Quantum Entanglement: Fundamentals, measures and application

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Week-04

Lec 17: Problem solving session-4

In this problem solving session, we are going to solve some problems related to quantum entanglement measure.

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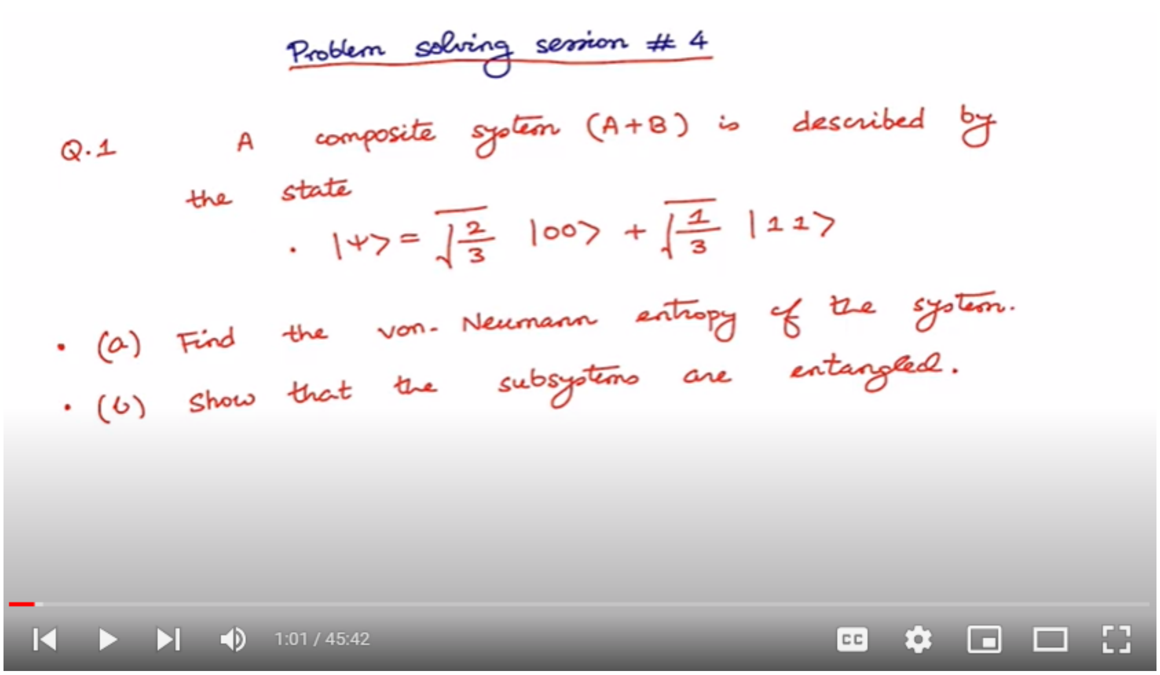

As a first problem, let us consider a composite system described by the state ket psi, which is written as a superposition of ket 0,0 and ket 1,1. You are asked to find the von Neumann entropy of the system and you are asked to show that the subsystem are entangled. So let us do that.

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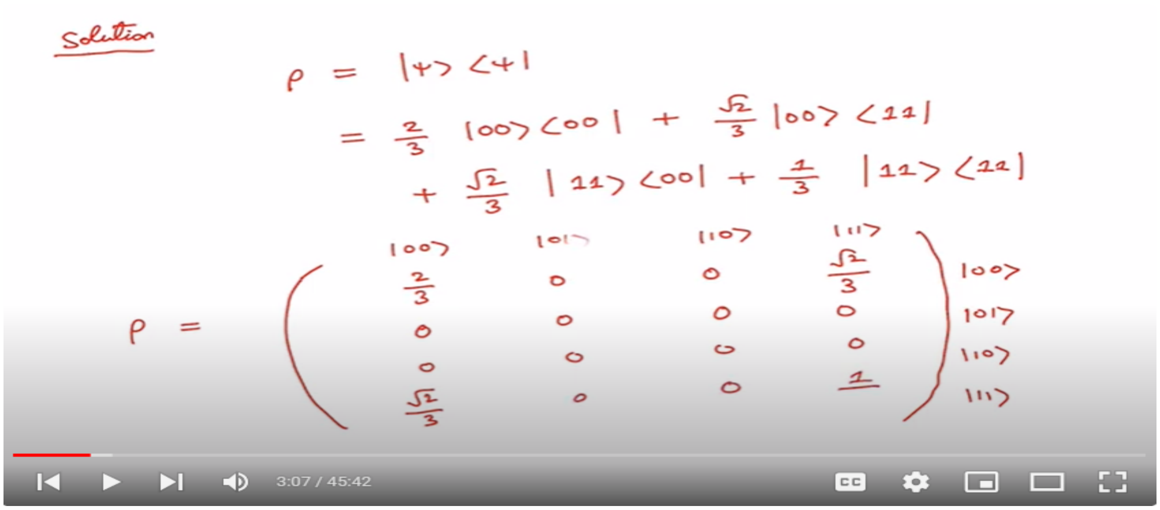

First of all, let me work out the density matrix. So density matrix would be ket psi bra psi and we have done so many problems on density matrices. So you can immediately write the density matrix as 2 by 3 ket 0,0 bra 0,0 plus root 2 by 3 ket 0,0 bra 1,1. And then you will have terms root 2 by 3 ket 1,1 bra 0,0 plus 1 by 3 ket 1,1 bra 1,1.

And this you can write in a matrix form. To write it in a matrix form, generally the trick is you arrange the row this way. Say it is ket 0,0, ket 0,1, ket 1,0 and ket 1,1. This method will be helpful for all other purposes as we will see and we know already. So the column are also let us say we put it this way ket 0,0, ket 0,1, ket 1,0, ket 1,1. Then we can write the first element as 2 by 3 0,0 root 2 by 3. Then we will have 0,0,0,0 and we will have 0,0,0,0 and we will have root 2 by 3 0,0,1 by 3.

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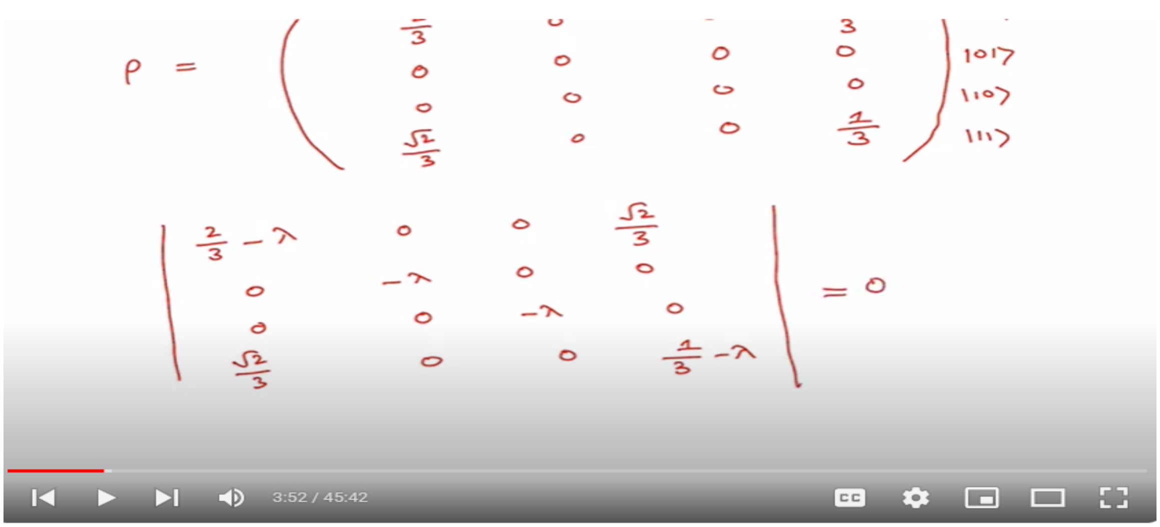

So this is the density matrix. Now to find the eigenvalues of rho let us set up the characteristic equation first. To set up the characteristic equation we know how to do that. So we just have to find the determinant of this. We have to set the determinant to be 0. 2 by 3 minus lambda 0,0 root 2 by 3 0 minus lambda 0,0 0,0 minus lambda 0 root 2 by 3 0,0 1 by 3 minus lambda. So this determinant is equal to 0 and if you actually open it up it is very straight forward.

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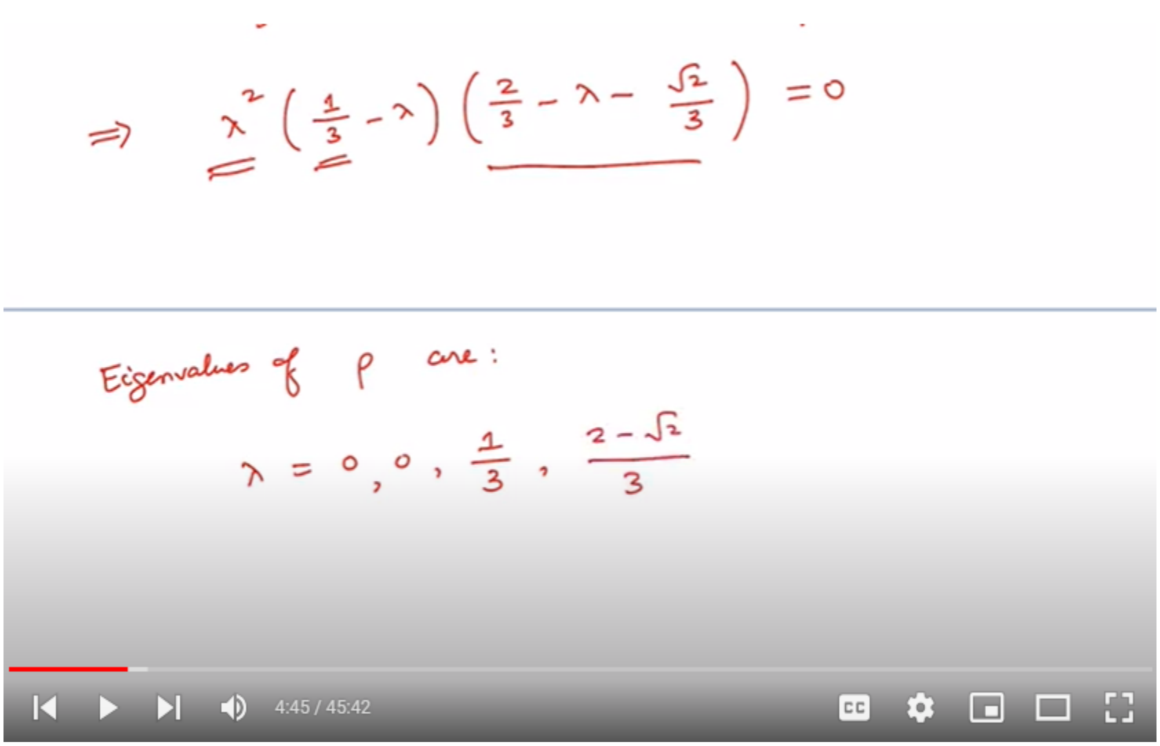

You will get this equation lambda square into 1 by 3 minus lambda. Please verify it yourself. 2 by 3 minus lambda minus root 2 by 3 is equal to 0.

So therefore the eigenvalues of rho are this density matrix will be lambda is equal to 0,0 from here. Then you will have one third from here and from the last part you will have lambda is equal to you will get 2 minus root 2 divided by 3. So these are the four eigenvalues you will obtain. Now you are asked to find out the von Neumann entropy of the system. So to do the von Neumann entropy we know work out the von Neumann entropy.

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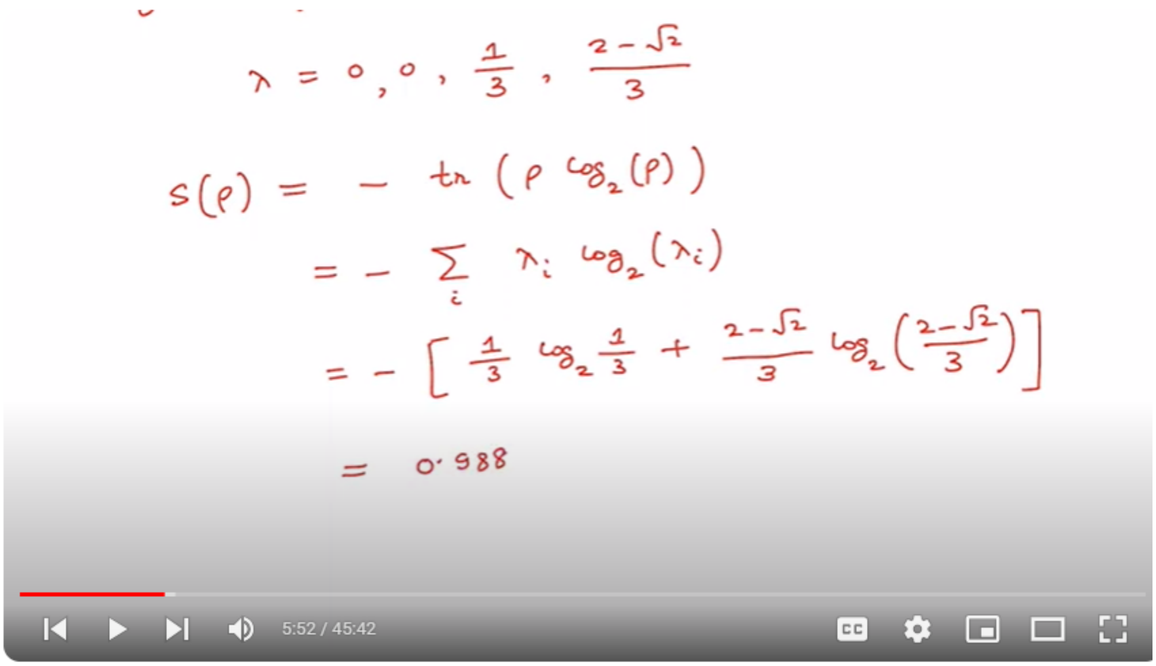

S rho is equal to minus trace of rho logarithm with base 2 rho and this is equal to minus summation of lambda i. Lambda i is the eigenvalues, ith eigenvalue log 2 lambda i. So you will get minus 1 by 3 log 2 1 by 3 plus 2 minus root 2 by 3 log base 2 2 minus root 2 by 3. In fact if you work it out using a calculator you will get it to be 0.988.

