Quantum Entanglement: Fundamentals, measures and application

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Week-01

Lec 5: Problem solving session-1

Hello, in this first problem solving session we are going to solve some problems based on whatever we have learned in module 1. We will try to have more problem-solving sessions as we go along in this course. In this problem-solving session we are going to solve some problems based on whatever we have learned in module 1. Before I do that let me first give you the detailed solutions of assignment 0.

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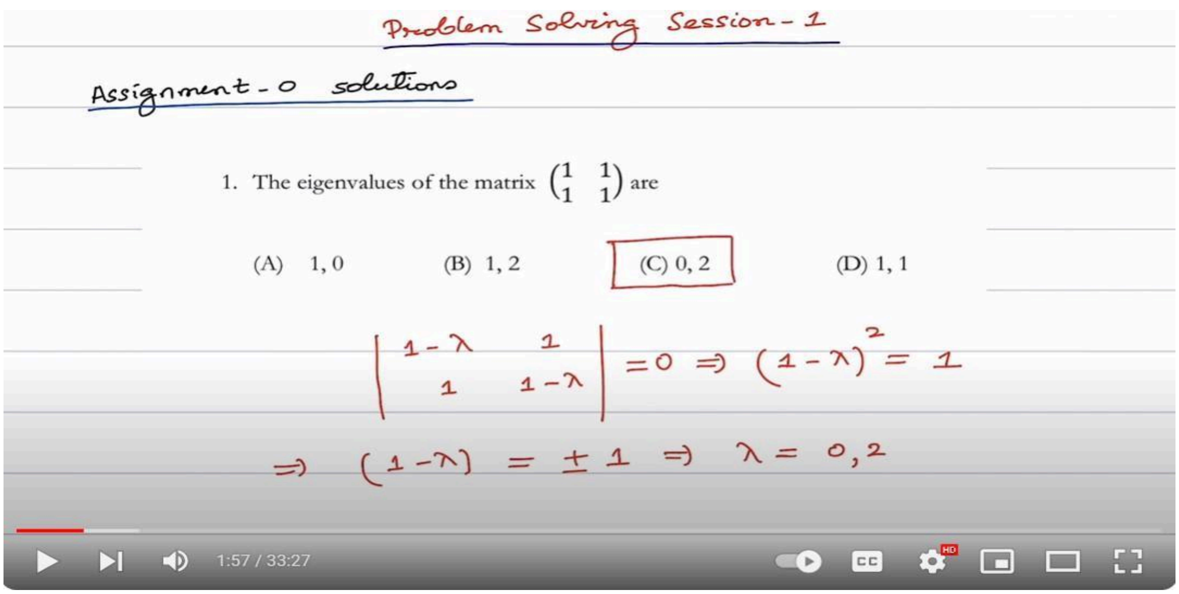

The first problem in the assignment 0 was this. You have to find out the eigenvalues of the given matrix 1 1 1 1. It's very easy you just have to first set up the characteristic equation and just find the roots. The characteristic equation would be 1 minus lambda 1 1 1 minus lambda. Its determinant is going to be 0. So, from here you will get 1 minus lambda whole square is equal to 1. And therefore 1 minus lambda would be equal to plus minus 1. So, this clearly gives you lambda is equal to either 0 or 2. So therefore the correct option of this problem is option C.

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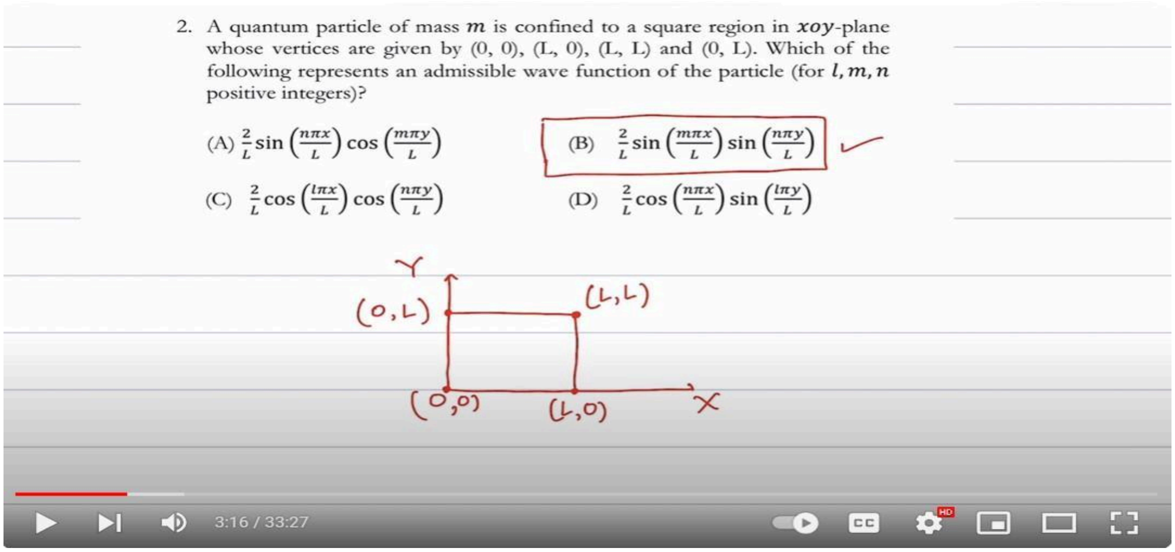

Then in the second problem a quantum particle of mass m is confined to a square region in X or Y plane whose vertices are given by 0 0 L 0 L 0 L which of the following represents an admissible wave function of the particle. This is a boundary value problem. So we are given a square well like this. So it is 0 like this the coordinates are so this is a x axis this is my y axis. This coordinate of this particular point is 0 0. Here this is 0 L sorry L 0 because x axis is along x direction. This is L 0 and this point is L L and here it is 0 L right. Now the wave function must have to varies at the boundaries. If you look at the options carefully you will find that only option where this boundary conditions are going to be satisfied is the option B. So correct option is B.

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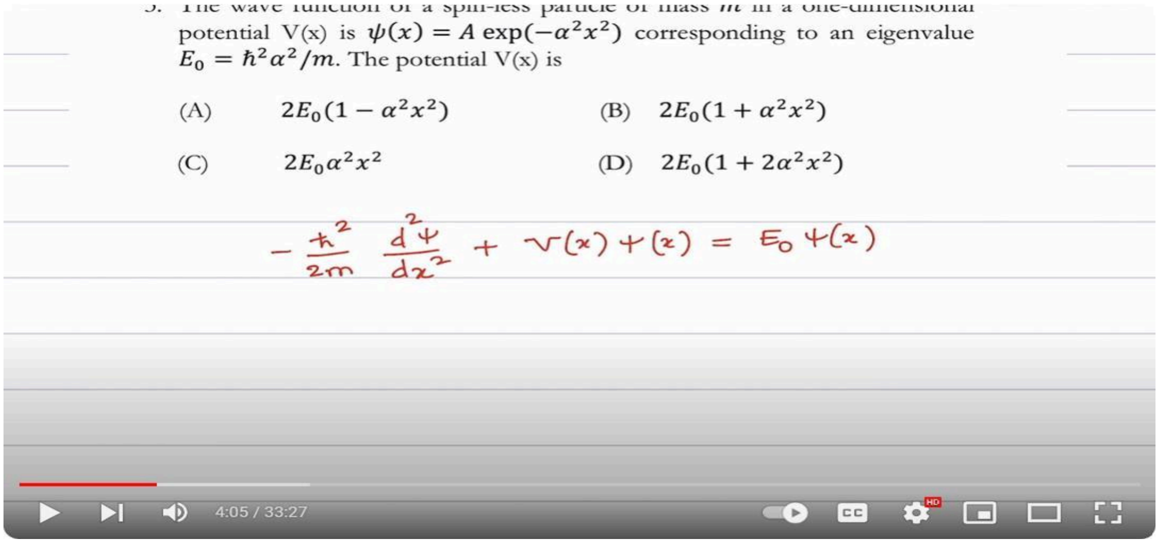

Now in this problem third problem the wave function of a spinless particle of mass in a one dimensional potential V of x is given to be psi of x is equal to A e to the power minus alpha square x square and it corresponds to an eigenvalue e 0 and you have to find out the potential. To solve it you just have to invoke the time independent Schrodinger equation that is minus h cross square by twice m. It is one dimension so therefore let me write the one-dimensional Schrodinger equation. So, this equation you have to utilize. Psi of x is equal to e 0 psi of x.

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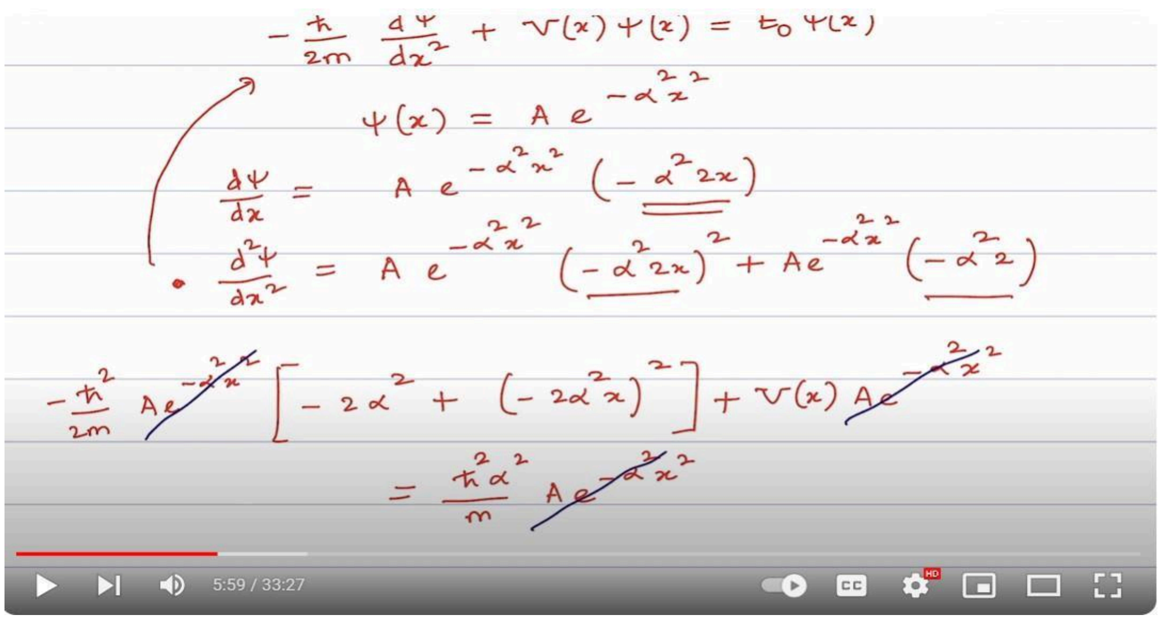

Now psi of x is equal to given to be A e to the power minus alpha square x square. So let me first find out d psi of dx. So, this is going to give me A e to the power minus alpha square x square. You will have minus alpha square twice x. If I take again another derivative second derivative if I take I will get A e to the power minus alpha square x square from it again I will have this one if I take the differentiation so alpha square twice x whole square plus A e to the power minus alpha square x square if I take the derivative I will have alpha square 2 right this is what I will have.

