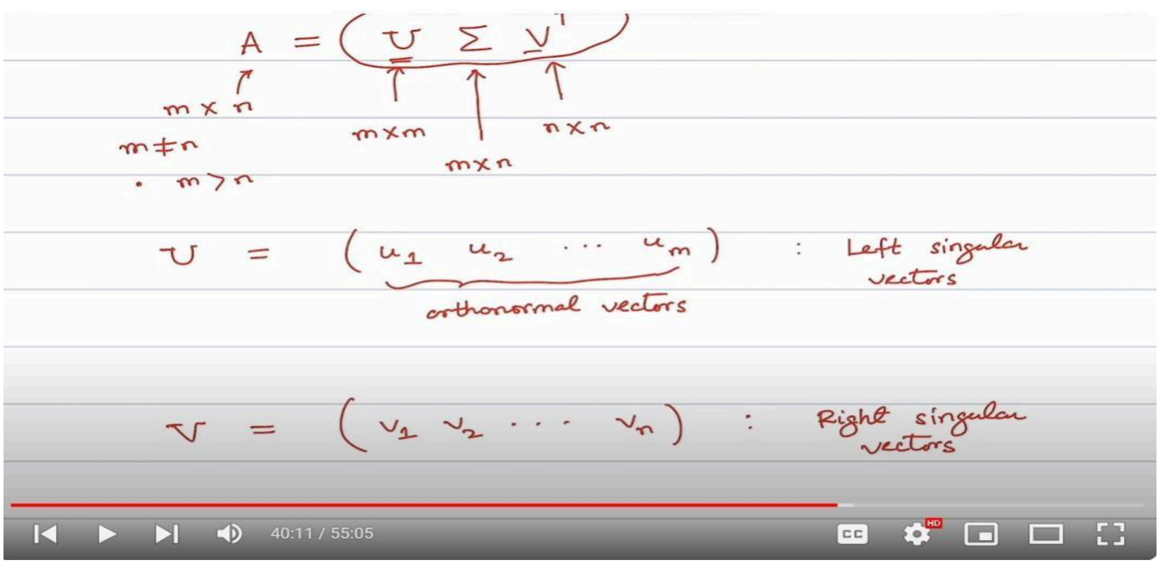

Now these matrices has to be of a particular type. For example, let me talk about all the components one by one now. Say this capital U matrix, this is a square matrix and it has to have this form where its components are say u1, u2, its elements are u1, u2 up to um. Where u1, u2, um these are orthonormal vectors. These are orthonormal vectors.

And I am going to discuss shortly how these orthonormal vectors can be obtained from the matrix A. And these are called left singular vectors. Left singular vectors because of the fact that this capital U matrix, you see this is situated at the left side of the whole expression here. Right? It is situated at the left. And this capital V also of the similar type, it is a square matrix with orthonormal vectors v1, v2 up to vn. And these are called right singular vectors for the same reason because they are situated at the right of this expression here as you can see.