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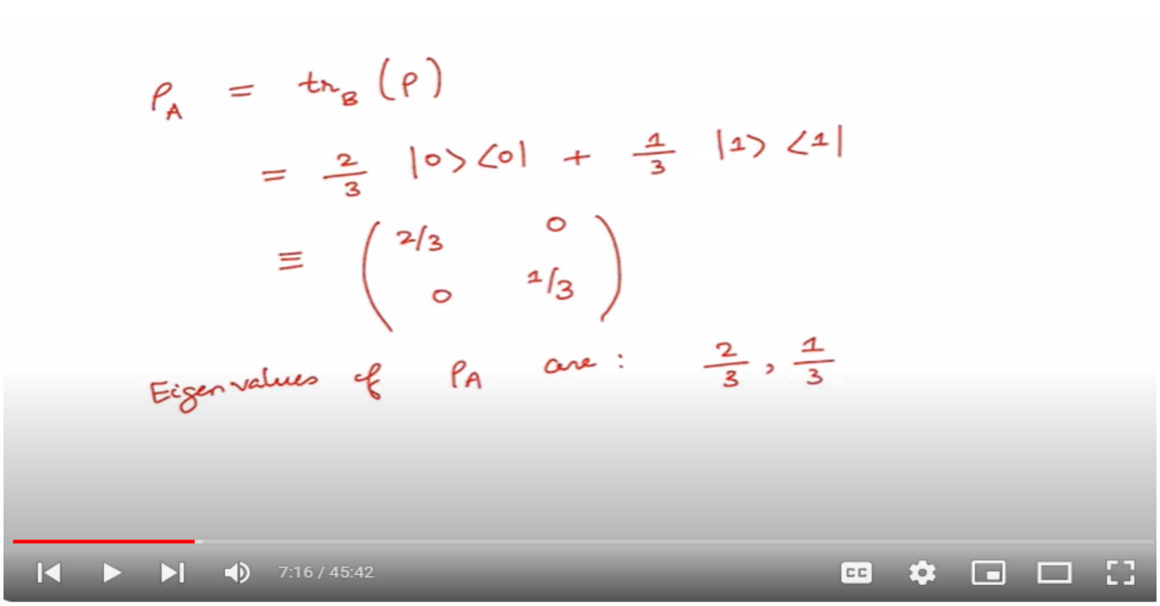

Now to do the next part, next part was to you are asked to show that the subsystems are entangled. To do that let us first calculate the density matrix of the subsystem A for example. Any subsystem you can pick up. So let us find out the reduced density matrix of A then you have to trace out B.

So reduced density matrix rho A would be if you trace out rho then you will be able to get that. And in fact we have done problems like this you can immediately get it to be 2 by 3 ket 0 bra 0 plus 1 by 3 ket 1 bra 1. And this you can write it in a matrix form. In matrix form it would be 2 by 3 0 0 1 by 3. Alright. So here it's very straightforward. Now the eigenvalues of rho A are you have only diagonal elements so it is 2 by 3 and 1 by 3.

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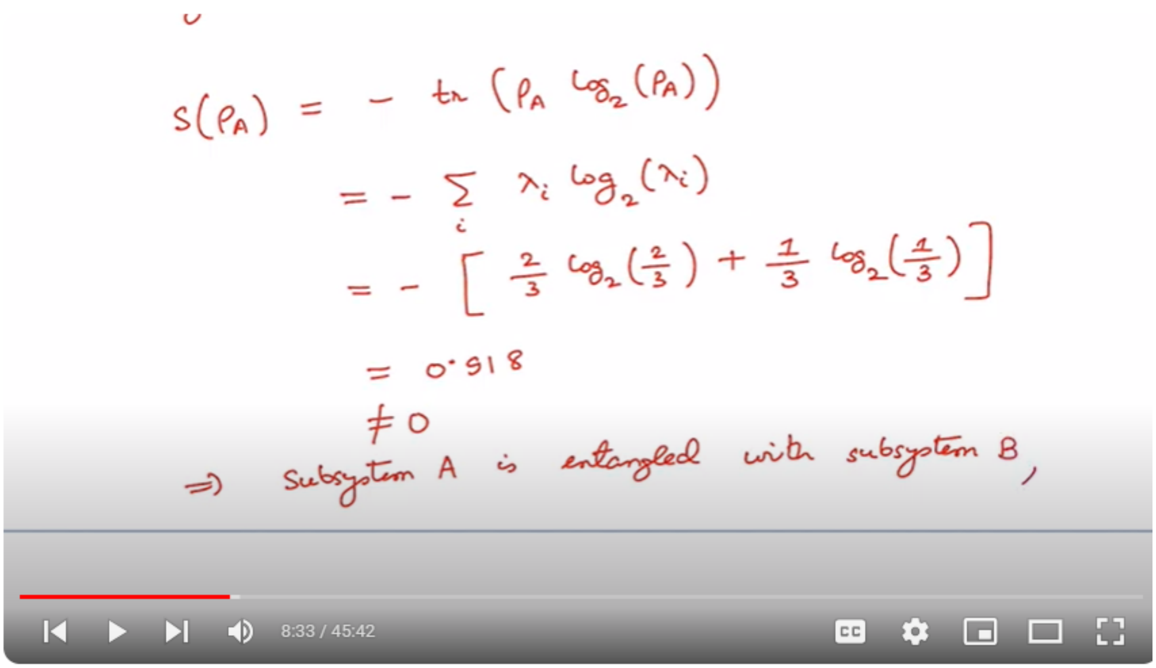

So to know whether the subsystems are entangled or not we just need to find out the reduced von Neumann entropy. So in this case let us work out as rho A that would be minus trace rho A log 2 rho A which is nothing but the summation of the eigenvalues of rho lambda I. Lambda I is the eigenvalue of rho A. There are two eigenvalues so base 2 log 2 lambda I and you will get minus 2 by 3 log 2 2 by 3 plus 1 by 3 log base 2 1 by 3. And if you work it out if you put it in a calculator you are going to get 0.918.

So you see as the reduced von Neumann entropy is non-zero this implies that subsystem A is entangled with subsystem B. Alright.

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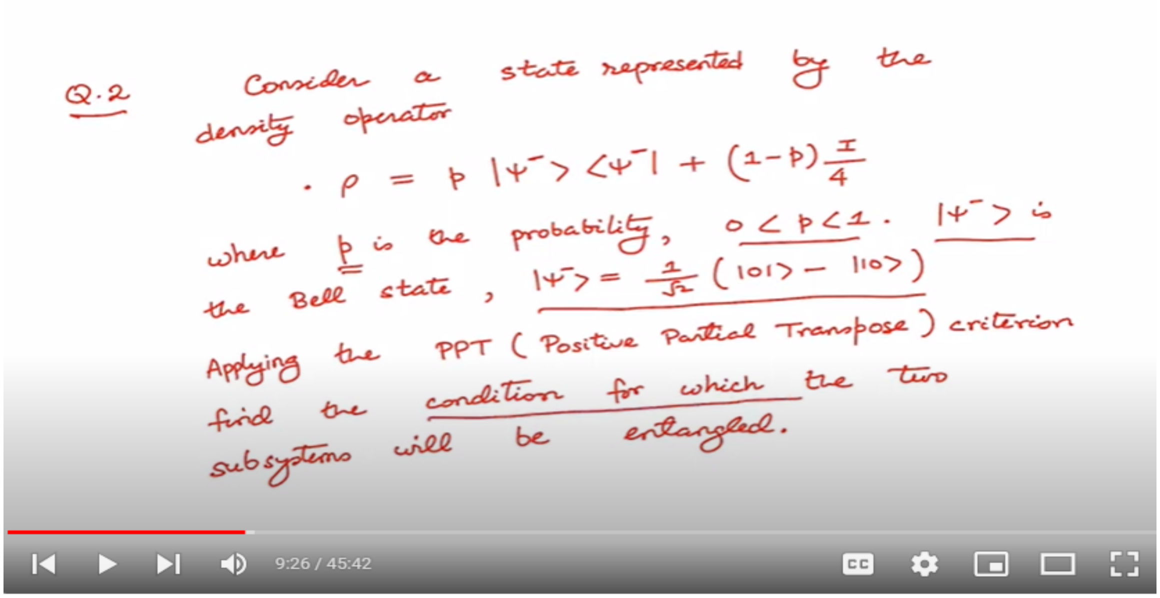

Now let us work out this problem. Consider a state represented by the density operator rho is equal to p ket psi minus bra psi minus plus 1 minus p I by 4. I is the identity matrix where p is the probability so it has to lie between 0 and 1. ket psi minus is the Bell state which is given as 1 by root 2 ket 0 1 minus ket 1 0. Applying the PPT that is the positive partial transpose criterion which is also known as Perry's Horodicki criterion. You are asked to find the condition for which the two subsystems will be entangled.

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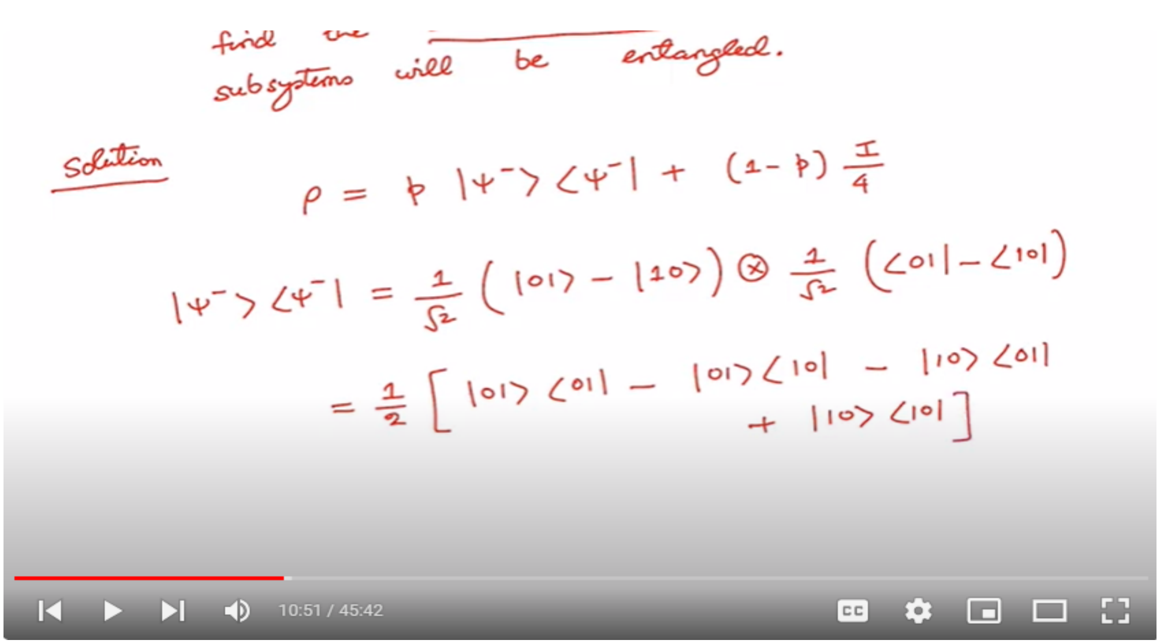

So let us work it out. This is an important problem and very interesting as well. So let us work it out. So rho is given as p ket psi minus bra psi minus plus 1 minus p I by 4. Let me work it out systematically.

First let me work out psi minus ket psi minus bra psi minus. This would be 1 by root 2 ket psi minus is ket 0 1 minus ket 1 0. Then you have 1 by root 2 bra 0 1 minus bra 1 0. So this will give me 1 by root 2 ket 0 1 minus ket 0 1 bra 0 1 minus ket 0 1 bra 1 0 minus ket 1 0 bra 0 1 plus ket 1 0 bra 1 0. So this is what I will get.