Now I will put this in my Schrodinger equation if I do that I have minus h cross square by twice m A e to the power minus alpha square x square and you will have terms like this minus 2 alpha square this one and then I will put this this would be plus minus twice alpha square x whole square okay. This is the first term, and we have this potential term V of x A e to the power minus alpha square x square and this is equal to e 0 e 0 is given to be h cross square alpha square by m and psi of x is A e to the power minus alpha square x square. Now it's easy to see that this gets cancelled out this goes out this goes out this goes out,

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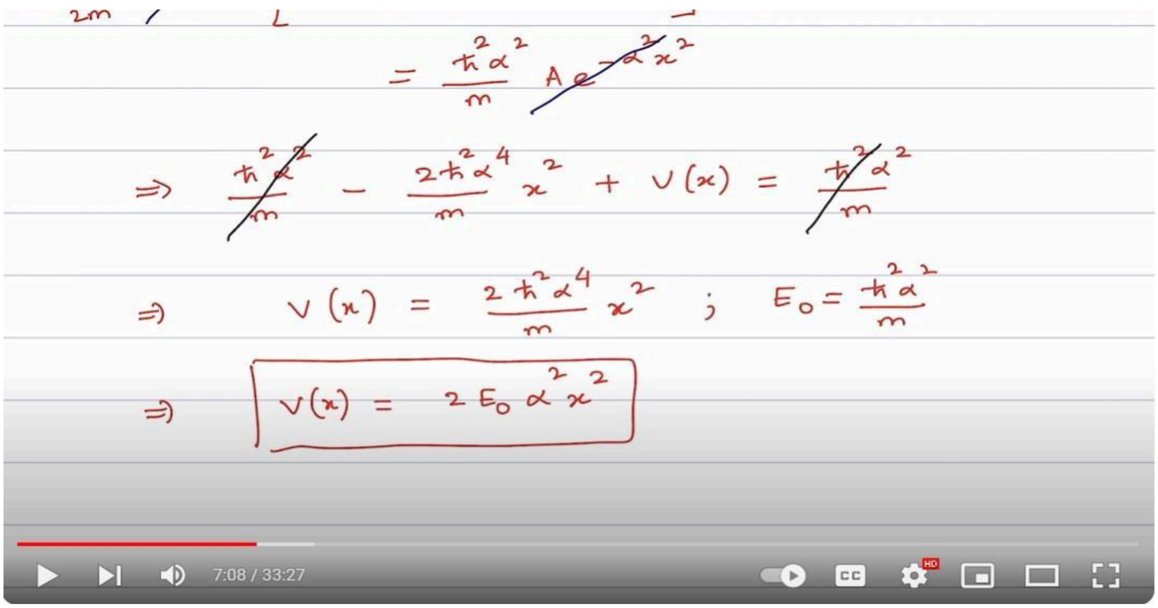

and from there I can have I will have h cross square first term would be h cross square alpha square by m then minus twice h cross square alpha to the power 4 divided by m x square plus V of x is equal to h cross square alpha square by m. Now again you see this term and this term goes out so therefore I get my V of x is equal to twice h cross square alpha to the power 4 by m x square now e 0 is given to be h cross square alpha square by m so utilizing it I can write V of x is equal to twice e 0 alpha square x square so this is the potential I have worked out and if you look at the option so option C is the correct one.

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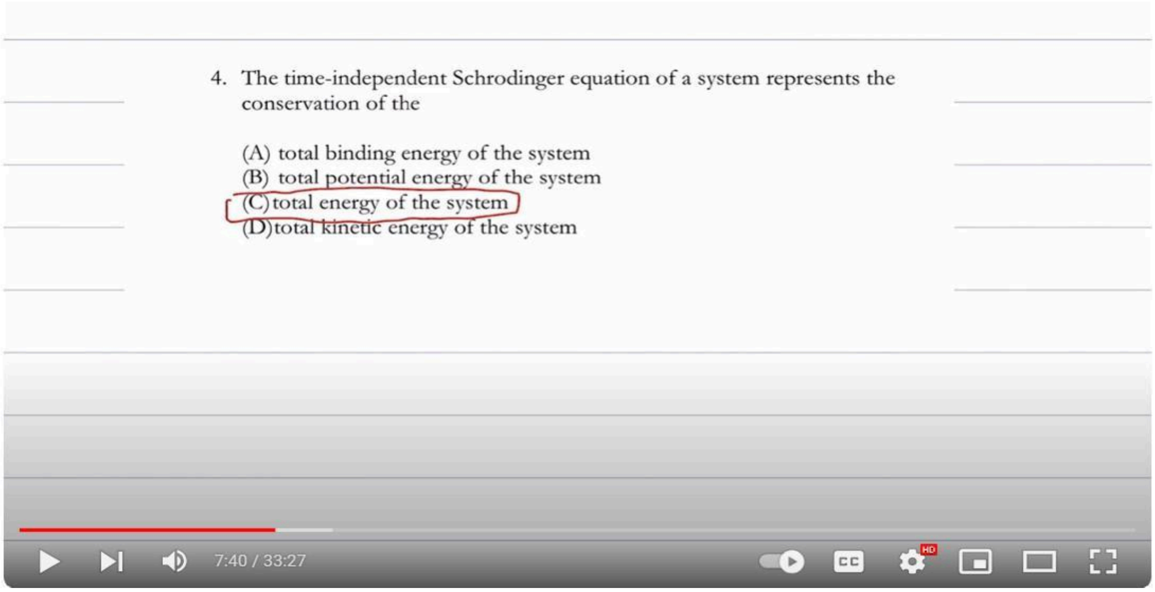

Okay now the next problem is this so time dependent independent Schrodinger equation let me take it here time independent Schrodinger equation of a system represents the conservation of the out of this option I think everybody knows it that the total energy of the system is conserved.

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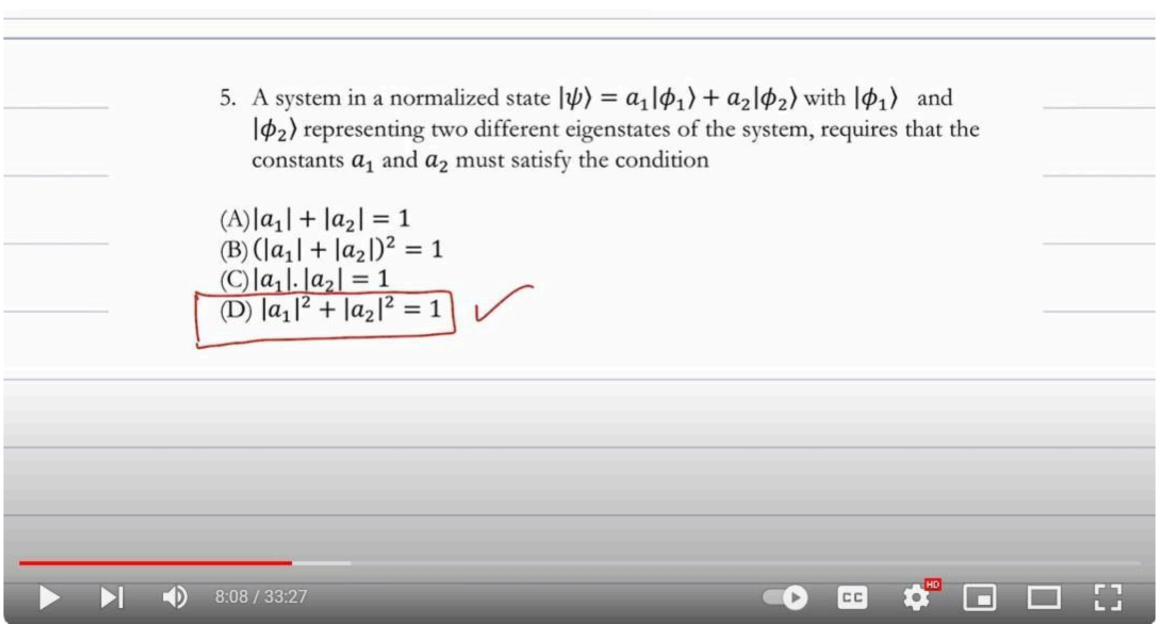

Problem number five a system in a normalized state is given to be this where phi 1 and phi 2 are the basis states or the eigenstates so what the what property is basically the constant a 1 and a 2 satisfy and it's a easy one the option correct option is option D everybody should be able to get it it's one of the postulate.

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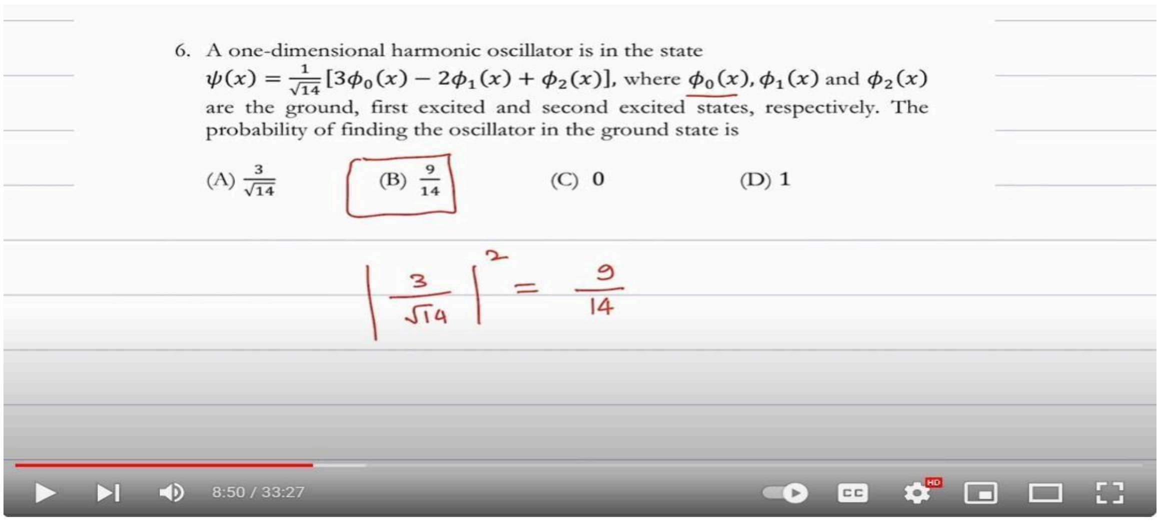

Problem number six a one-dimensional harmonic oscillator is in this state where phi 0 phi 1 phi 2 are ground first excited second and second excited state respectively the probability of finding the oscillator in the ground state is ground state wave function corresponds to phi 0 so therefore associated coefficients here is 3 by root 14 so it's mod square is the correct option so it would it should be 9 by 14 so correct option is obviously option B is the right answer in this particular problem right.