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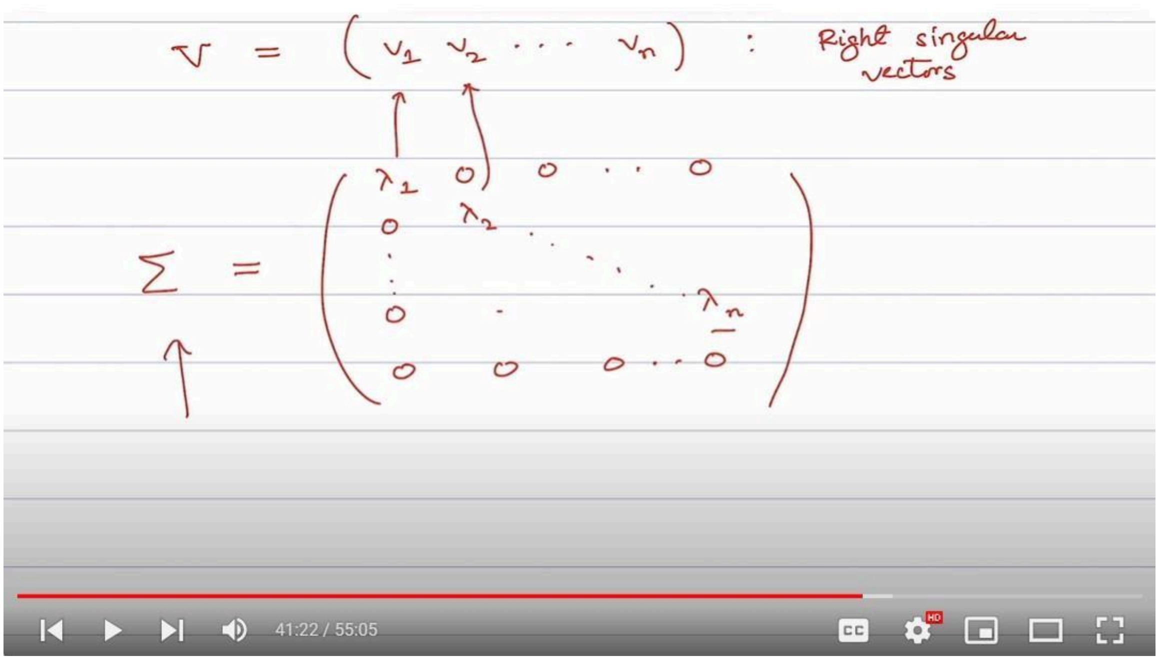

Now what about this matrix in the middle that is capital sigma. Capital sigma is a diagonal matrix and it is a m by n matrix. It is a non-square matrix. And its elements, diagonal elements are arranged this way. Say its diagonal elements are lambda1, lambda2. All of them are arranged in the main diagonal up to say lambda n. And rest of the elements are zero. So all the other elements are zero. Like this. Okay.

Also it has this property that it's a positive matrix. So lambda1, lambda2, lambda n are positive numbers. And lambdas are arranged in such a way that lambda1 has the highest value and lambda n has the lowest value. This matrix capital sigma is called singular value matrix. It is called singular value matrix. And this decomposition, if we can write A as capital U, capital sigma, v dagger with all the matrices having the form that I have just discussed is called singular value decomposition method or in short it is called SVD.

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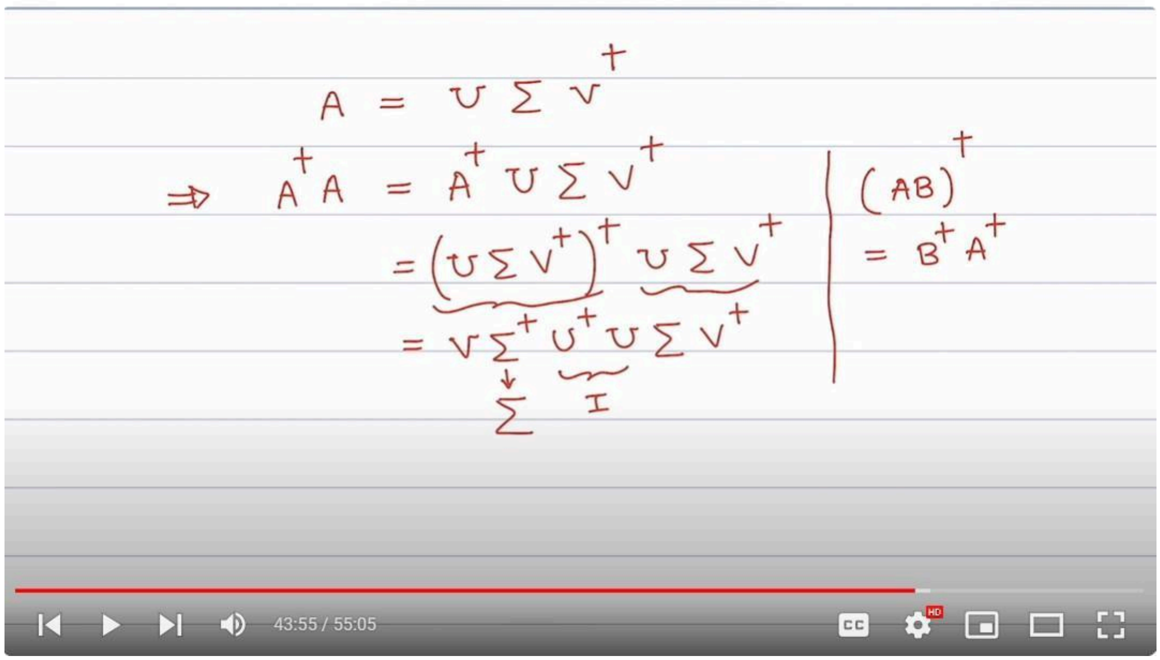