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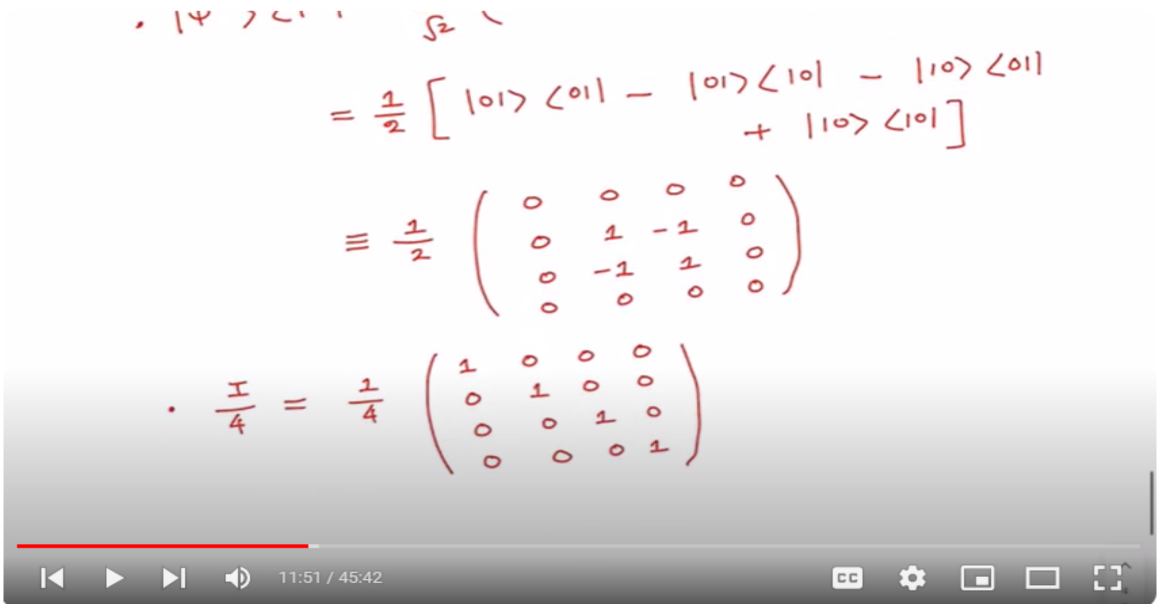

In fact I can write it in the matrix form. That would be 1 by root 2 ket 0 0 0 0 1 minus 1 0 0 minus 1 1 0 0 0 0 0. So this is what I will have. Now again we are having another term in the density matrix operator. I by 4 would be 1 by 4. I by 4 is a 4 by 4 identity matrix. So you will have 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1. So now if I combine this with this one. If I put it here then I can finally write rho.

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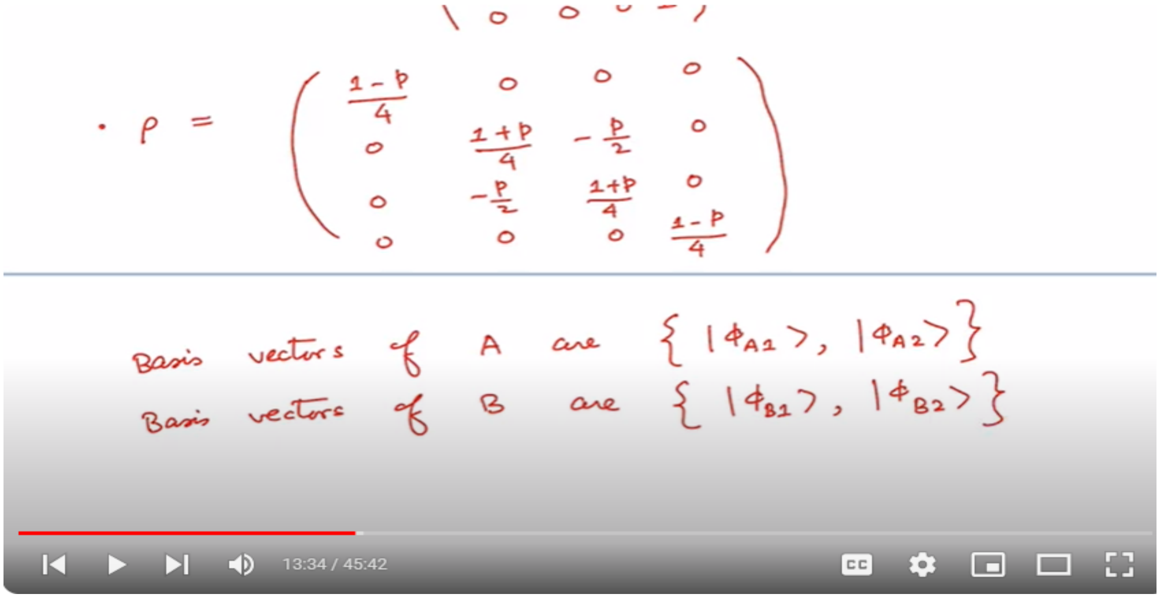

So rho would be equal to the density matrix would be equal to. I have 1 minus p by 4 0 0 0 0 1 plus p by 4 minus p by 2 0 0 minus p by 2 1 plus p by 4 0. And last rho would be 0 0 0 1 minus p by 4. So this is what I will have. OK.

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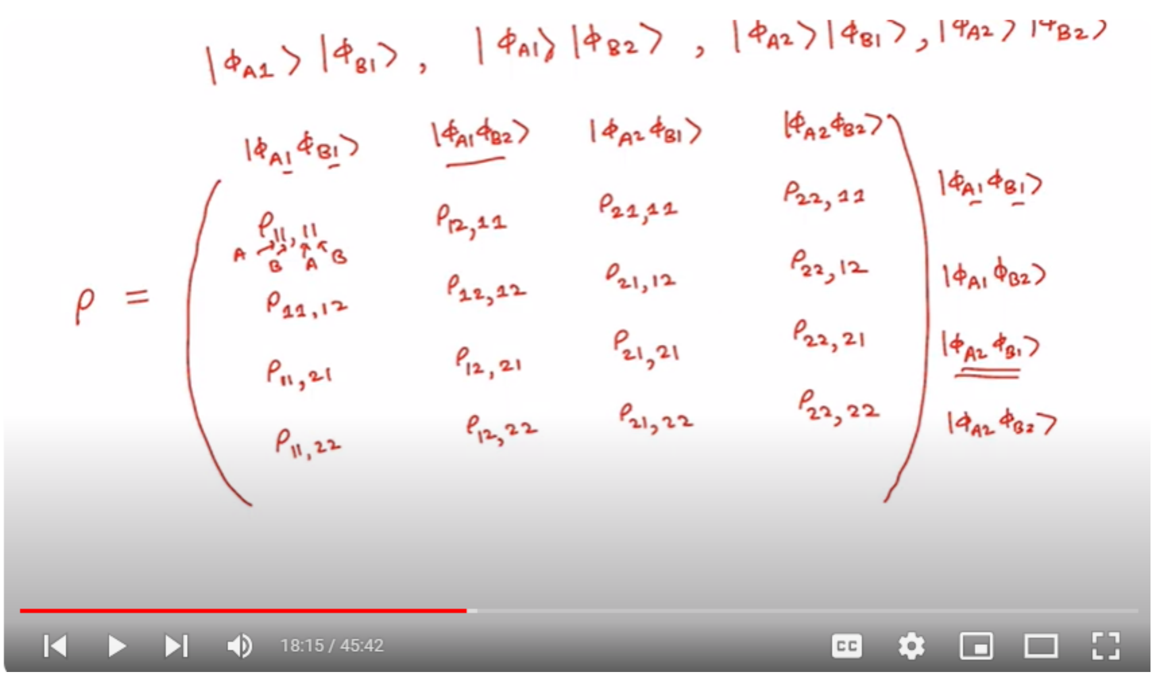

Now before we proceed further. I think it will be useful to understand how the matrix element of rho are written in the basis vectors of subsystem A and subsystem B.

So let me explain that carefully. You see the basis vectors. Let us say basis vectors of A are in the Hilbert space of A are say phi A1 and phi A2. And basis vectors. Vectors of B are. ket phi B1 and ket phi B2. OK. The basis vectors of the composite systems in this matrix is going to be arranged in this form. Phi in this order.

Let us say it is phi A1. Because now the this is a composite system. This density matrix represents a composite system of A and B. And basis vectors. This Hilbert space composite Hilbert space is spanned by the basis vectors of A and B. And these basis vectors are going to be arranged in this order. Phi A1 phi B1 and you will have phi A1 ket phi B2. Then you will have phi A2 phi B1 and you will have phi A2 phi B2. Let me explain it little bit more clearly.

Let me write the density operator. Density matrix. And it is the similar line I am going to discuss it the way I have done it in the previous problem. So let me arrange it in the row. Row would be in this order. Phi A1 phi B1. Then you have phi A1 phi B2. And you have here phi A2 phi B1. And you have phi A2 phi B2.

Similarly the column. Let me write it this way. Let me take it little bit this side. Let me put a gap here. Similarly this one. And let me. The column are also. Phi A1 phi B1. Phi A1 phi B2. Phi B1. Phi A2 now. You have to be careful. So phi A2 phi B1. Then you have finally phi A2 phi B2.

Then the density matrix elements would be. First element would be. Let me put the indices 1 1 from here and 1 1 from here from the column. So first element would be rho 1 1 1 1. This first element belongs to A. This first one belongs to A. This one belongs to B. This indices belong to A. And this indices belong to B. Then you will have in the second term. You are going to have rho 1 2 1 1. Rho 2 1 1 1. Rho 2 2 1 1. Then the second row would be. Rho 1 1 1 2. Rho 1 2 1 2. Rho 2 1 1 2. Rho 2 2 1 2. And then third row. You will have rho 1 1 2 1. You see. I hope you are getting it. Then rho 1 2. Rho 1 2 2 1. These indices are very critical. Then you will have. You will have here rho 2 1. 2 1. And rho 2 2. 2 1. And finally. In the final row you will have. Rho 1 1. 2 2. Rho 1 2. 2 2. Rho 2 1. 2 2. And rho 2 2. 2 2. Right. So this is how we can write the. All the matrix elements of density operator.

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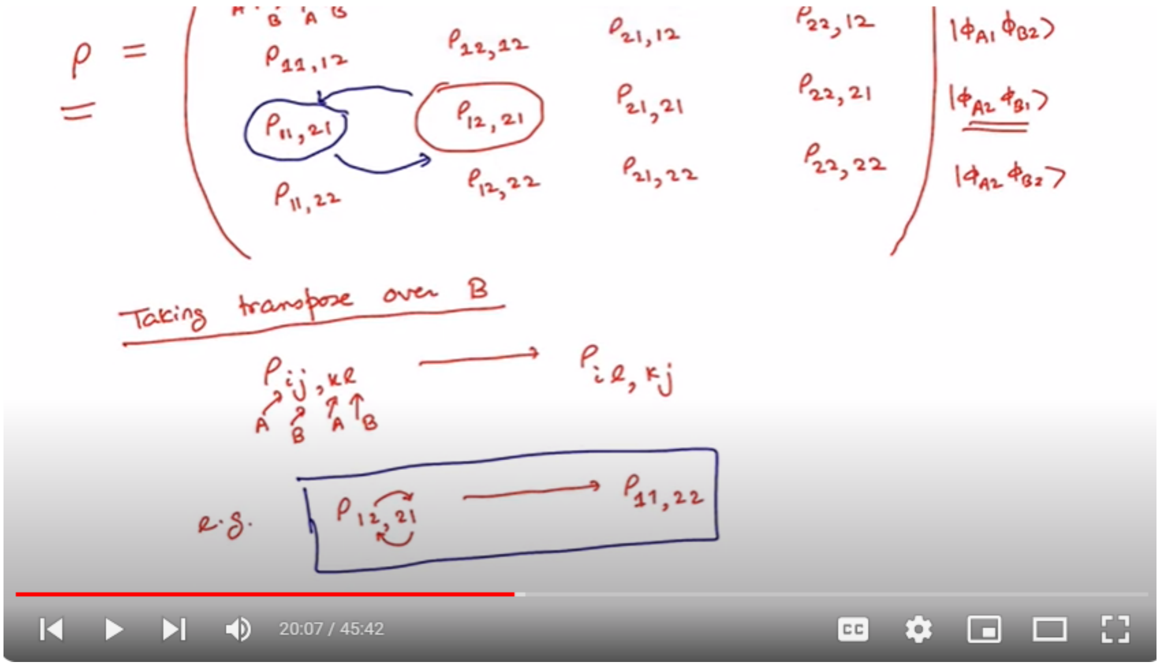

Now. Taking transpose over B. Means. Interchanging the indices associated with B. So all and keeping indices belonging to A intact. So. Say I have the indices I J K. I J K L. Where I belongs to A. J belongs to B. And K belongs to A and L belongs B.

Here I mean to say. Taking you know. Taking transpose. Taking transpose. Over. B. Which means. I will keep the indices belonging to A intact. And I will interchange the indices belonging to B. So I will have I L. K J. This is what we have.

For example. In the density operator here. If I have the element is say. Density matrix element is say rho 1 2 2 1. Then taking transpose means I will have rho 1. 1. 2 will go here and 1 will go here right. It will be rho 1 1. And. You will have 2. 2. So I will have the elements here. If I look at the. Matrix here. So I have this one. And if I want to take the transpose over B. This will go. This side. And this will go this side. This would get. Interchanged. If I just talk about this one only.

I hope you get this idea. And this is very important. Now. To do that. Let us see. Because I have to take the transpose over B.Of the given density operator. I am given the density operator.