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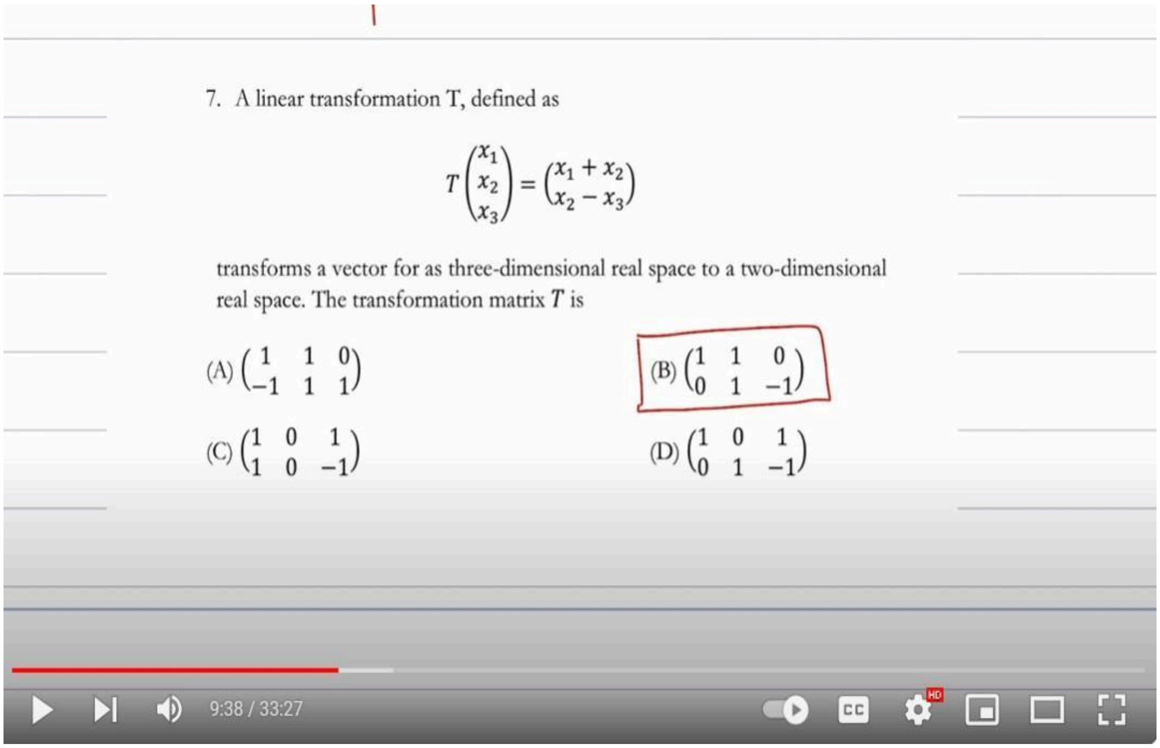

then finally okay we have this also the linear transformation is defined let me take it up here okay linear transformation T is defined as by this relation you have to find out the transformation matrix it's also easy one you can why I ask this because it is based on linear algebra and linear algebra is the mathematical basis of quantum mechanics its mathematical formalism is intricately related to linear algebra various transformations so if you work it out carefully it's very easy to straightforward you will find that option B is the correct one okay.

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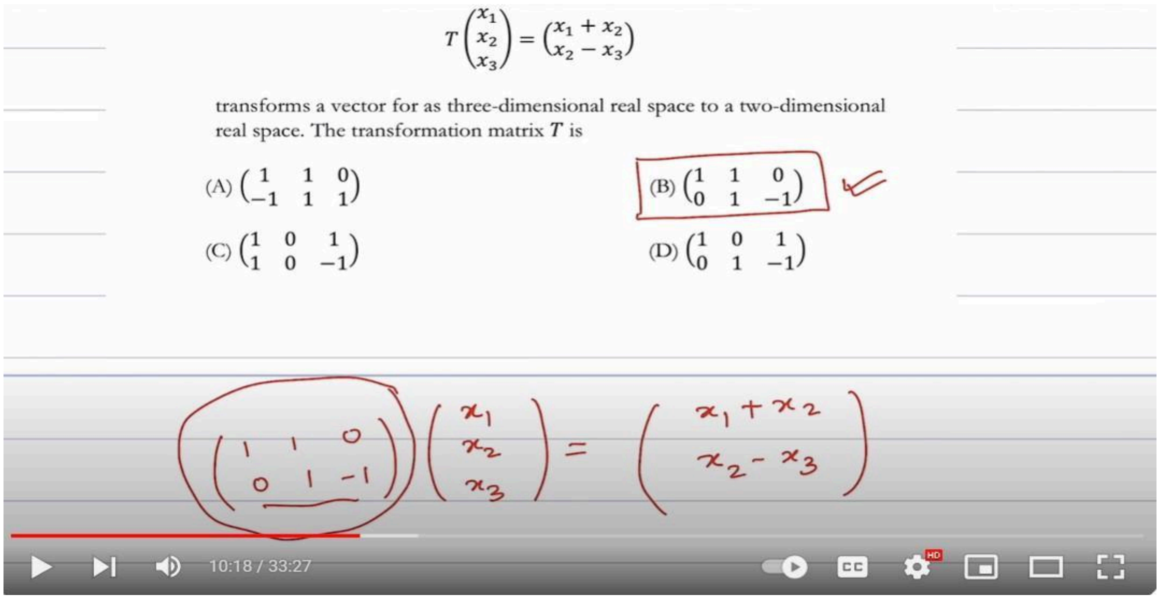

let me quickly show you if you write B is equal to 1 1 0 0 1 minus 1 if you pick it up and you multiply with this one matrix multiplication if you do it you will see that you will get from the first expression you will get for first row and the first column of this matrix it is going to give you X 1 plus X 2 and this one is going to give the X 2 minus X 3 so correct transformation matrix is obviously this one so option B is the correct one here.

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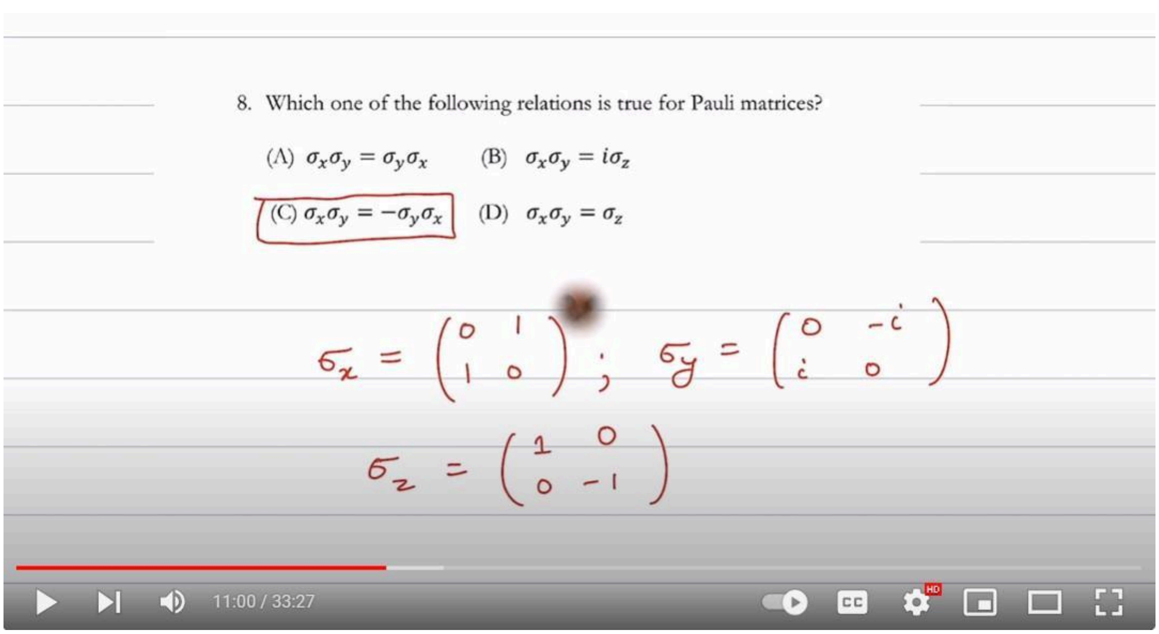

last one is which one of the problem number 8 which one of the following relation is true for Pauli matrices now the Pauli matrices basically you know that they anti commute they don't commute so correct option is C just to remind you you can do it verify it also Sigma X refers to the Pauli matrix 0 1 1 0 Sigma Y is 0 minus I I 0 and Sigma Z is equal to 1 0 0 minus 1.

Now let us do some problems based on module 1 particularly we will solve problems related to density matrix formalism.

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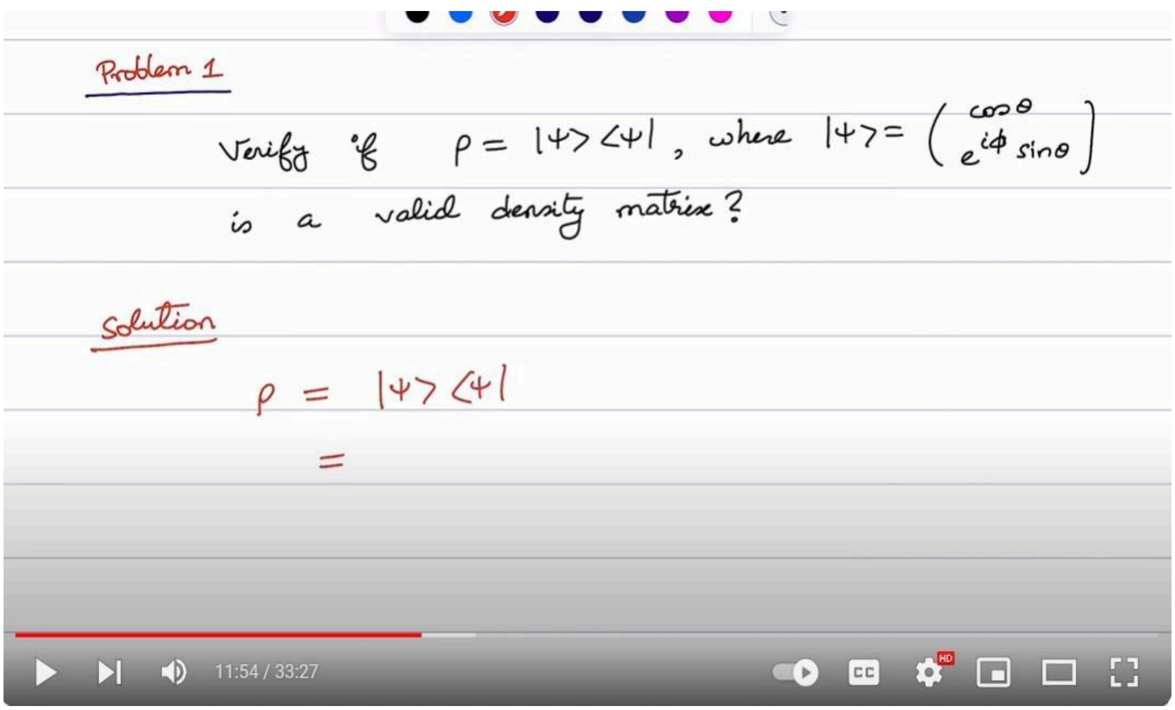