Now let us see how we can obtain these various components. First let me discuss how we can obtain this matrix capital V and this matrix capital sigma given a non-square matrix A. To do that let me multiply this expression.

So let me write it separately here. So let me write once again our SVD. A is equal to capital U. From SVD method, say we can write A is equal to like this. If I multiply both sides of this equation by A dagger, Hermitian conjugate of the matrix A, I have here A dagger capital U sigma v dagger. Now let me put A again here in this expression. A is equal to capital U sigma v dagger. And then here I have this dagger here. U capital sigma v dagger.

Let me now exploit this particular property from linear algebra. You may know that A, the product of two matrices A and B and if I take the Hermitian conjugate, then I am going to get B dagger A dagger. So exploiting this, the first term, this term I can write as V sigma dagger U dagger and this expression let me write as it is capital U capital sigma v dagger. Now U dagger U is an identity matrix and capital sigma is simply sigma dagger is simply sigma because sigma dagger is a diagonal matrix. So sigma dagger is also a diagonal matrix.

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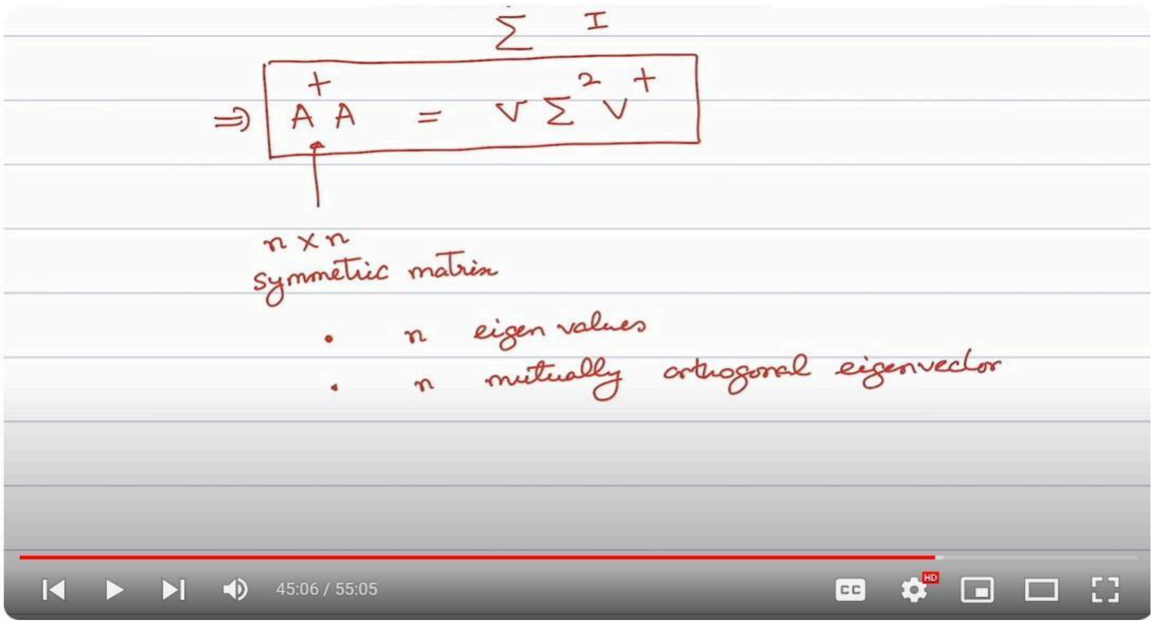