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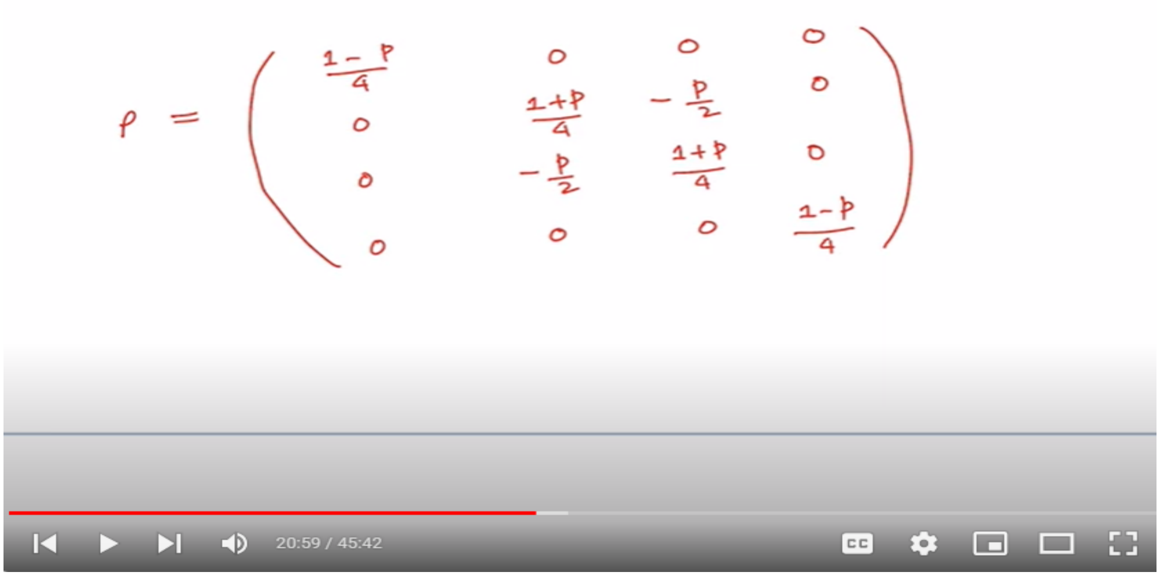

Let me write it. This is what we have worked out. Let me write it once again here. 1 minus P by 4. I have. 0. 0 0. And then I have 0. 1 plus P by 4. Minus P by 2. 0. 0. Minus P by 2. 1 plus P by 4. 0. 0. 0. 0. 1 minus P by 4. Let me. Take it to the other base. Let me take it here. Now. Okay.

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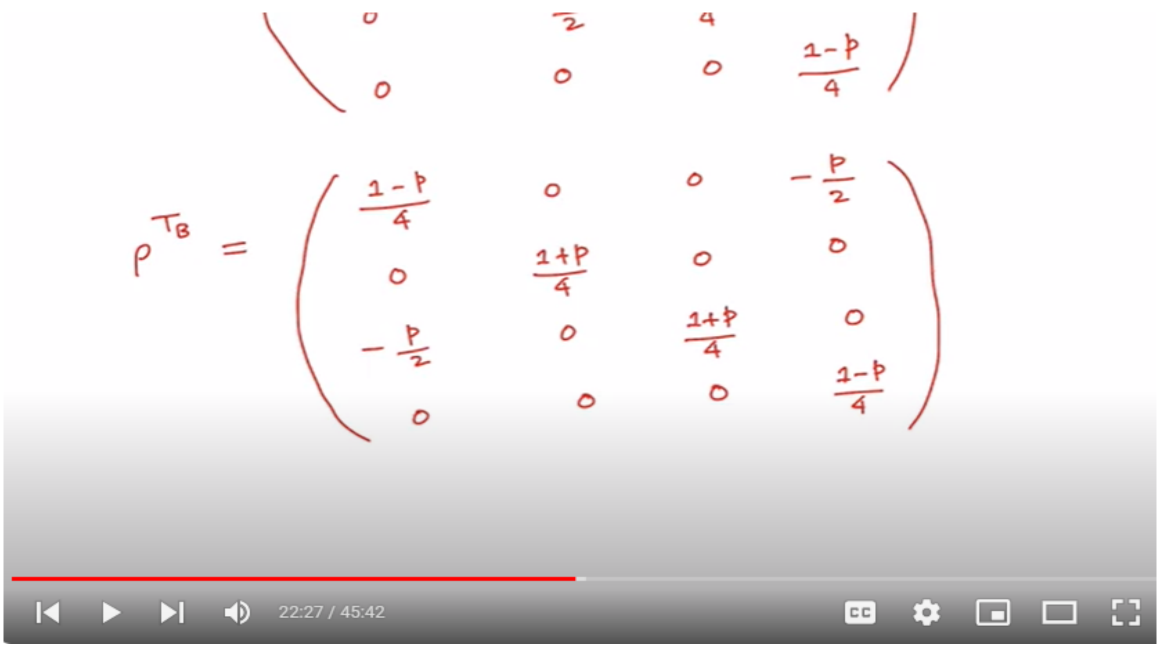

Now. Let us find out. Partial transpose. Transpose over. B. If you follow the scheme. Then you will get this. Please do that yourself. You will get it very easily. Diagonal elements. Please do that yourself. Diagonal elements are going to remain unaffected. It will remain It will remain unchanged.

But here you will see. You will have this will now become Because of the transpose Over B. It will become minus P by 2. And you will have The elements will be 0. 0. 1 plus P by 4. 0. 0. Minus P by 2. 0. 1 plus P by 4. 0. 0. 0. 1 minus P by 4. So this is the Transposed Partially transposed matrix We are having.

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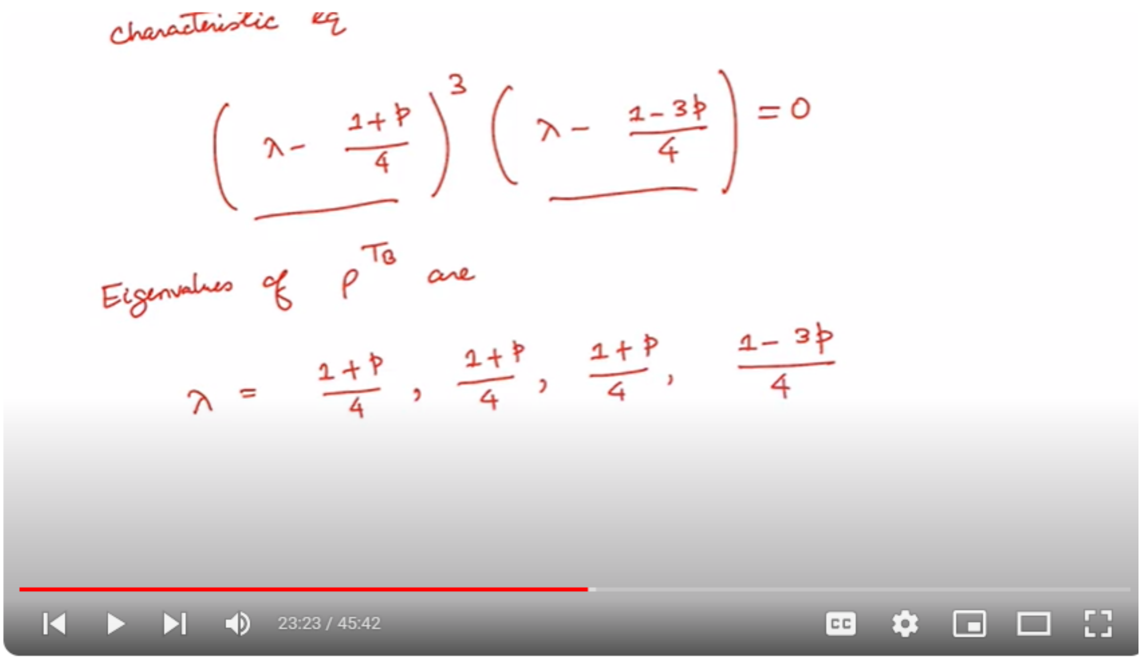

Now let us work out the Eigenvalues of this matrix Rho T B. And to do that we have to Set up the usual Characteristic equation. The characteristic equation You can set up easily. And if you do that Characteristic equation would be Lambda minus 1 plus P by 4 Whole cube Into Lambda minus 1 minus 3 P by 4 Is equal to 0.

So therefore it implies That the Eigenvalues Would be Eigenvalues of Rho T B are Lambda is equal to It is you see threefold So you will get 1 plus P by 4 1 plus P by 4 1 plus P by 4 And from the Last one you will get 1 minus 3 P by 4 So this is what you will obtain So these are the four Eigenvalues out of which Three are Degenerate and one is Non-degenerate the last one.

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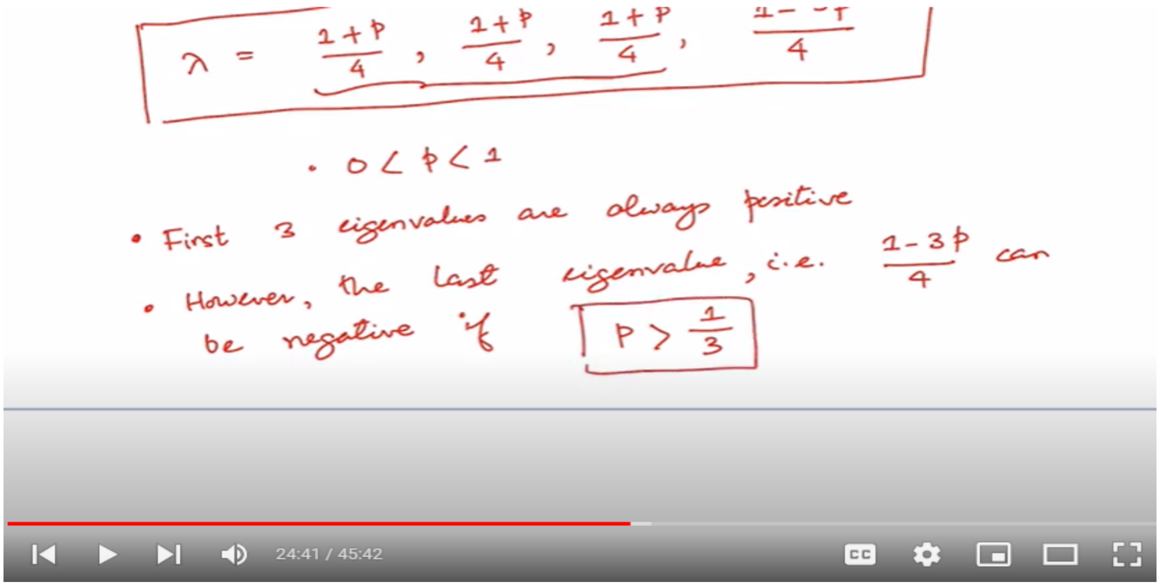

Now you see P lies between 0 and 1 Right? So the first three Eigenvalues are First three First three Eigenvalues are positive Eigenvalues Are always positive Are Always Always positive But if you look at The This is the first three one However if you look at the last one However However The last Eigenvalue It's very easy to see The last Eigenvalue Which is 1 minus 3 P by 4 Can be negative Can be negative Can be negative If You can easily see If P is greater than 1 by 3.

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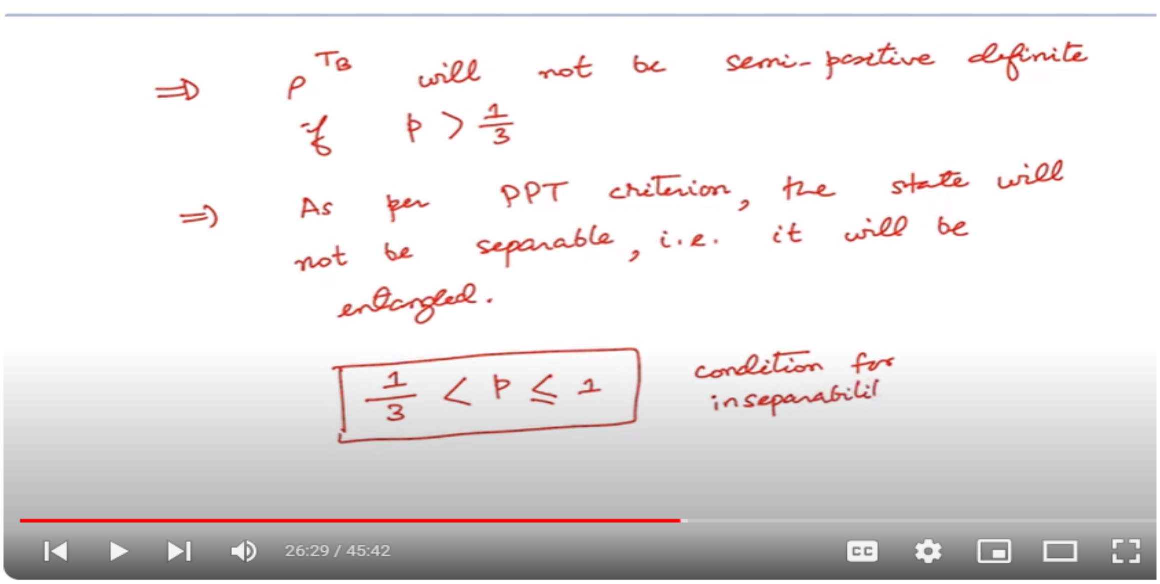

So this is important If one of the One of the Eigenvalues become negative We say that rho T B This is Not density matrix any longer than This matrix Will not be semi positive definite Will not be Will not be Semi positive Semi positive Definite.

That means one of the Eigenvalues are going to be negative If P is greater than 1 by 3 This implies As per Positive partial transpose Criterion Or PPT criterion What you are going to have is that The state Will The state will not be Separable Not be Separable What does it mean? It will be The state will be Entangled So this is the condition you are asked To find out in this particular Problem.