Let us work out this problem you are asked to verify if rho this is the form is of density operator whether for the given state ket psi written as a column vector cos theta e to the power I phi sine theta is a valid density matrix or not so we can work it out if it has to be a valid density matrix so density operator must have to be a hermitian first of all let me write down the density matrix so we have this ket psi bra psi,

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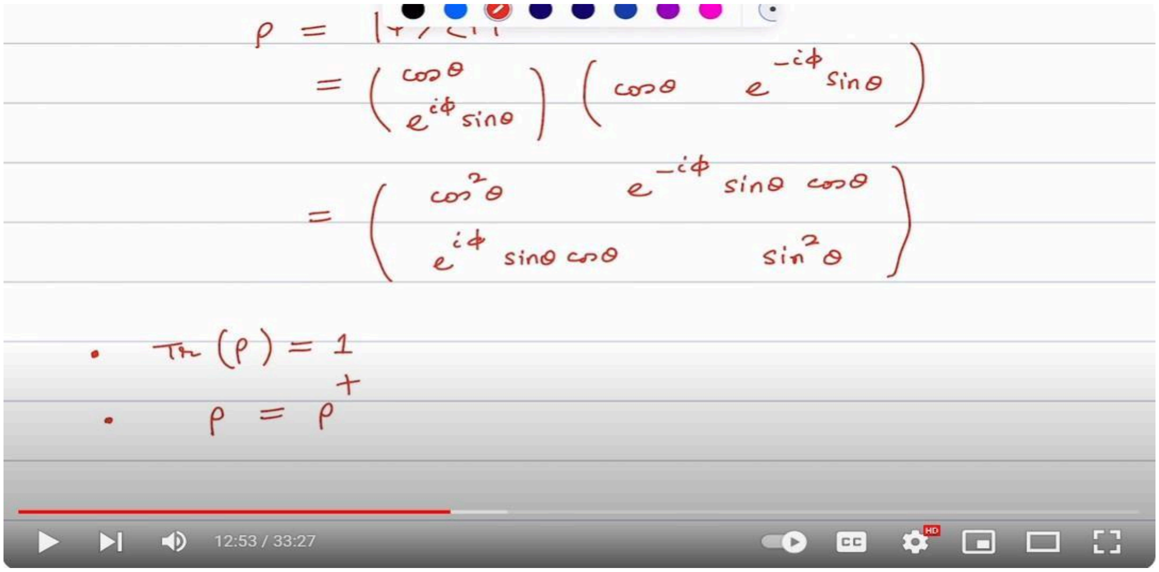

ket psi is cos theta e to the power I phi sine theta the bra psi would be cos theta it would be rho vector e to the power minus I phi sine theta so from here I get the density operator to be cos square theta density matrix because square theta e to the power minus I phi sine theta cos theta e to the power I phi sine theta cos theta and here I would have simply sine square theta as you can see immediately the trace of rho is equal to 1 that is one of the required property and another one you can easily see that rho is a hermitian it's a hermitian matrix so that is also another requirement also you can quickly check that rho square if you work it out you will find out that rho square is simply rho this implies that the given density matrix or the given state is a pure state okay.

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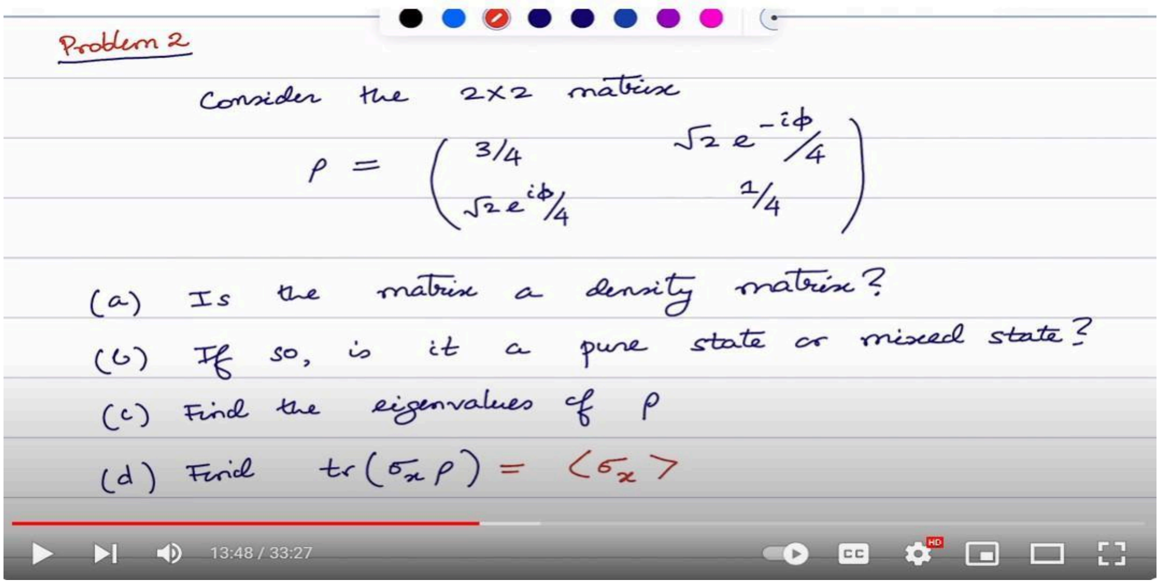

Now let us work out this problem you are given a 2 by 2 matrix again the question is whether this is a density matrix or not if so is it a pure state or mixed state then you are asked to find the eigenvalues of rho and finally work out the trace of the product of sigma X rho in fact you may recognize that this is basically nothing but you are in a way you are asked to find out the expectation value of sigma X

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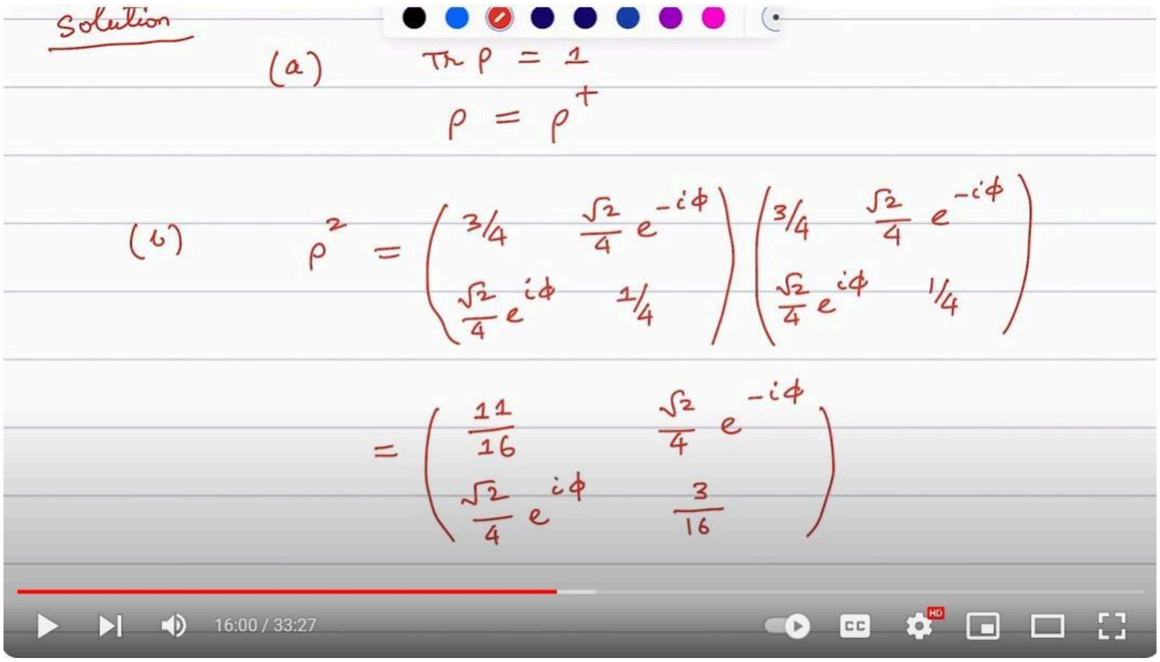

Okay let us do it so first of all immediately one thing you can observe from the matrix itself from here you see that the trace of rho trace of rho is equal to 1 okay and what next it has to be hermitian whether it is hermitian or not you can make it out you have to take the transpose and then if you take the complex conjugate and from there it is very easy to see that indeed this is a hermitian matrix so rho is equal to rho dagger so these two properties are satisfied also you will find that rho is a semi positive definite that means the eigenvalues are non-negative.

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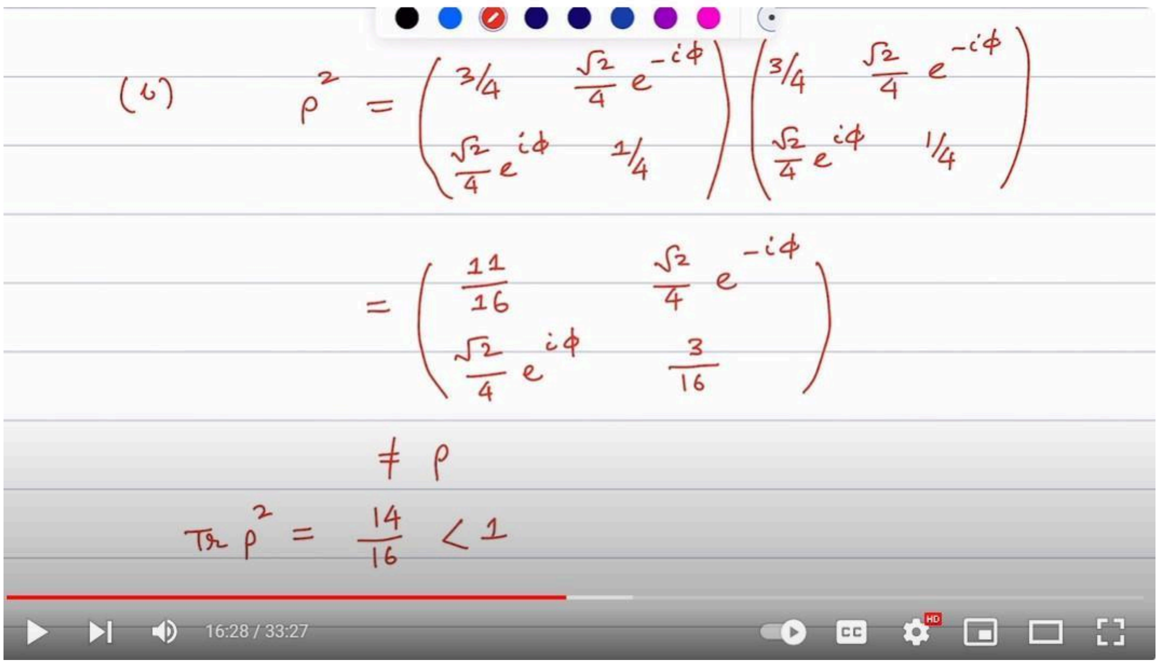

To do the second part whether it is a pure state or mixed state let me quickly work out what is rho square if you can work out rho square is basically do the matrix multiplication so you will have say you have here rho is 3 by 4 root 2 by 4 e to the power minus i Phi here you are having root 2 by 4 e to the power plus i Phi and you are 1 by 4 and you have to multiply with the same matrix again root 2 by 4 e to the power minus i Phi root 2 by 4 e to the power plus i Phi 1 by 4 if you do the matrix multiplication you will find that you should get if you do it you will get 11 by 6 16 and you will get here root 2 by 4 e to the power minus i Phi I urge you to work it out yourself it's very simple straightforward simple mathematics matrix multiplication and here you will get 3 by 16 okay now you see that obviously rho square is not equal to rho so immediately you see that this is a mixed state in fact you can also find out what is trace rho square trace rho square would be simply 14 divided by 16 which is less than 1 and this confirms the fact that the given matrix is a density matrix is from it because trace rho square is less than 1 it is a mixed state trace rho square is equal to 1 is for pure state if you remember.