So now utilizing these results, I can write A dagger A is equal to V sigma square v dagger. Okay. This is an important relation I have obtained and we are going to exploit it. But before that Now I have one important point to note here that this matrix A dagger A is a square matrix now. It's a n by n matrix and it's a symmetric matrix. It is a n by n symmetric matrix. That a symmetric matrix has exactly n eigenvalues it has. It has n eigenvalues and it has n mutually orthogonal eigenvectors. These are two important properties of a symmetric matrix.

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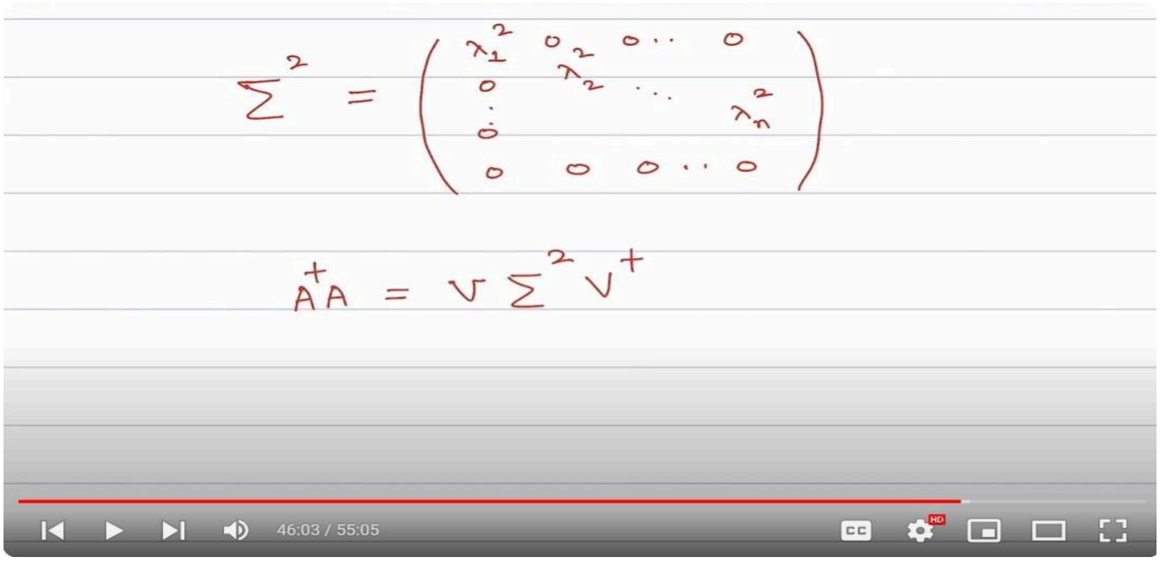

And again this guy sigma square, it is a diagonal matrix. It's a n by n diagonal. It's again a square matrix now. It has a, and its diagonal elements are now lambda one square in the main diagonal. The elements would be lambda one square, lambda two square up to lambda n square and rest of the elements are going to be zero. Okay. Rest of the elements are going to be zero here. And therefore, I can write finally A dagger A is equal to V capital sigma square v dagger.