So the condition that you have basically Arrived The required condition of inseparability One P should be P cannot be If P is greater than 1 by 3 Because P cannot be greater than 1 It has to be less than 1 But if P is greater than 1 by 3 If this is the condition This is the condition This is the Condition for Inseparability For entanglement Inseparability.

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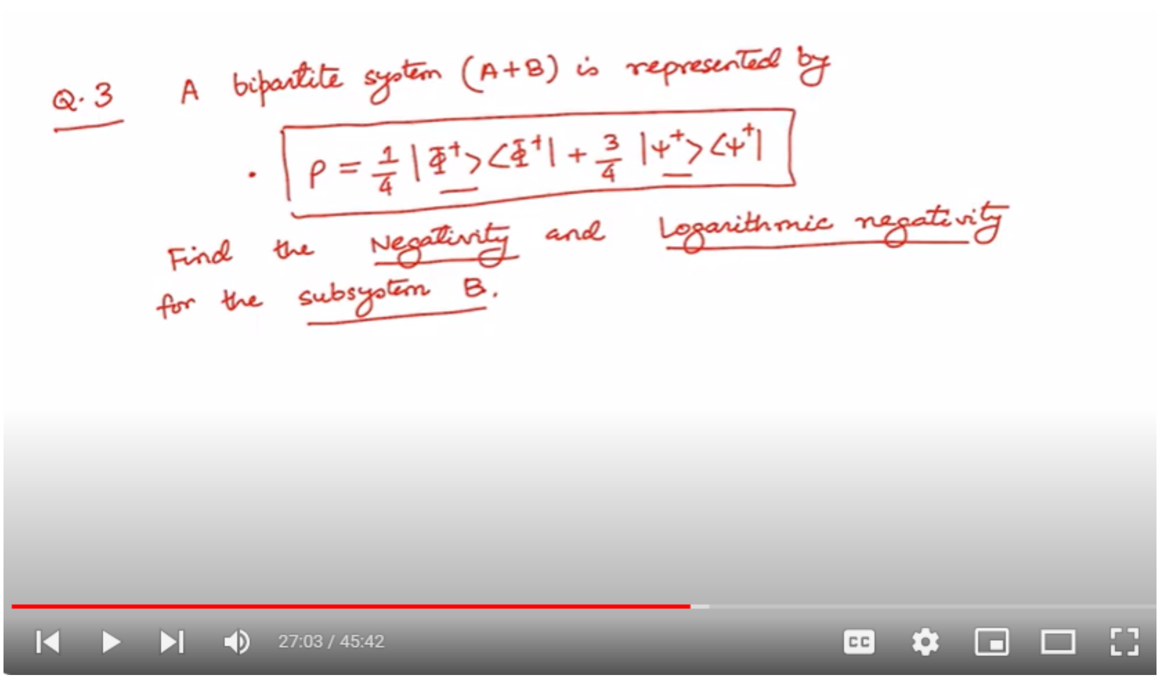

Let us now work out this problem. A bipartite system B is represented by this density operator You are asked To find out the negativity And logarithmic negativity For the subsystem B As you can see By this density operator That this is a mixed Step and it's a collection Of Bell state phi plus and Bell state psi plus With probability 1 by 4 and 3 by 4 Respectively.

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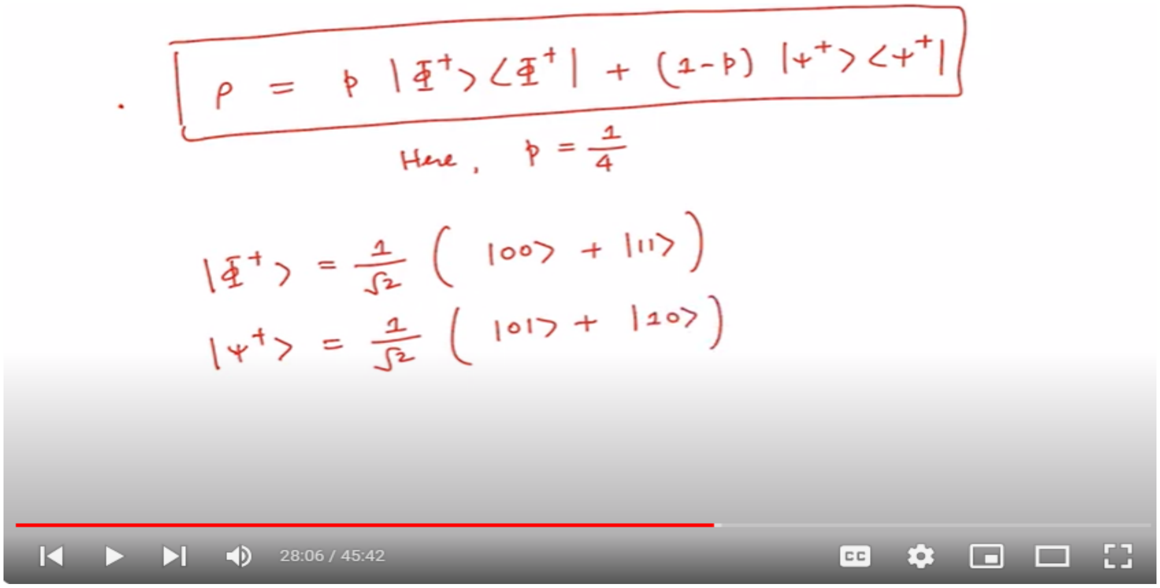

To do this problem Let me write it in a General form This density operator Rho is equal to let me write it as P phi plus ket phi plus bra phi plus And 1 minus P ket psi plus bra psi plus ket psi plus bra psi plus ket psi plus bra psi plus And here And here We have P is equal to 1 by 4 Okay But first let me Set up the density matrix for this In this form And we know that Phi plus is equal to 1 by root 2 ket 0, 0 plus ket 1, 1 And psi plus ket psi plus ket psi plus is 1 by root 2 ket 0, 1 plus ket 1, 0 Okay

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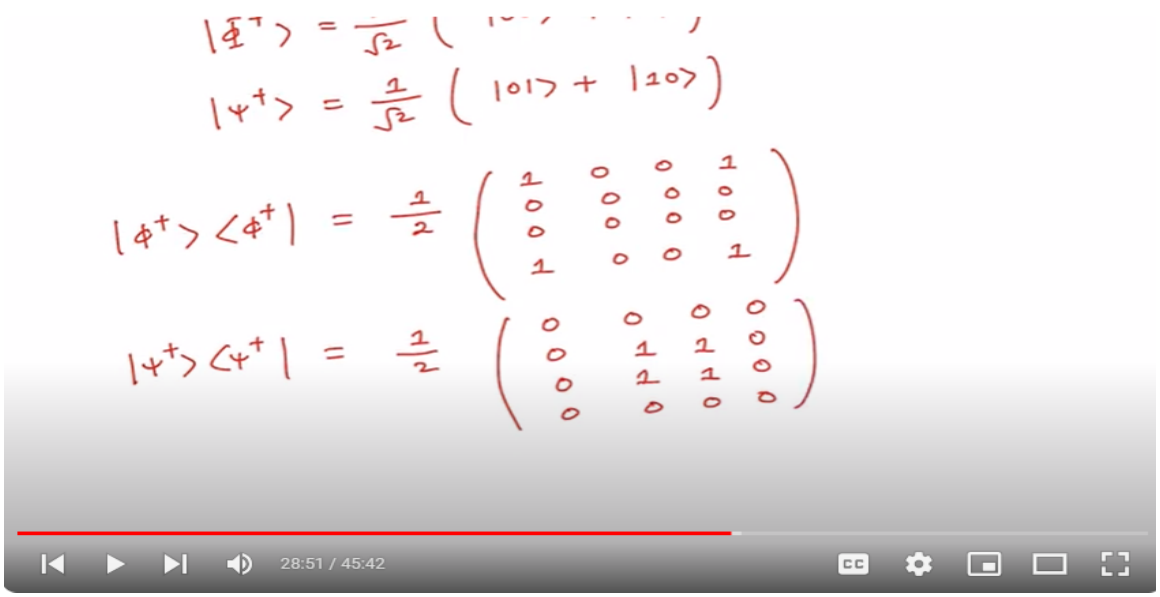

So I can write ket phi plus bra phi plus In a matrix form It would be We already know it It would be 1 half 1, 0, 0, 1 0, 0, 0, 0 0, 0, 0, 0 1, 0, 0, 1 On the other hand Psi plus ket psi plus bra psi plus That would be equal to 1 half 0, 0, 0, 0 0, 1, 1, 0 0, 1, 1, 0 0, 1, 1, 0 0, 0, 0, 0, 0 0, 0, 0, 0 .

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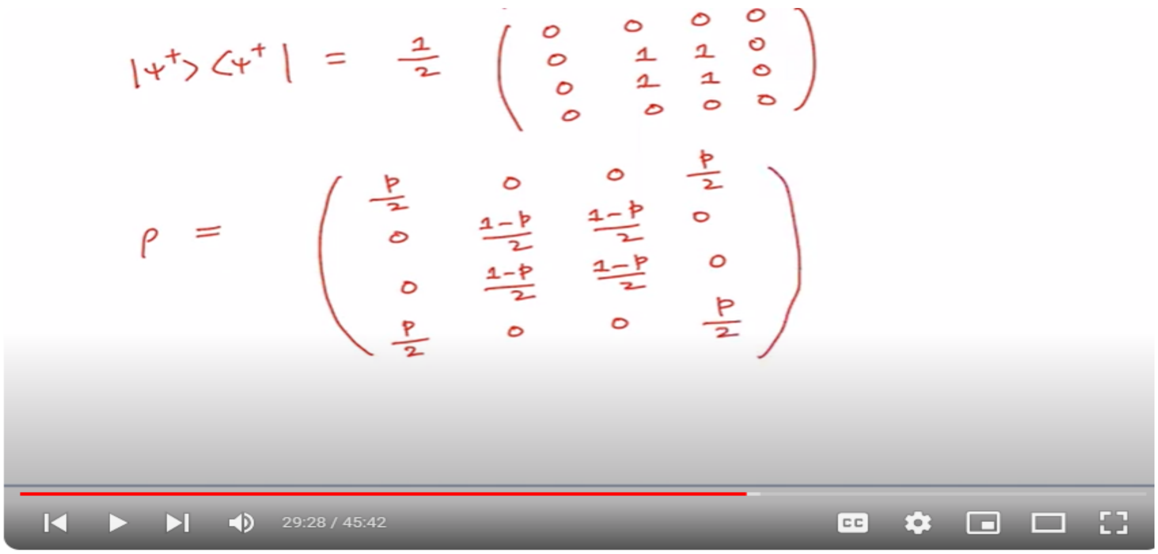

So using this I can now write our density operator in this matrix form Rho would be equal to P by 2 0, 0 P by 2 0, 1 minus P by 2 1 minus P by 2 0 0, 1 minus P by 2 1 minus P by 2 Please verify it yourself It is very straight forward P by 2 0, 0 P by 2

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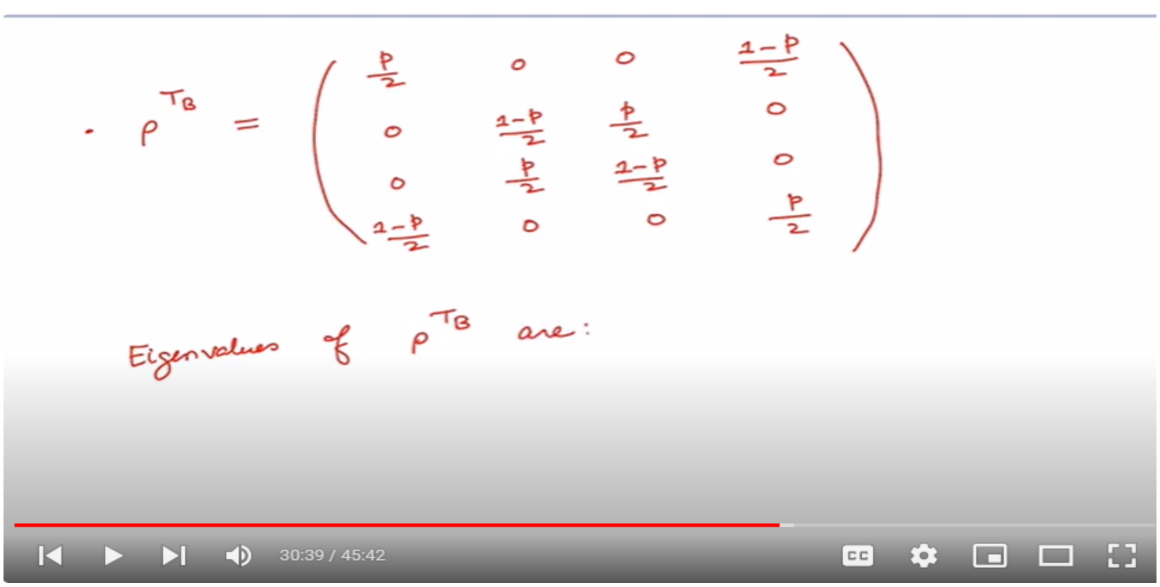

Now let us take the partial transpose over the subsystem B So if I follow the same procedure as we have done in the last problem, previous problem partial transpose over B you will get the matrix in this form, you will get P by 2 0, 0 1 minus P by 2 0 1 minus P by 2 P by 2 0 Diagonal elements actually would remain as it is 0 P by 2 1 minus P by 2 0 1 minus P by 2 0, 0 P by 2 So this is what you will obtain, you can set up the characteristic equation for rho Tb and solving the characteristic equation you can find out the eigenvalues for rho Tb