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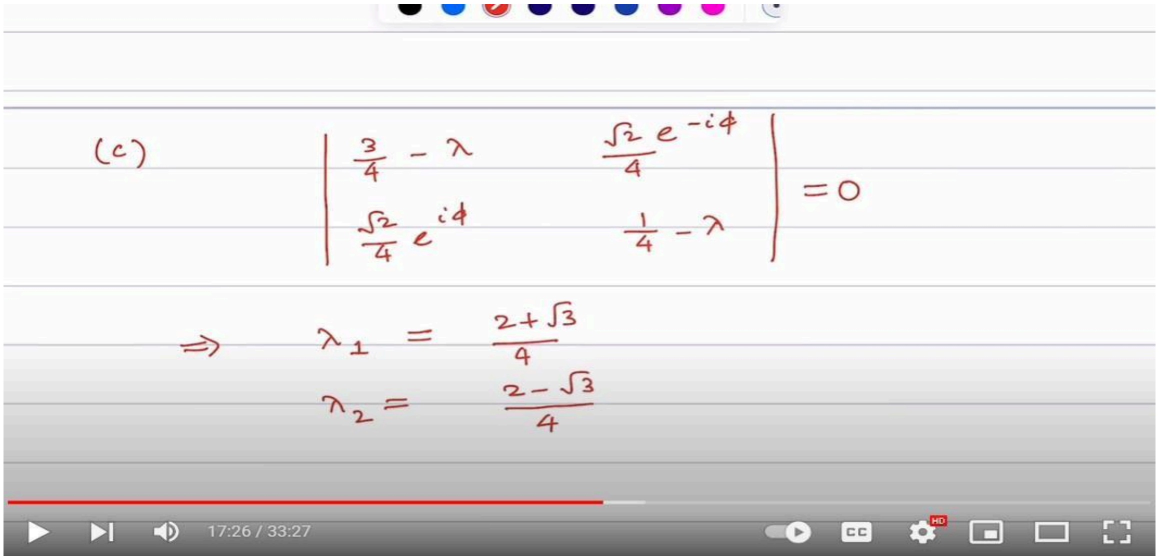

Finally you are asked to okay third part of the problem says that you have to find out the eigenvalues of rho to do that you just have to set up the characteristic equation 3 minus 4 by lambda this determinant you have to solve root 2 by 4 e to the power minus i Phi here you have root 2 by 4 e to the power plus i Phi 1 by 4 minus lambda this equation you have to solve you will get a characteristic equation you can please verify that you should get 2 roots 1 would be lambda 1 is equal to 2 plus root 3 by 4 and lambda 2 the other eigenvalue you will get it as 2 minus root 3 by 4 all right .

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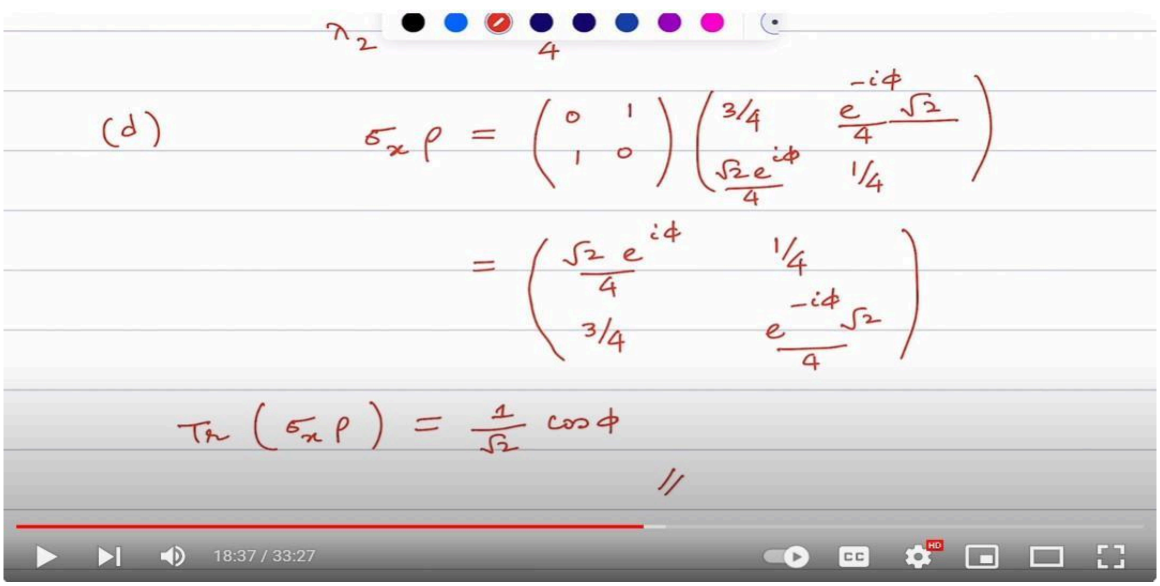

Finally final part you are asked to find out the trace of the product of sigma x and rho and sigma x you know is 0 1 1 0 and rho is given to be 3 by 4 e to the power minus i phi by 4 okay so 2 root 2 is also there and then you have root 2 e to the power plus i Phi by 4 1 by 4 so if you multiply this is going to just exchange of rhos is going to takes place because of that you will get root 2 e to the power i Phi by 4 1 by 4 3 by 4 e to the power minus i Phi root 2 by 4 and what about the trace trace would be simply the sum of the diagonal elements so you will see you will simply get 1 by root 2 cos Phi so that is going to be the answer for this problem.

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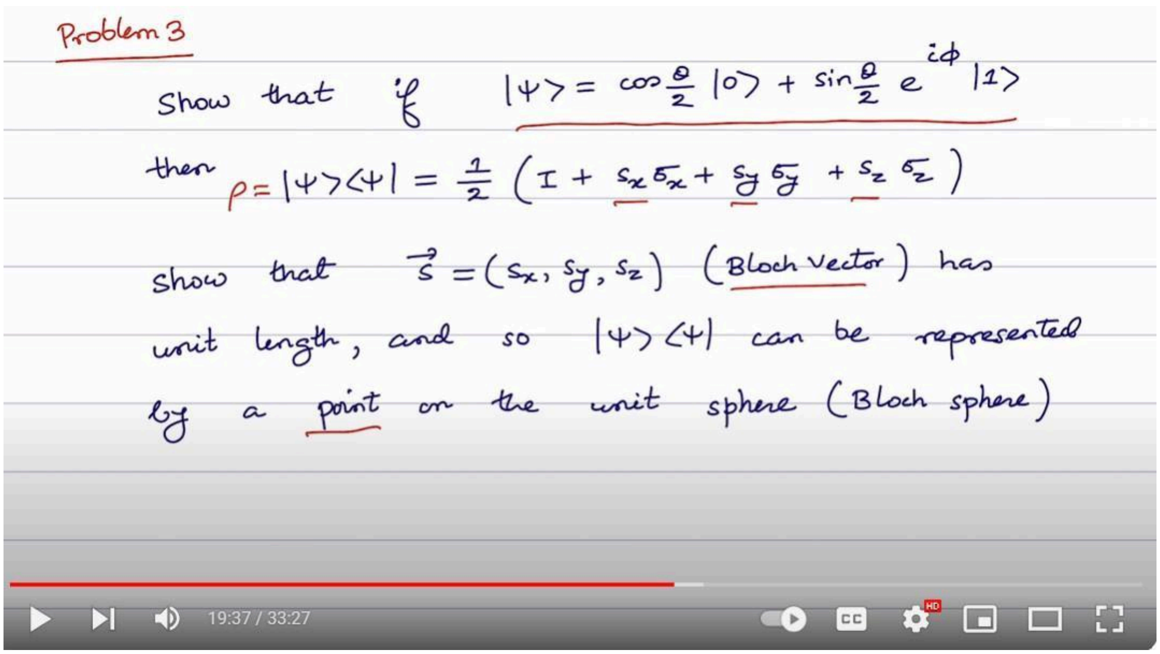

Let us now work out this problem in this problem you are asked to show that given this ket psi is equal to cos theta by 2 ke t0 plus sine theta by 2 e to the power i Phi ket 1 you are asked to find the so that the corresponding density operator or density matrix can be written in this form in fact this particular state if if you remember this already we have encountered in our brief discussion related to bloch sphere in an earlier class now here this this s x s y s z are the components of a vector s which is known as the bloch vector you are asked to show that it is it has unit length and thereby this density operator can be represented by a point on the unit sphere.

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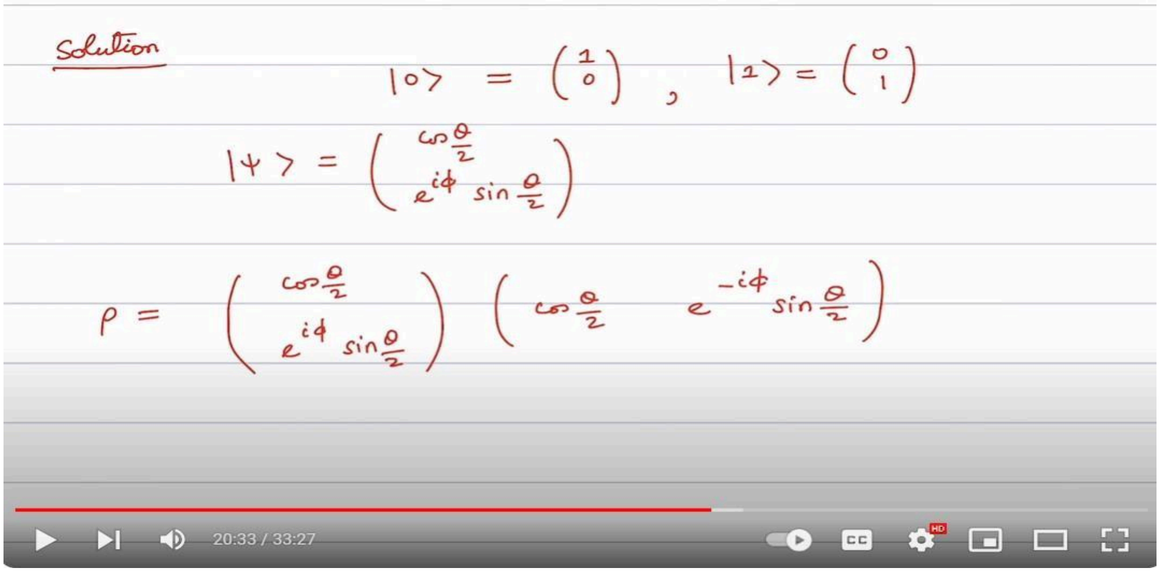