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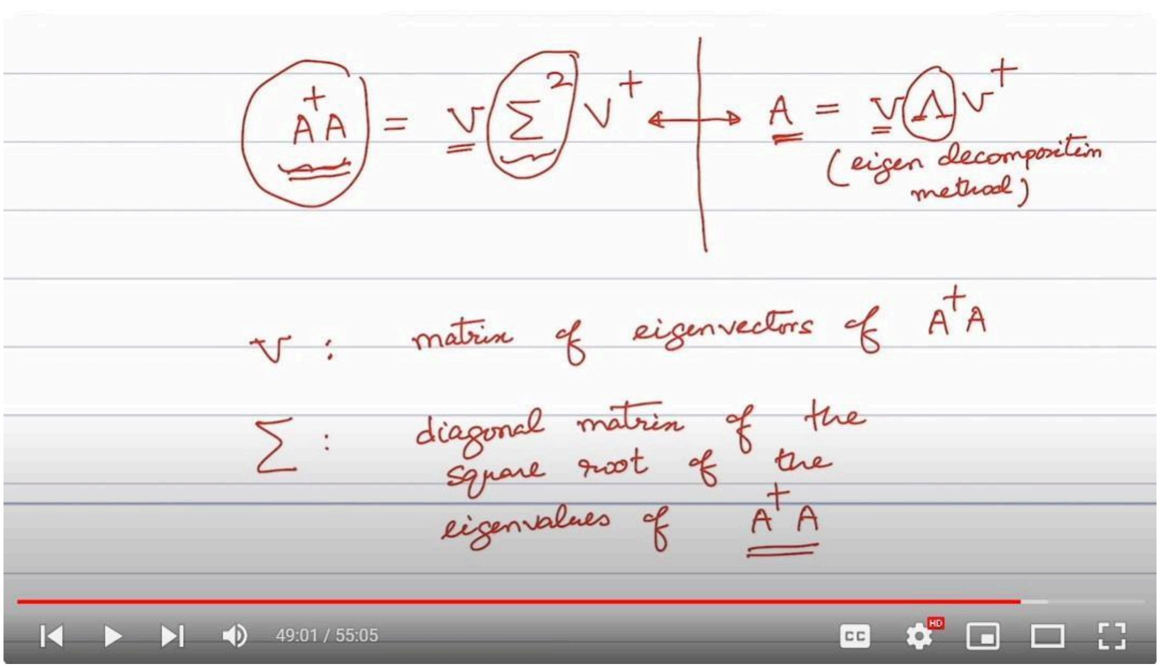

Now A dagger A is a n by n square matrix. Sigma square is a capital sigma square is also a square matrix. Now this expression should remind you about another expression which we have discussed in the context of eigen decomposition method where A was a, in eigen decomposition method we got this expression if you remember. We got V capital lambda capital V dagger, right? And here A is a square matrix, capital lambda is a diagonal matrix, alright? And this we actually got from eigen decomposition method.

We got this from decomposition method. Now certainly we can have a one to one correspondence between these two expressions and if we do the one to one correspondence then you will see from here that this capital V is the, is a matrix. It's a matrix of eigenvectors, eigenvectors, matrix of eigenvectors of A dagger A. You got the idea? Just like here if you recall this capital V, it consists of vectors.

It was basically how it was constructed if you recall. It was constructed by the eigenvectors of the matrix A. So in the similar way this matrix capital V, it is built up from the eigenvectors of this matrix A dagger A overall. This capital sigma is the diagonal matrix of the square root of the square root of the eigenvalues of the eigenvalues of A dagger A.

I hope you get the idea because again using the same correspondence, here you see this capital lambda, its diagonal elements are the eigenvalues of this matrix A. Similarly here capital sigma square, its diagonal elements are the eigenvalues of this matrix A dagger A.
So capital sigma is going to have the elements which are the square root of the eigenvalues of the matrix A dagger A.