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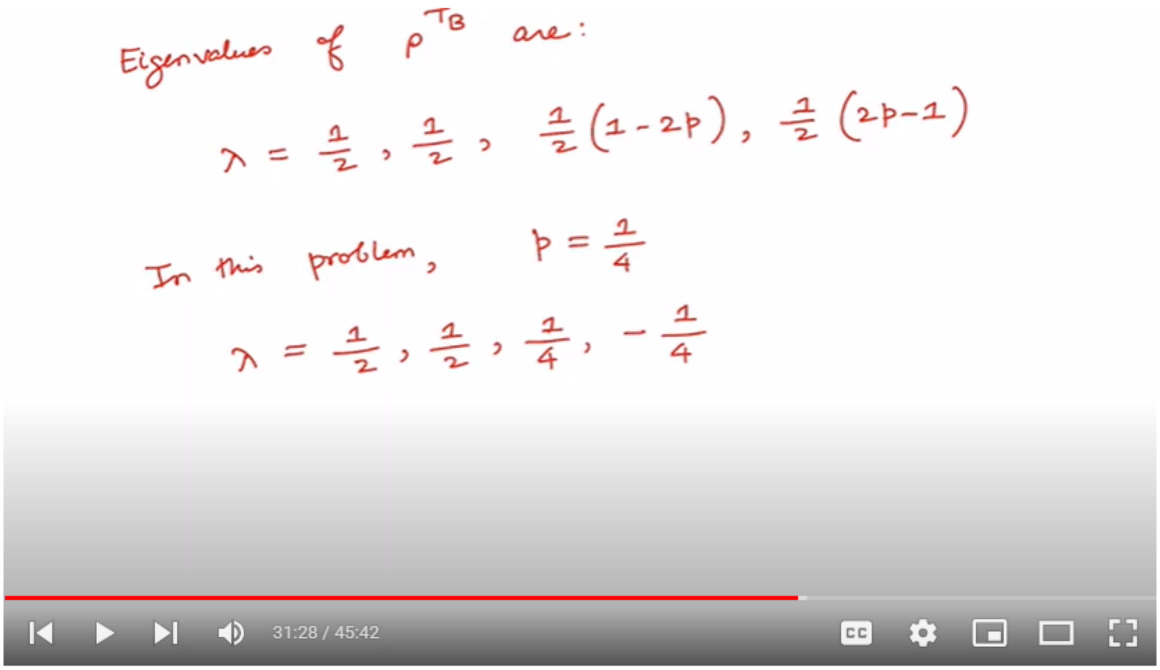

and if you do that you will obtain the eigenvalues of rho Tb as 1 half 1 half 1 half into 1 minus 2 P and 1 half 2 P minus 1 Now here in this problem in this problem we are given P is equal to 1 by 4, so therefore the eigenvalues for the given problem would be 1 half for rho Tb here would be 1 half 1 half 1 by 4 and minus 1 by 4 So as you can see that one of the eigenvalues is negative so therefore the state is basically entangled and rho Tb is not semi positive definite so as per PPT criteria rho is not separable, it is inseparable

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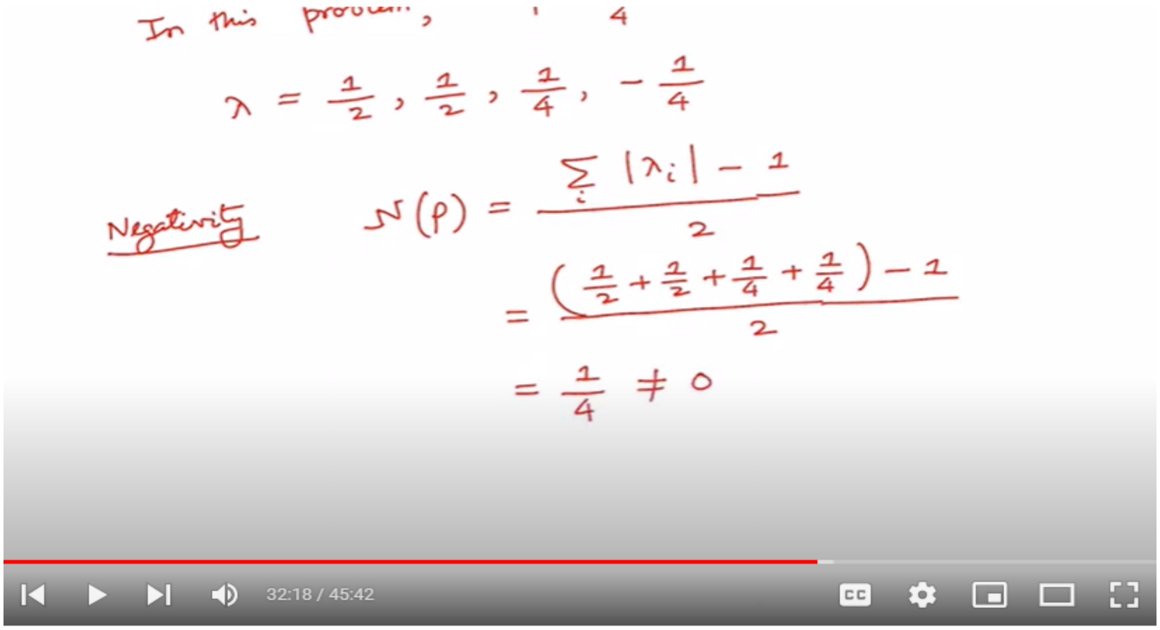

and negativity you can work it out easily, negativity for the sub-system B negativity and of rho is equal to sum of our modulus of the eigenvalues minus 1 divided by 2 and if you do that we have 1 half plus 1 half plus 1 by 4 modulus we are taking so it would be 1 by 4 sum of all this minus 1 divided by 2 so this is going to give us 1 by 4 so you see the negativity is non-zero that also ensures the sub-systems are entangled

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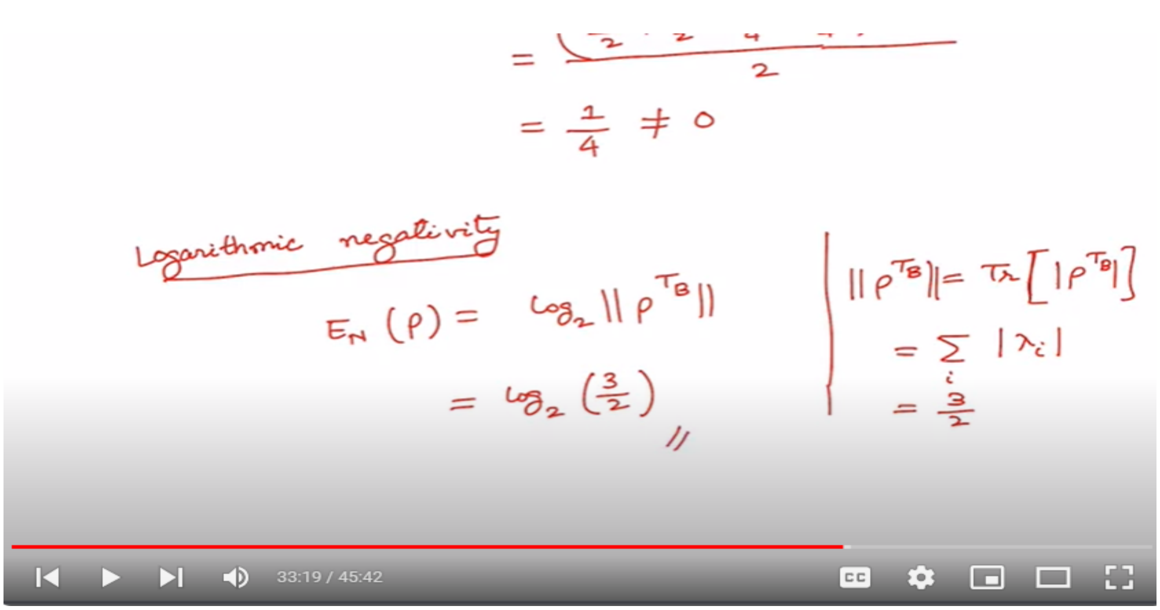

and the logarithmic negativity we can easily work out, logarithmic negativity the formula for logarithmic negativity is En of rho is equal to logarithm of 2 you have to take the trace norm of our rho Tb so in fact trace norm rho Tb is equal to you have to take the modulus of sum of the modulus of all the eigenvalues of rho and because the eigenvalues already we have worked it out if you add all this if you take the modulus then you are going to get 3 by 2 so this implies that the logarithmic negativity would be log 2 3 by 2 so this is the answer that is required.

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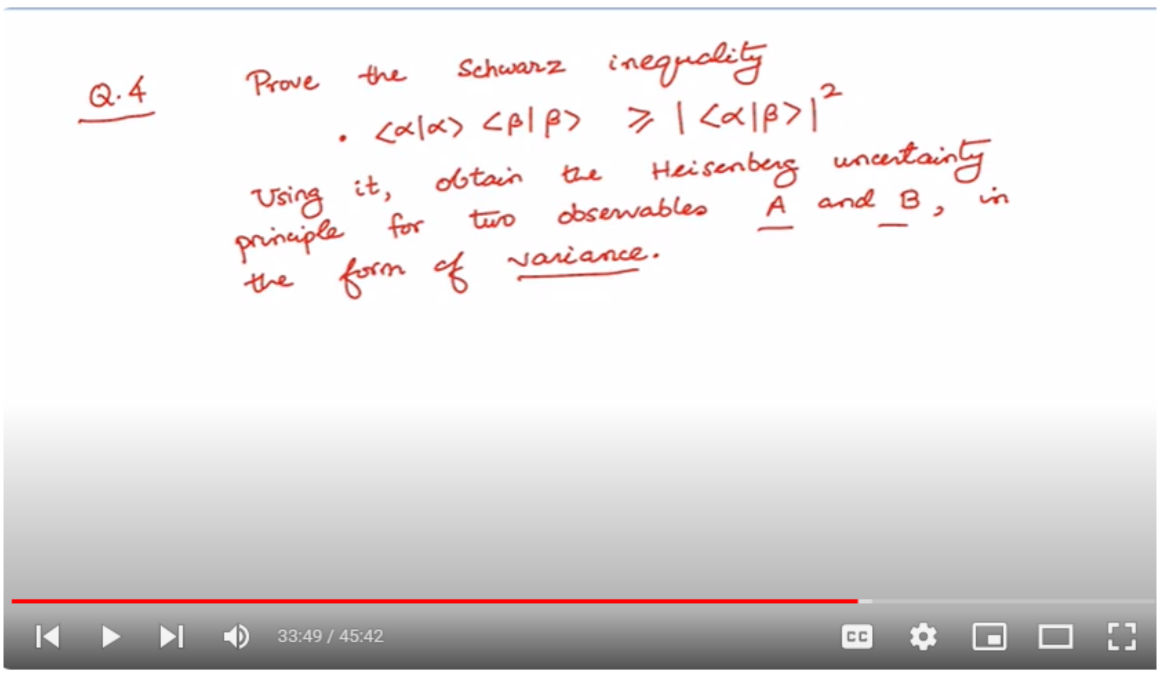

As a final problem let us work out this problem you have to prove the Schwarz inequality and using it obtain the Heisenberg uncertainty principle for two observables A and B in the form of variance we encountered variance in the context of Duan criteria that we have discussed Heisenberg uncertainty principle actually plays very important role when we discuss about continuous variable entanglement. So let us work it out.

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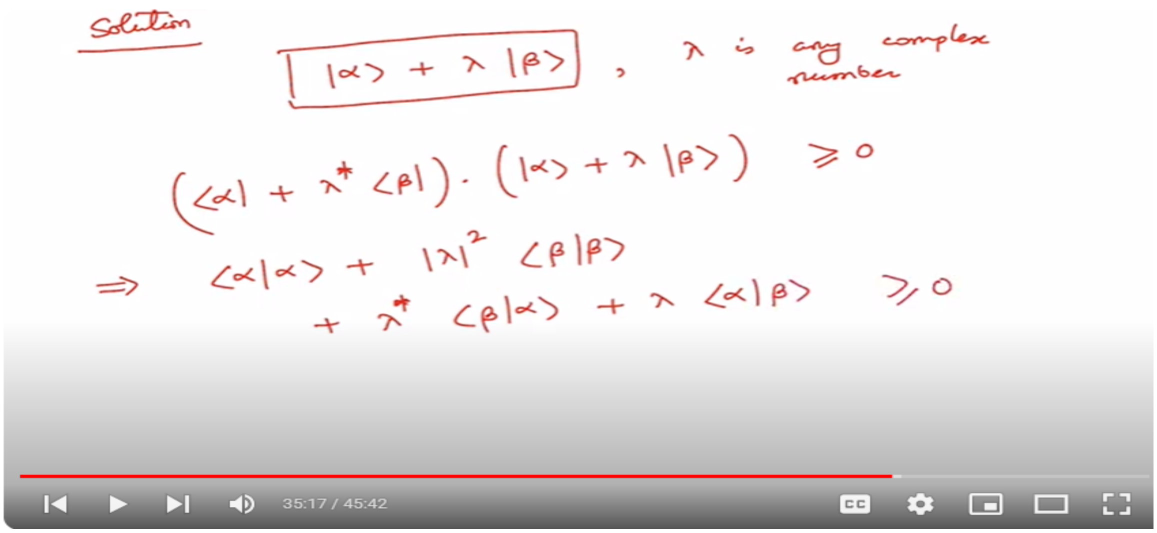

First of all let me consider a state vector let us say ket alpha plus lambda ket beta where lambda is any complex number and now let us take the scalar product of this vector with itself and if we do that we will get bra alpha plus lambda star beta scalar product with ket alpha plus lambda beta and you know that the scalar product is always positive it is greater than or equal to zero. So we have this inequality now from here let me open it up we will have bra alpha scalar product of alpha with itself then mod lambda square scalar product of beta with itself plus lambda star scalar product of beta and alpha plus lambda into scalar product of alpha and beta and this is greater than or equal to zero.