So let us do it so to do that let me do the calculations in details first of all i know that this is written in the basis state ket0 and ket1 and ket0 is represented by the column vector 1 0 and ket1 is represented by the column vector 0 1 so therefore this state psi i can write it as cos theta by 2 e to the power i Phi sine theta by 2 so therefore the corresponding density operator will be simply a multiplication of this column vector cos theta by 2 e to the power i Phi sine theta by 2 with the rho vector cos theta by 2 e to the power minus i Phi sine theta by 2,

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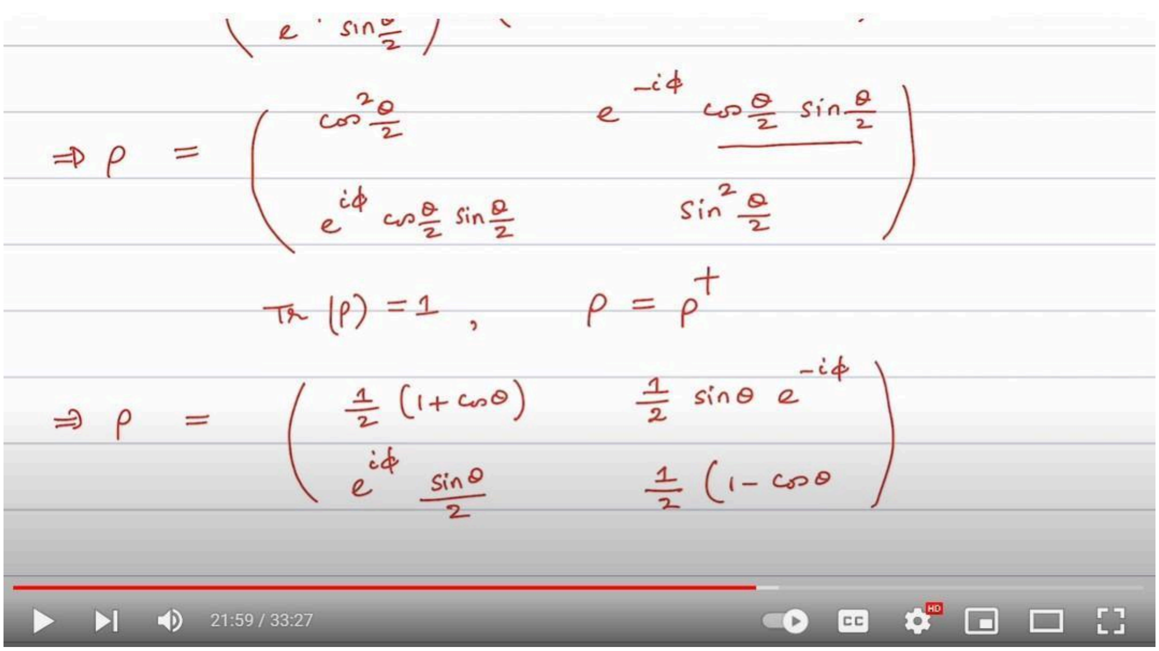

Okay and from here i get my density matrix or the operator rho density operator as cos square theta by 2 e to the power minus i Phi cos theta by 2 sine theta by 2 and here i will have e to the power i Phi cos theta by 2 sine theta by 2 and sine square theta by 2 immediately you can see that the trace of rho is equal to 1 here and also this rho is a hermitian matrix so therefore it is definitely a valid density matrix and it represents a two-state system okay this matrix can be further written in a different form using trigonometric relations cos square theta by 2 i can write it as half 1 plus cos theta okay and cos theta by 2 sine theta by 2 i can write it as sine theta if i just divide it by half so it would be sine theta and e to the power minus i Phi similarly here e to the power i Phi i will have it as sine theta divided by 2 and this is i can write as half 1 minus cos theta okay,

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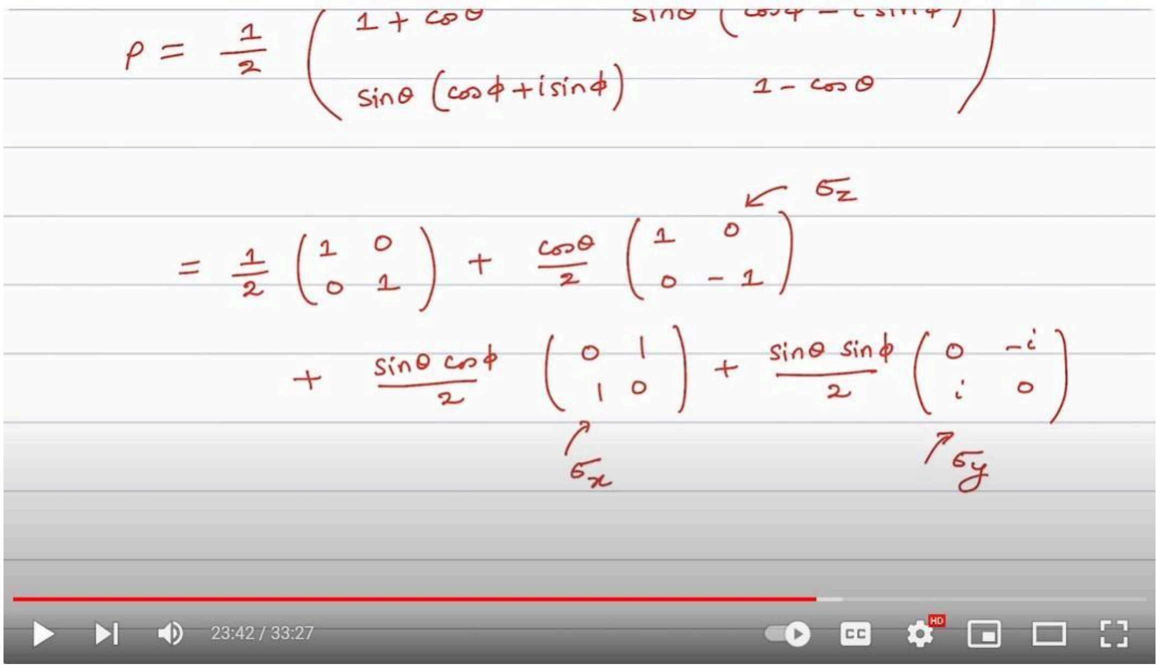

in fact let me expand e to the power i Phi as well then if i do that i can write my density operator rho as half 1 plus cos theta sine theta e to the power minus i Phi i can write as cos Phi minus i sine Phi here i will have sine theta cos Phi plus i sine Phi and 1 minus cos theta this can be further simplified i can use the so-called Pauli matrices to write it in in that form so okay let me show you you can write it as let me basically break it up in terms of the Pauli matrices first one 1 0 0 1 this is identity or unit matrix then i have cos theta by 2 i can write it as 1 0 0 minus 1 and then i have sine theta cos Phi divided by 2 that would be 0 1 1 0 and i have sine theta theta sine Phi divided by 2 0 minus i i 0 you can immediately see that this guy is nothing but the z component of the Pauli matrices sigma z this one is sigma x and this one is sigma y,

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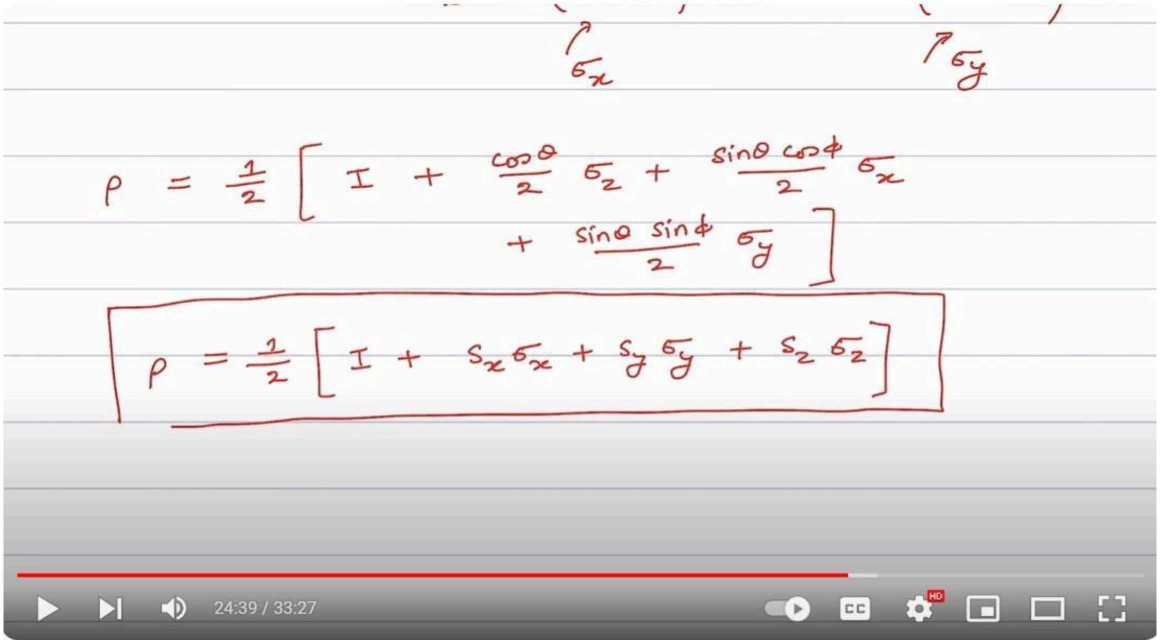

so therefore i can write my density operator rho as half the identity matrix 1 0 0 1 and it is cos theta by 2 sigma z plus sine theta cos Phi by 2 sigma x sine theta sine Phi by 2 sigma y in fact i now recognize my components sx sy and sz i can write it as i plus sx sigma x sy sigma y and sz sigma z so this is the form this is the form that was asked in the question,

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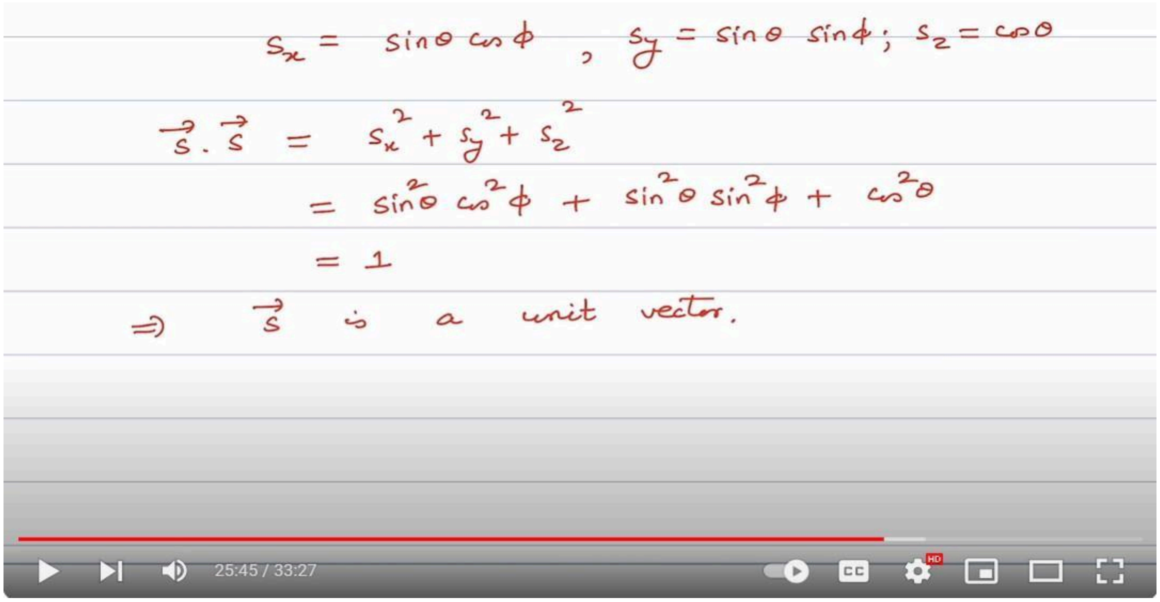

to show where sx is equal to sine theta cos Phi sy is equal to sine theta sine Phi and sz is equal to simply cos theta what about the dot product of this vector s x square sy square plus h z square and you can easily see sx square is sine square theta cos square Phi sy is sy square is sine square theta sine square Phi and sz square is cos square theta immediately you see that this guy is equal to s dot s dot is equal to s s dot is equal to 1 so it is immediately apparent that s is a unit vector s is a unit vector and it connects the origin with a point on the unit sphere.