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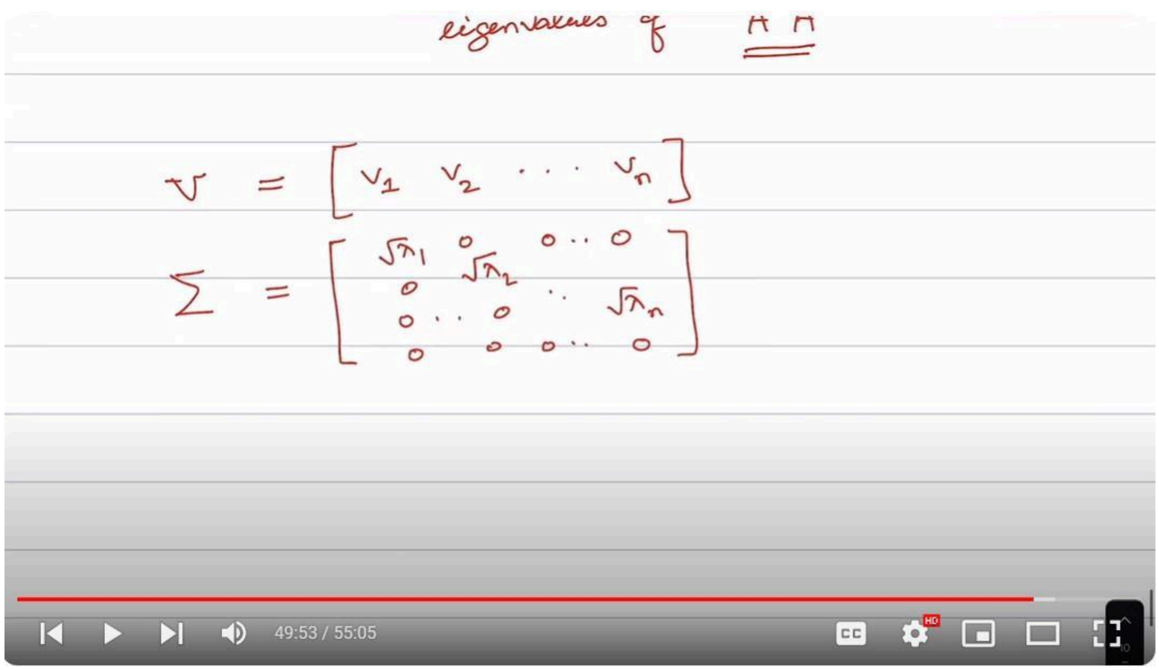

So that means what we are going to have, how we can find V and capital sigma. V is going to be from the eigenvectors of the matrix A dagger A. So you will have say its vector should be, orthonormal vector should be say V1, V2 up to Vn. And this capital sigma is going to have the elements say square root of lambda 1, square root of lambda 2 up to square root of lambda n in the main diagonal.

So we will do some example also to understand the whole thing later on. So this is what you are going to get. Now how to obtain capital U? It can also be obtained in the similar way, only with a slight difference.

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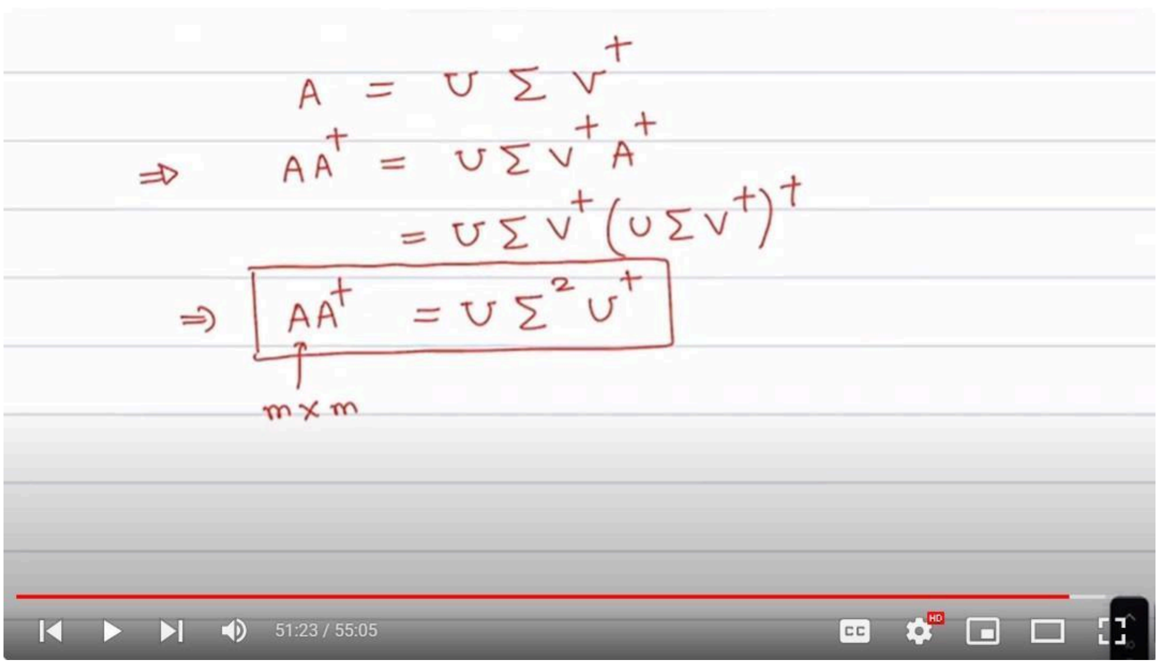