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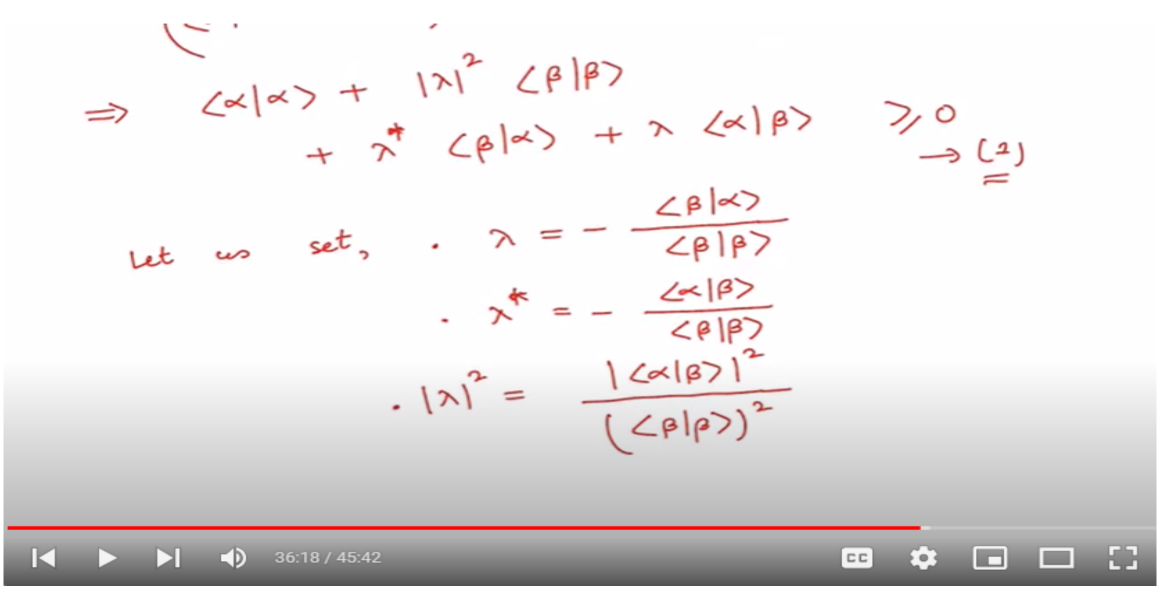

Let's say this is my equation number one. let me take let us set now because lambda is a complex number let us set lambda is equal to it is just a complex number I am taking say beta alpha scalar product of beta alpha divided by scalar product of beta with itself and then you will have lambda star would be equal to minus alpha beta here beta beta and mod lambda square would be equal to you will get it as mod of alpha beta whole square and beta scalar product of beta with itself whole square this is what you will get as mod lambda square.

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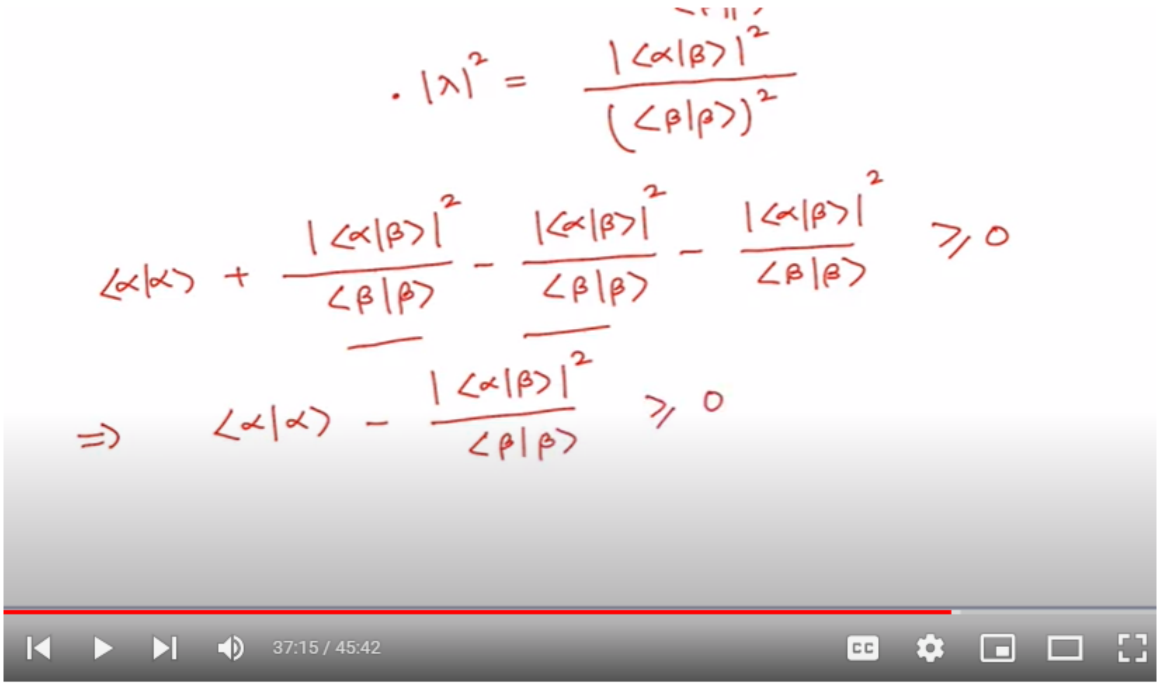

Now let me put all these lambdas lambda lambda star and mod lambda square in this equation one if I do that then I will get scalar product of alpha with itself plus modulus of scalar product of alpha beta whole square divided by beta beta minus modulus of alpha beta it's very easy to do this it's a simple algebra beta beta and finally I will have modulus of alpha beta mod square beta beta this is greater than equal to zero so this term get cancelled so from here you will immediately get alpha alpha minus modulus of alpha beta whole square divided by beta beta greater than or equal to zero

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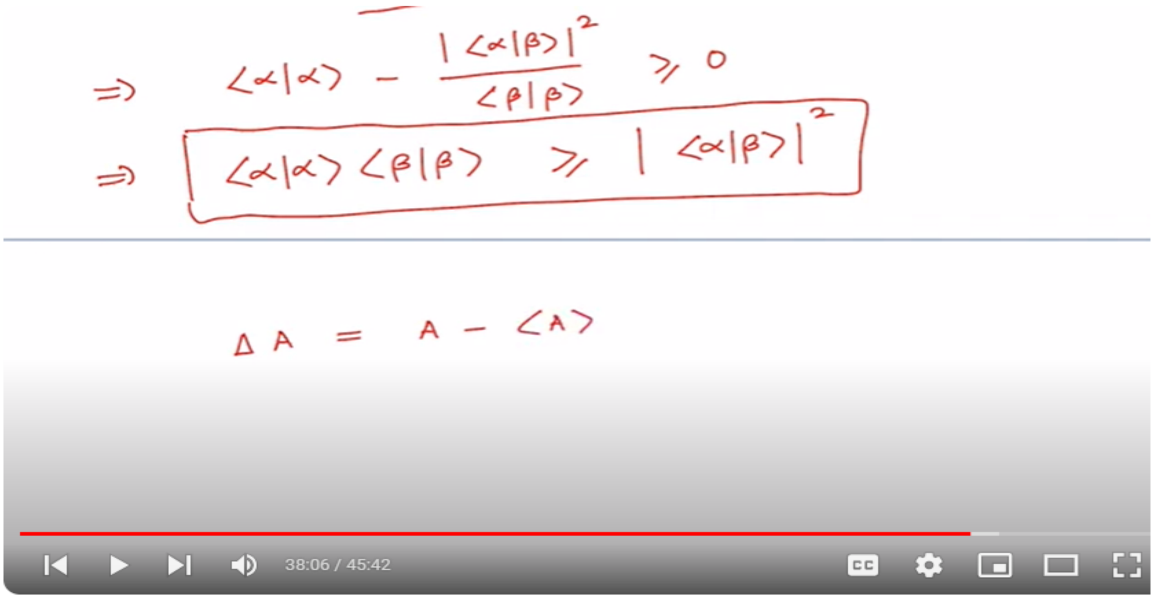

so very trivially you obtain the required inequality alpha alpha beta beta is greater than or equal to modulus of alpha beta scalar product of alpha beta whole square so this is the famous Schwarz inequality and this is what is required.

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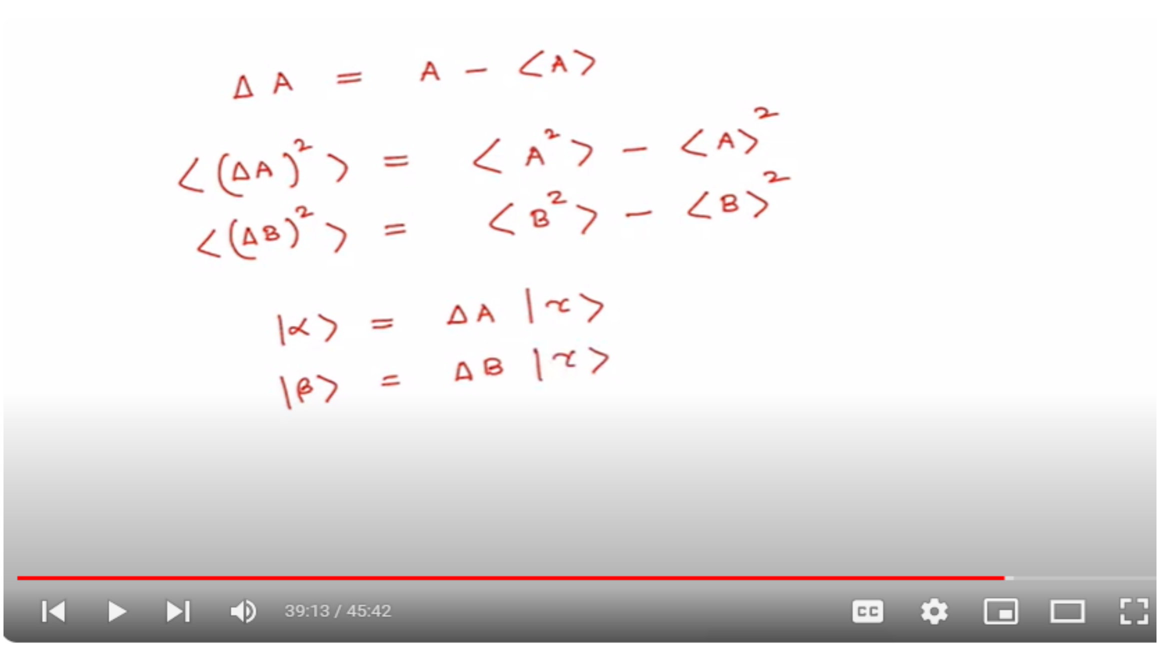

Now let us do the second part and obtain the Heisenberg's uncertainty principle for observables A and B let us say we have delta A A is operator here, A and B are now operators I am not going to use the operator sign however but please understand that I am now talking about operators let's say delta A is equal to A minus expectation value of A then expectation value of delta A whole square which you can very trivially work out and show that this would be equal to this is the variance actually A square expectation value of A square minus expectation value of A whole square right? The square of the expectation value of A similarly delta B expectation value of delta B square this is the variance of B would be expectation value of B square minus average or expectation value of B whole square.