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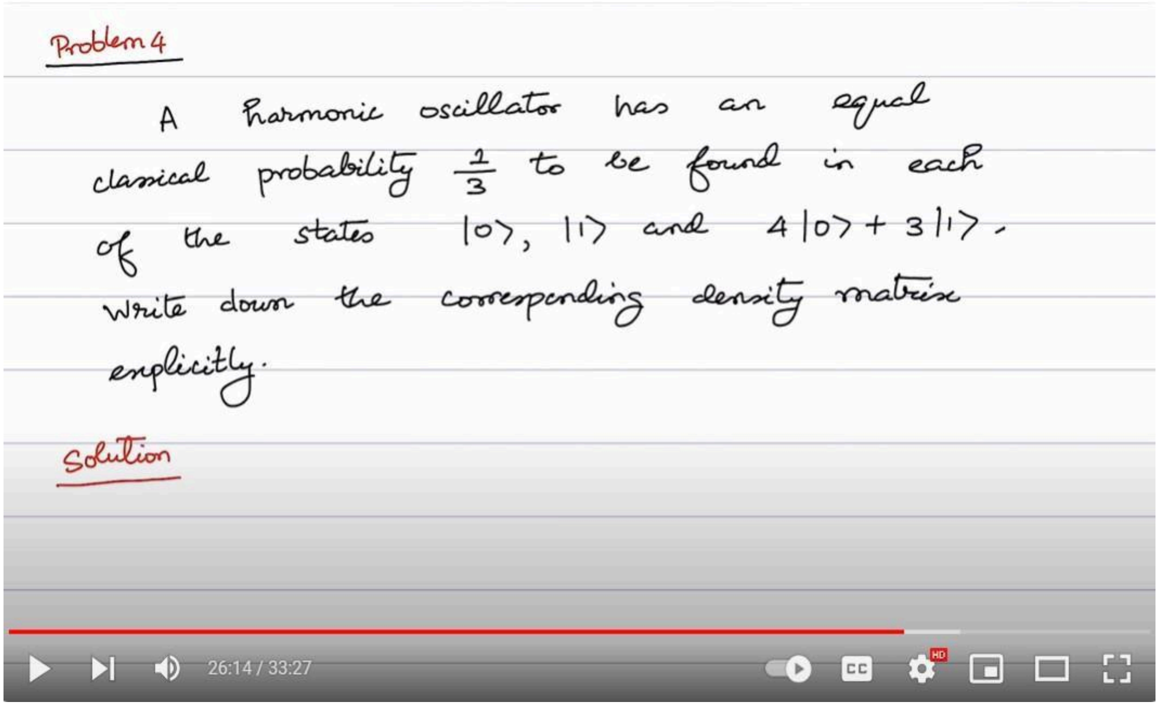

Now let us work out this problem a harmonic oscillator has an equal classical probability of one third to be found in each of the state ket 0 ket 1 and superposition of ket 0 and ket 1 that is 4 ket 0 plus 3 ket 1 write down the corresponding density matrix explicitly so let us work it out,

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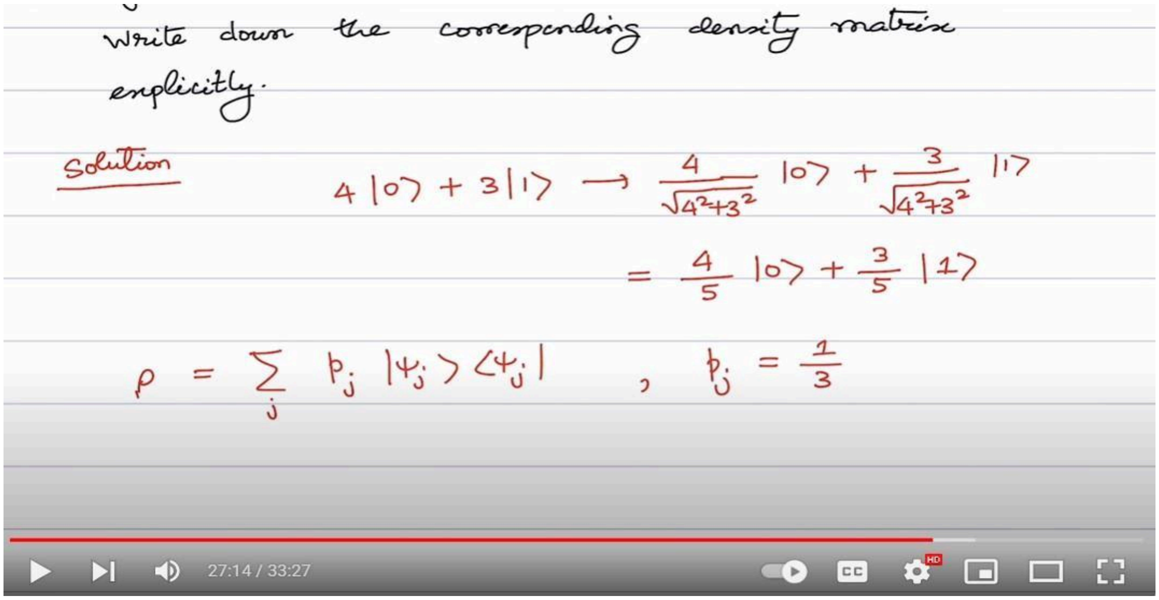

First of all we need all the states to be normalized in particular we have to normalize the third state that is 4 ket 0 plus 3 ket 1 it is not normalized if we normalize it it will be 4 divided by square root of 4 square plus 3 square ket 0 plus 3 divided by square root of 4 square plus 3 square ket 1 which means that we will have the normalized state as 4 by root 5 ket 0 plus 3 by sorry 4 by 5 ket 0 plus 3 by 5 ket 1 okay now we can build up the density matrix as follows because rho is equal to the original definition of density operator that is invoked that is pj psi j psi j right now here pj is equal to one third for all the states all the three states,

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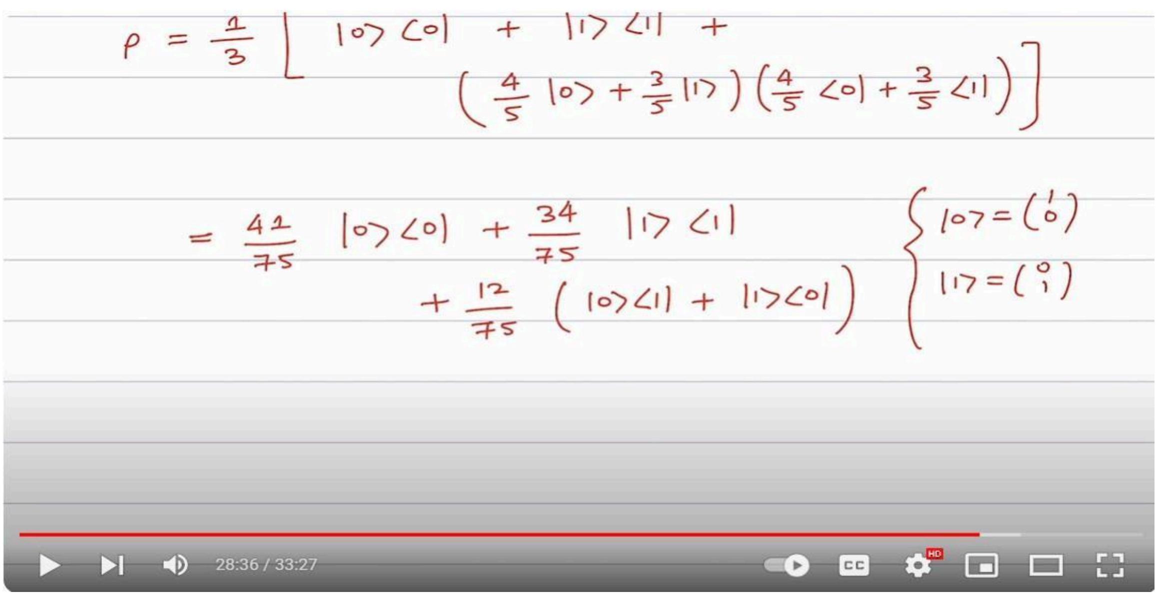

so if i open it up rho is equal to one third and then i have ket 0 bra 0 plus ket 1 bra 1 and plus the third state it would be 4 by 5 ket 0 plus 3 by 5 ket 1 okay ket 1 and their corresponding bra part that would be 4 by 5 bra 0 plus 3 by 5 bra 1 and if i do the mathematics you will find that you will get 41 divided by 75 ket 0 bra 0 plus 34 divided by 75 ket 1 bra 1 and plus 12 divided by 75 ket 0 bra 1 plus ket 1 bra 0 now you can utilize the fact that ket 0 is 1 0 the matrix representation and ket 1 is 0 1