Let me explain that. We will start again with this expression. Capital A is equal to capital U, capital sigma, capital V dagger. Now multiply both sides of this equation by A dagger from right. So I have A dagger on the right if I multiply. So I will have U capital sigma V dagger A dagger. Now let me simplify this. Capital U capital sigma V dagger A I am going to put capital U capital sigma V dagger then dagger. Now from here you can show it very easily that you are going to get this final expression capital U capital sigma square capital U dagger.

So this is what you are going to get. Now there is a difference in the sense that this matrix is again a square matrix but this time it has a dimension of M by M. Unlike the one that we had discussed earlier for the other case here this one you see it has the dimension of N by N. And original matrix A has the dimension of M by N because that was a non-square matrix that was a rectangular matrix.

What we are now getting because of this matrix operation we are getting a square matrix. A A dagger is a square matrix. And A dagger A is also square matrix but having different dimensions. Now if you do the same kind of analysis for this one also you will find that this capital U is going to be a matrix consisting of the eigenvectors of the matrix A A dagger with the elements. Eigenvectors would be U1 U2 up to Um. These are the eigenvectors of the matrix A A dagger. And capital sigma square it is also going to have the same non-zero eigenvalues as that of A dagger A.

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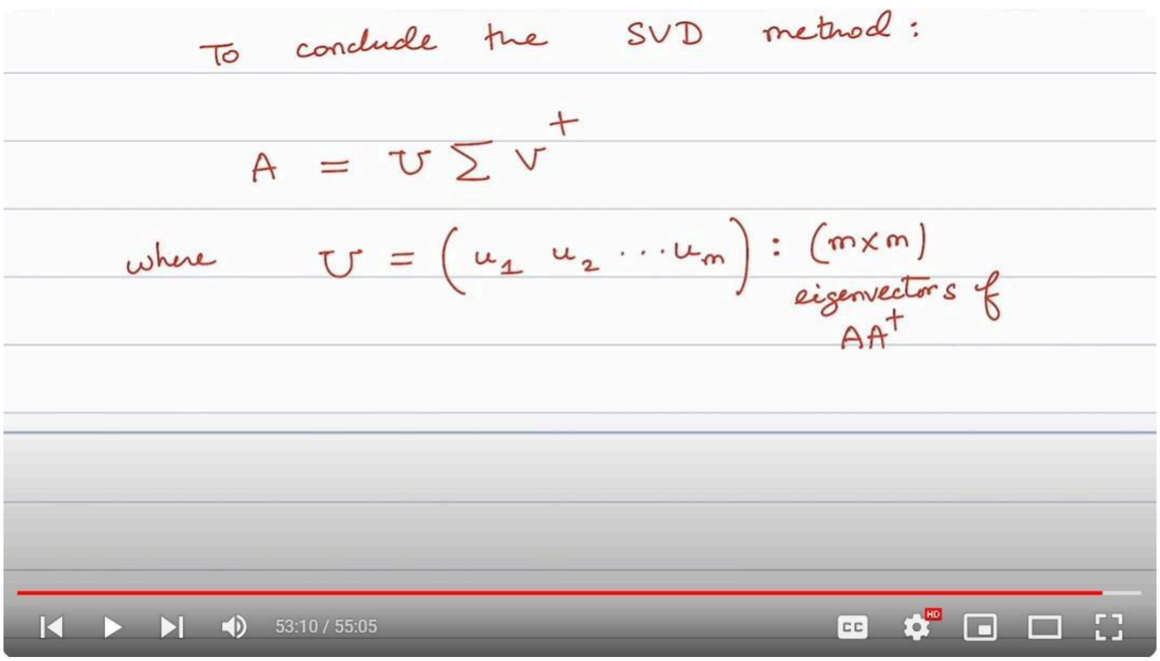