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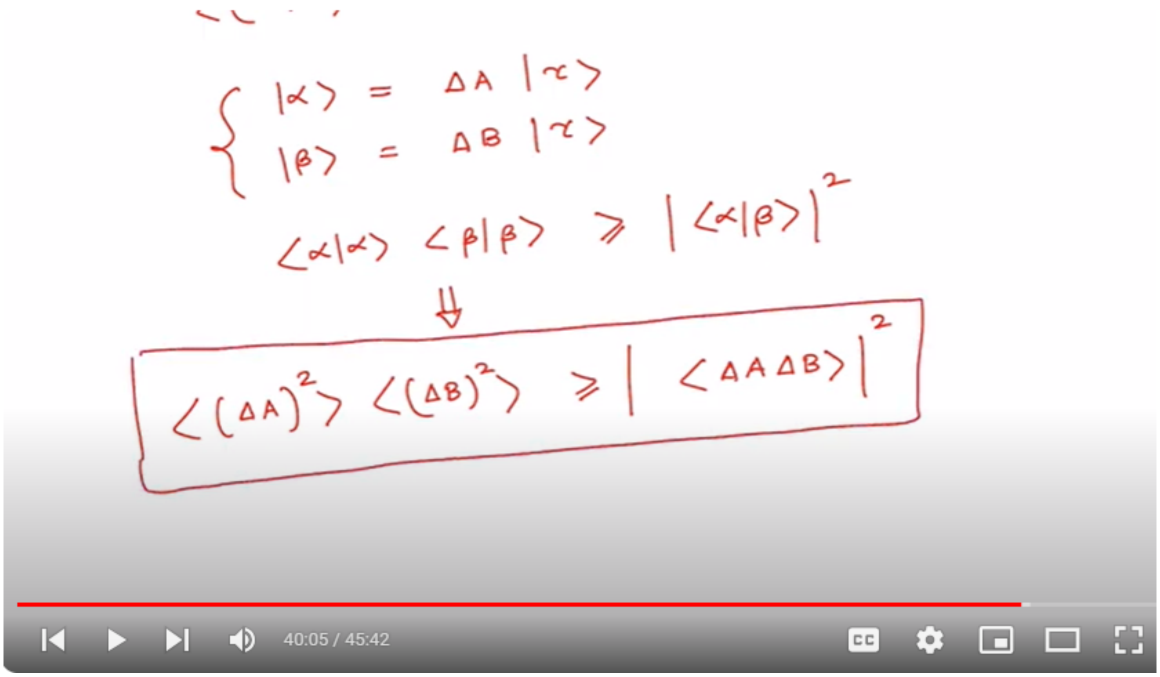

Okay now in the Schwarz inequality let me put ket alpha is equal to delta A this is an operator operating on some arbitrary ket say ket gamma and then ket beta is equal to which is the result of the operation of the operator delta beta on the arbitrary ket gamma and if I put this in the Schwarz inequality if I put these things in this Schwarz inequality you can very easily show let me write the Schwarz inequality from here if you use this here then you will obtain very easily this equation expectation value of delta A square and expectation value of delta B square is greater than or equal to modulus of the product of expectation value of the product of delta A delta B then you take the modulus square. So this is you are getting from the so called Schwarz inequality

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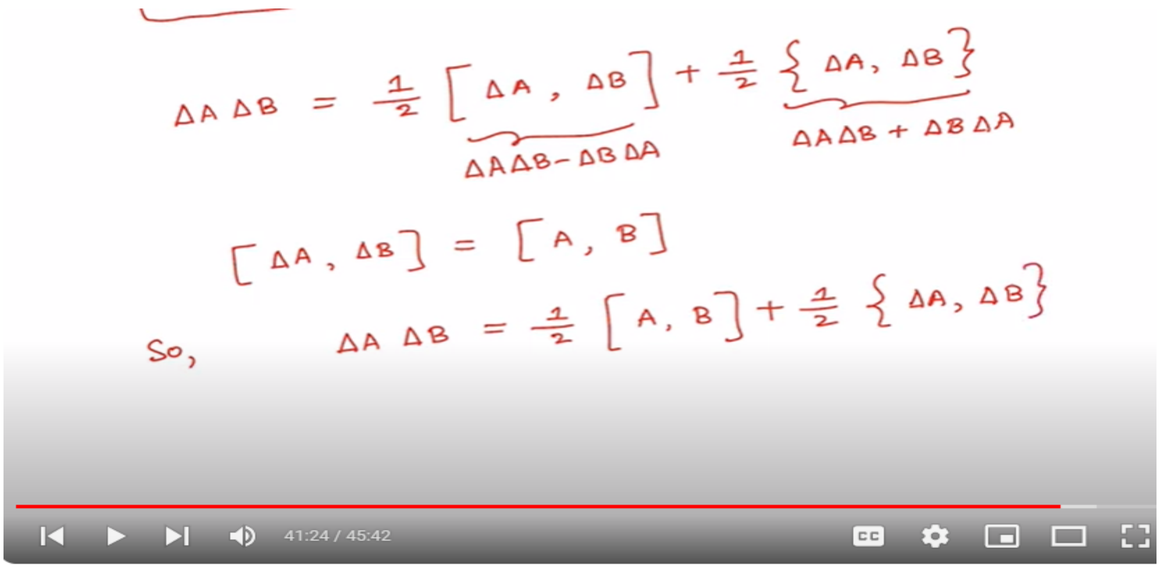

Now you see to get the Heisenberg uncertainty principle let me write the product of these two operators delta A and delta B in this form I can write it in terms of the commutator delta A delta B and the anti-commutator one half delta A delta B and commutator you know this is the commutator this is delta A delta B minus delta B delta A and this anti-commutator is delta A delta B plus delta B delta A now you can you can also show that this commutator delta A delta B is equal to commutator of the operator A and B so we have the product of delta A delta B is equal to one half commutator AB plus one half delta A delta B this is anti-commutator

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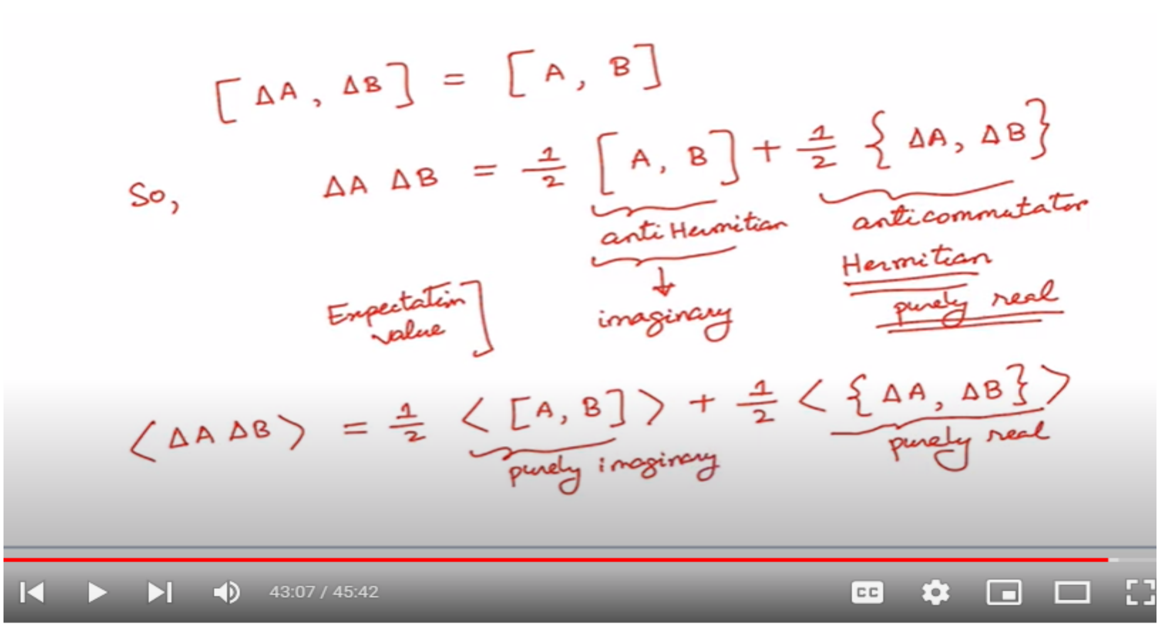

now you see this anti-commutator is this is anti commutator as I have already explained this anti-commutator is Hermitian this is Hermitian and this commutator is anti-Hermitian this is anti-Hermitian because it is anti-Hermitian we know that the expectation value of a Hermitian operator this is Hermitian a Hermitian operator is purely real expectation value is real on the other hand for an anti-Hermitian operator expectation value expectation value of anti-Hermitian operator is imaginary this is imaginary and this is real for a Hermitian operator it is real

so let me write put it here delta A delta B expectation value of the product of delta A delta B is equal to one half expectation value of commutator AB plus one half expectation value of anti-commutator delta A delta B and here this is as I said this is because it is Hermitian it will be purely real and this is going to be purely imaginary this is going to be purely imaginary

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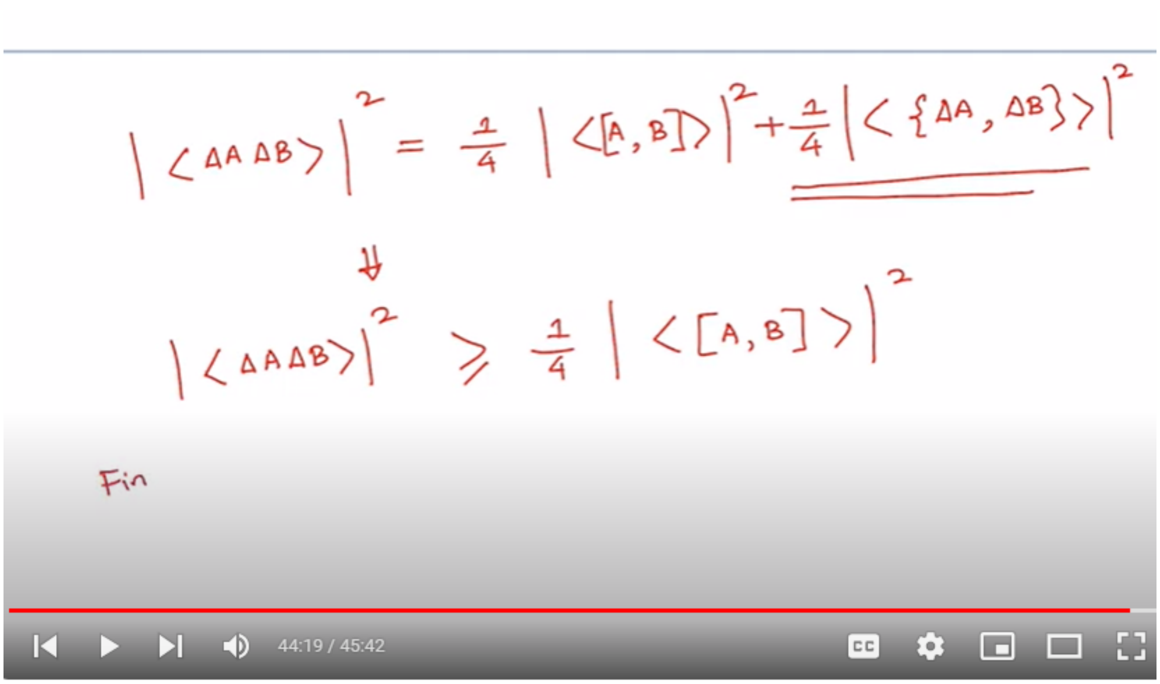

so if I take now the modulus of the expectation value of delta A delta B then we will get one by four this modulus of expectation value of the commutator AB plus one by four because I am now taking the modulus I am going to get a positive number positive number here delta A delta B mod whole square right since this last term is a real number positive real number this implies I can now write modulus of the expectation value of delta A delta B whole square is greater than or equal to one by four modulus of the expectation value of the commutator AB whole square

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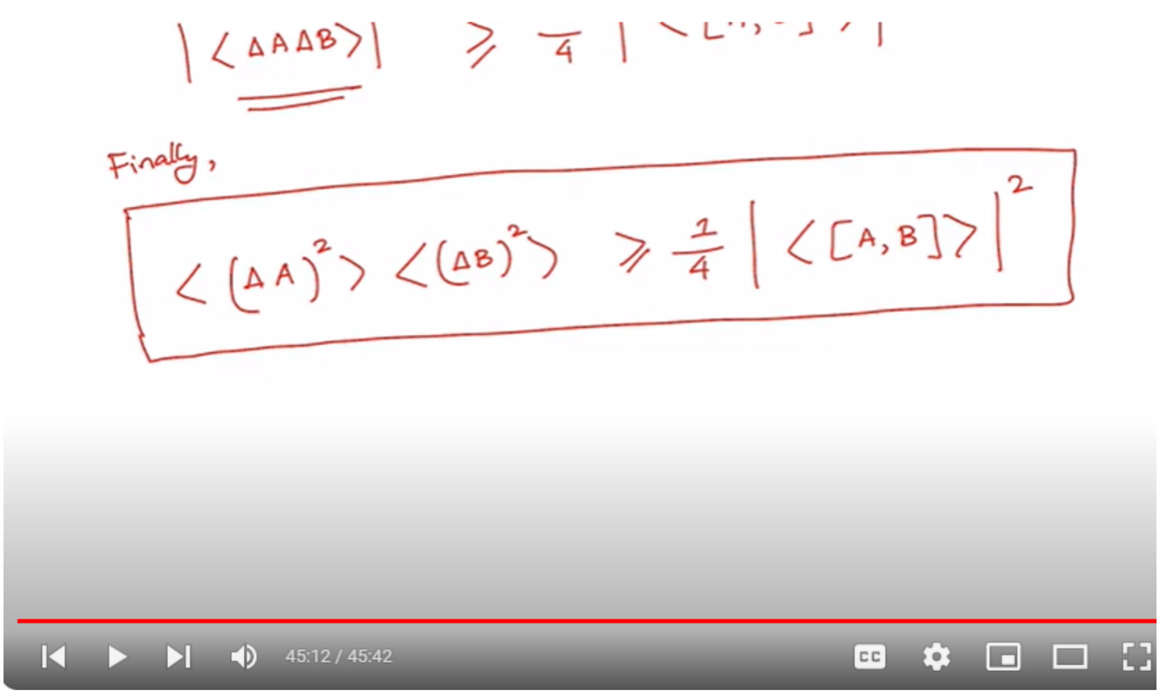

and therefore finally from here from here finally therefore we can write I have this guy with me so if you from this Schwarz inequality I have this expression I have worked out worked this quantity out so if I put it in the Schwarz inequality then I will finally obtain expectation value of delta A square into expectation value of delta B square is greater than or equal to one by four expectation modulus of the expectation value of AB commutator AB whole square right so this is the so called Heisenberg uncertainty relation written in a general form for any two observable A and B.