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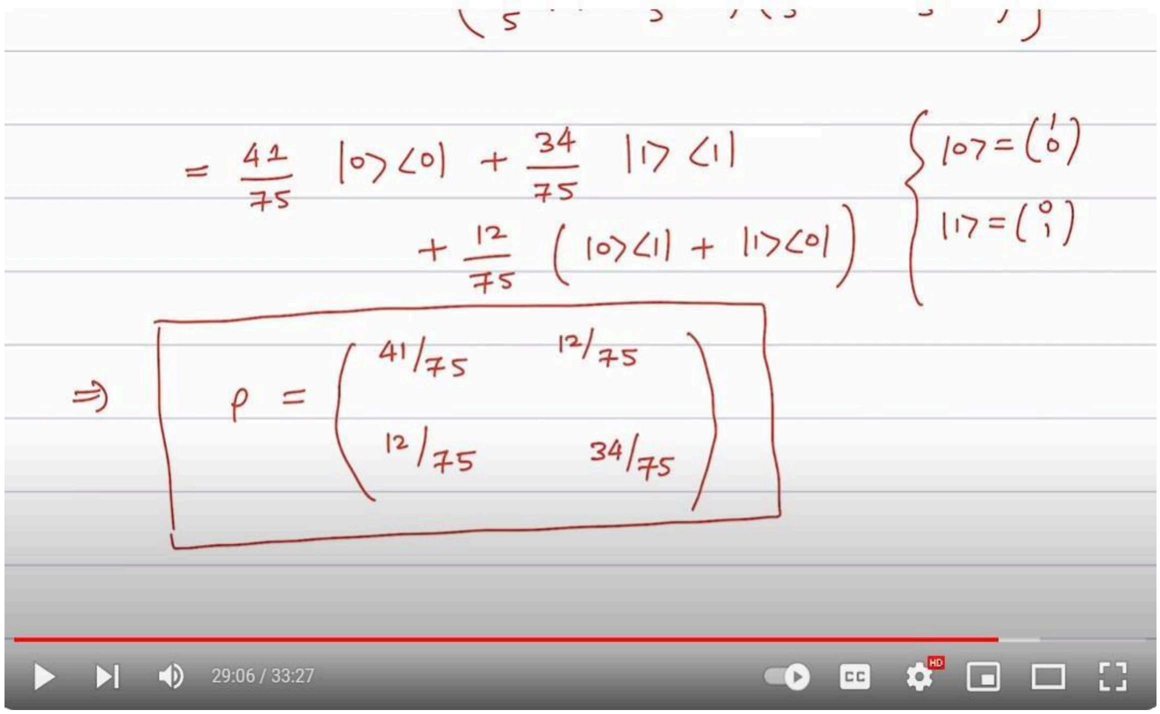

utilizing it you can write the explicit form of the density matrix for the given problem as the diagonal elements would be 41 divided by 75 and 34 divided by 75 and the off diagonal elements would be 12 divided by 75 12 divided by 75 as you can see trace of rho is equal to 1 here so it's a valid density matrix and the density matrix are the diagonal elements are positive so it is positive semi-definite as well.

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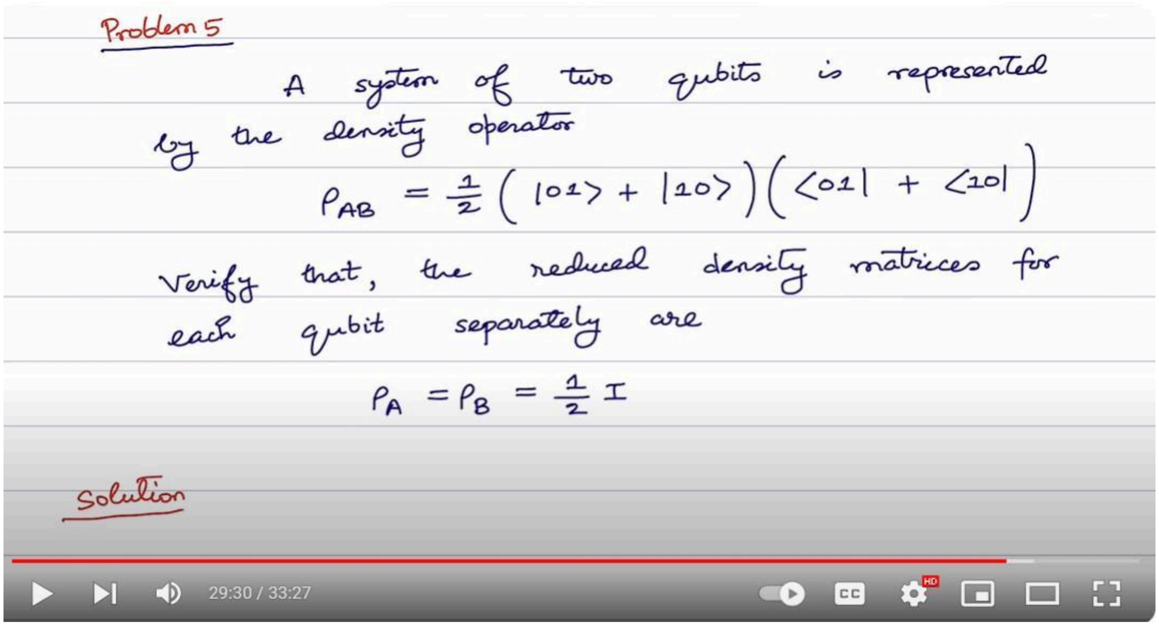

Now let us work out this problem given that a system of two qubit is represented by the density operator you have to verify that the reduced density matrices for each qubit separately are this okay so we have done similar stuff earlier so let me do it again here

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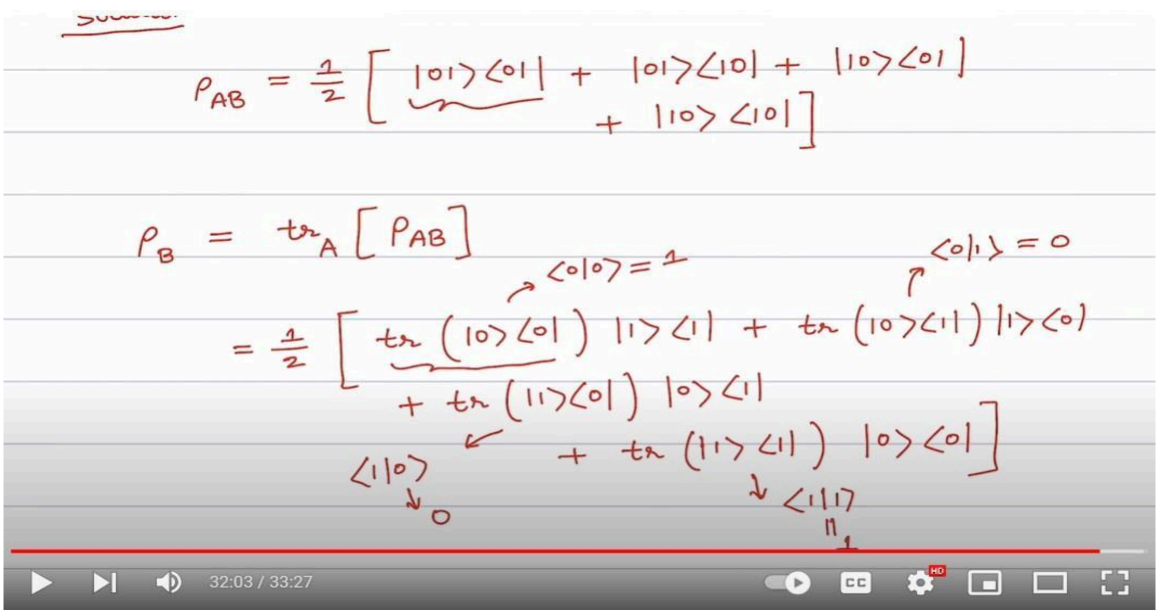

first of all let me write rho ab in full form that would be half we'll have 0 1 0 1 plus 0 1 1 0 plus 1 0 0 1 right and we'll have 1 0 1 0 so this is the full form of the density operator for the composite system now if i want to find out density operator for system a or system b i have to do the tracing out operation for example if i want to find out say density operator of the system b then i have then i have to trace out a from rho ab so let me do this okay so what i will get it be half the system b is going to remain unaffected so i have to take the trace operation over the first system that would be trace of this 0 0 and we'll have the system b is going to remain unaffected then from the second i have just deal with this one and the second one would be trace 0 1 and 1 0 then i have trace 1 0 and then i have the second part is going to remain unaffected is going to give me trace uh you know it would be 1 1 and the outer product of the second one is anyway so it remains unaffected already we know what is the result of this this is going to give us the scalar product of 0 0 this is going to give us the scalar product of 0 1 and this one is going to give the scalar product of 1 0 and this is going to give us the scalar product of 1 1 and we know that 0 0 scalar product is going to give me 1 these are orthogonal 0 1 is orthogonal so therefore i'll be having 0 here here also it is going to give me 0 and here i'm going to get 1

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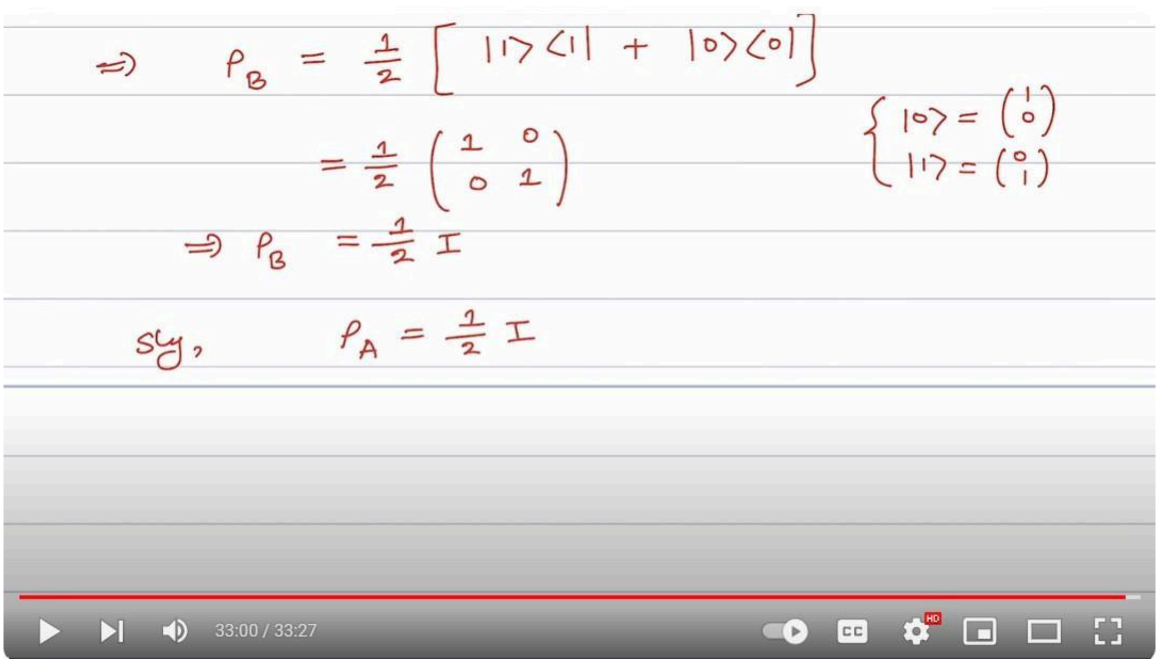

so from here you see that density operator or the reduced density matrix for the system b i get it as half and you will have 1 1 plus 0 0 now using the fact that again the operator representation or the sorry matrix representation of ket 0 is 1 0 and ket 1 is 0 1 utilizing it i can simply write it as half 1 0 0 1 so i have therefore rho b is equal to half into the identity matrix similarly we have done it in in the class also earlier rho a you can show that it will be half i.