So to conclude the SVD method we can express a non-square matrix A in the form capital U capital sigma V dagger. Where this capital U has elements it consists of the eigenvectors U1 U2 up to Um which is a M by M square matrix. It is a M by M square matrix. And these are the eigenvectors of the matrix A A dagger.

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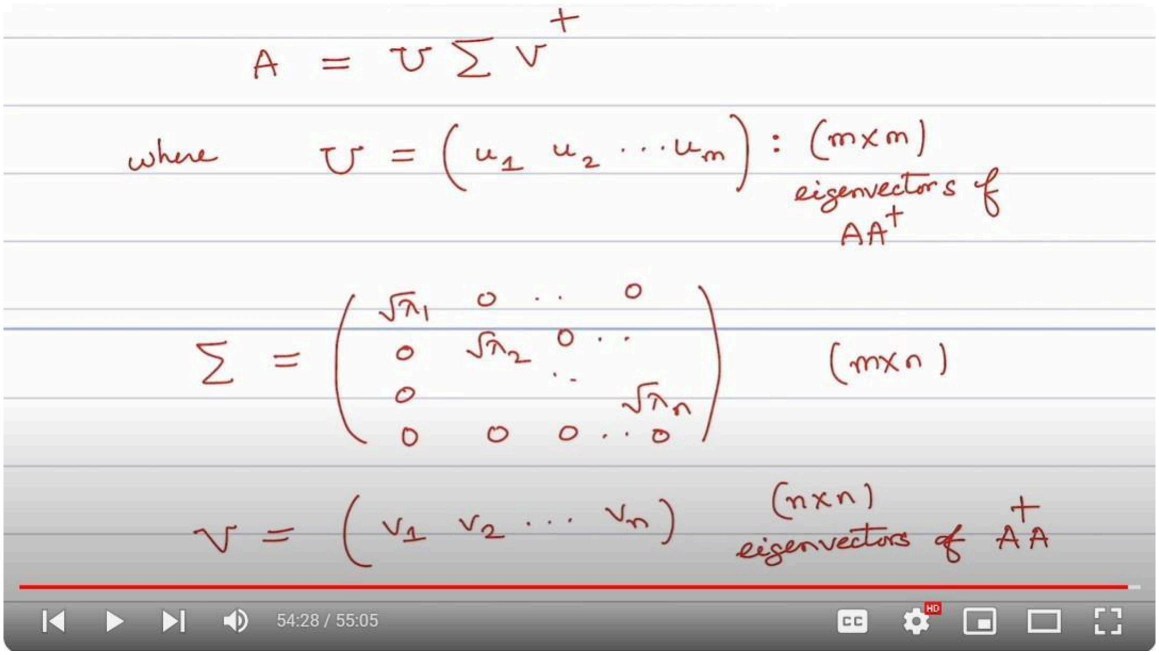

And capital sigma capital sigma has the diagonal elements square root of lambda 1 square root of lambda 2 and up to square root of lambda n. And rest of the elements are zero. All the elements are zero of diagonal elements and this is going to be M by n matrix. And these are eigenvalues of the matrix A dagger A or A A dagger. And V is going to be a matrix formed by the eigenvectors V1 V2 up to Vn. This is a N by N square matrix and these are the eigenvectors of A dagger A.

So now we will do some example in the next lecture to understand this SVD method little bit more clearly. Let me stop for now. We will continue our discussion in the next lecture. Thank you so much.