Quantum Entanglement: Fundamentals, measures and application

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Week-04

Lec 15: Applications of Quantum Entanglement-I

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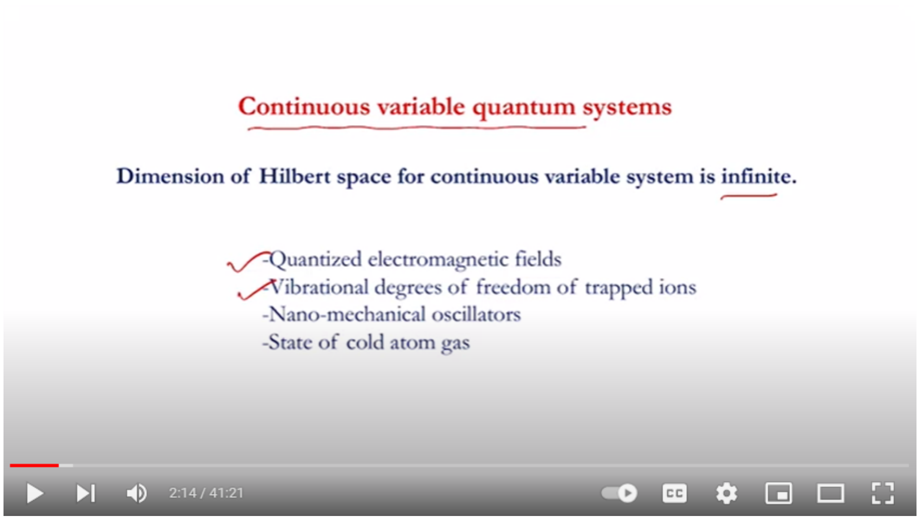

Hello, welcome to lecture 11 of this course. This is lecture number 1 of module 4. In this lecture, we will discuss some applications of quantum entanglement. However, before I do that, in continuation with the previous class on quantum entanglement measure, in this lecture, I will discuss very briefly about continuous variable entanglement. So let's begin. In the last two classes, we have discussed quantum entanglement measures in the context of discrete variables.

Discrete variable systems such as spin-half systems are bit difficult to realize experimentally for quantum information science applications. On the other hand, continuous variable systems are pretty easy to realize experimentally or handle experimentally. However, there are difficult issues involved with continuous variable systems. We know that the dimension of Hilbert space is infinite for continuous variable system.

Some examples of continuous variable systems are say quantized electromagnetic fields. When quantized, you know, electromagnetic fields act like a collection of independent harmonic oscillators having amplitude and phase quadrature as its variables. Then we have vibrational degrees of freedom of trapped ions. You know trapped ion is one of the prominent candidate for quantum computer. Then we have nanomechanical oscillators, then state of cold atom gas and so on.

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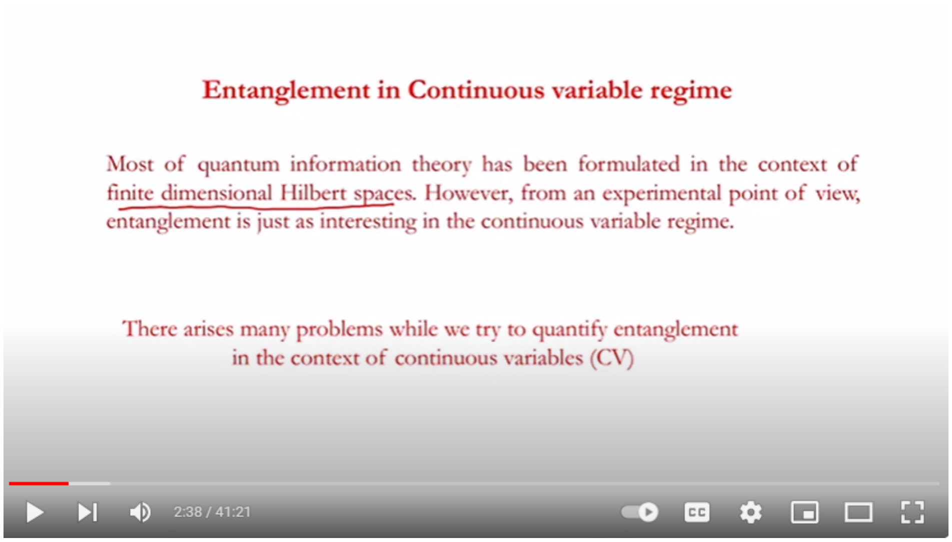

Now, as regards entanglement in continuous variable regime is concerned, so far what we have seen is that most quantum information theory has been formulated in the context of finite dimensional Hilbert space. But there arises many problems if we extend it to continuous variable system.

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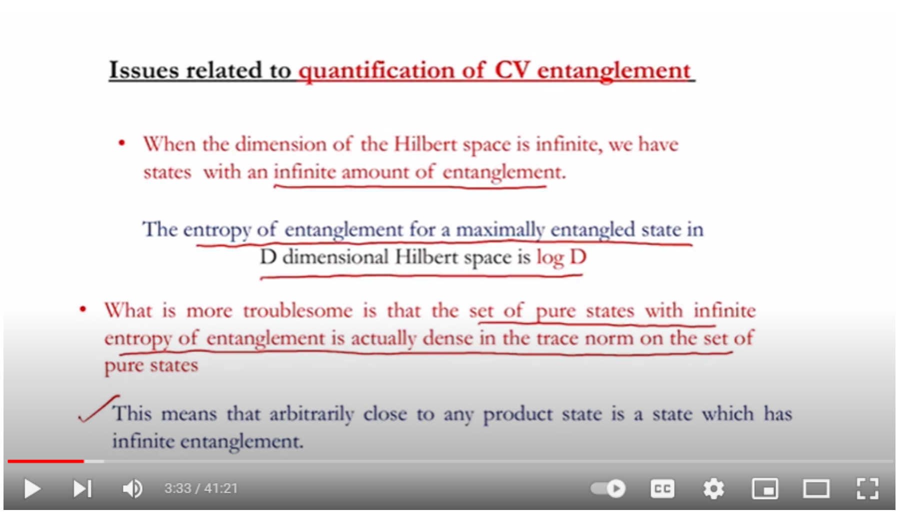

One problem is that there are many issues. Some of the prominent one are that the dimension of the Hilbert space here in continuous variable system is infinite as I said. And as a result, we have states with an infinite amount of entanglement.

And this should not surprise you because in earlier class I told that the entropy of entanglement for a maximally entangled state in d dimensional Hilbert space is log d. And now we have to deal with infinite dimensional Hilbert space, right? But what is more troublesome is that the set of pure states with infinite entropy of entanglement is actually dense in trace norm on the set of pure state. It is basically a bit technical but the point here is that this basically means that the arbitrarily close to any product state is a state which has infinite entanglement.

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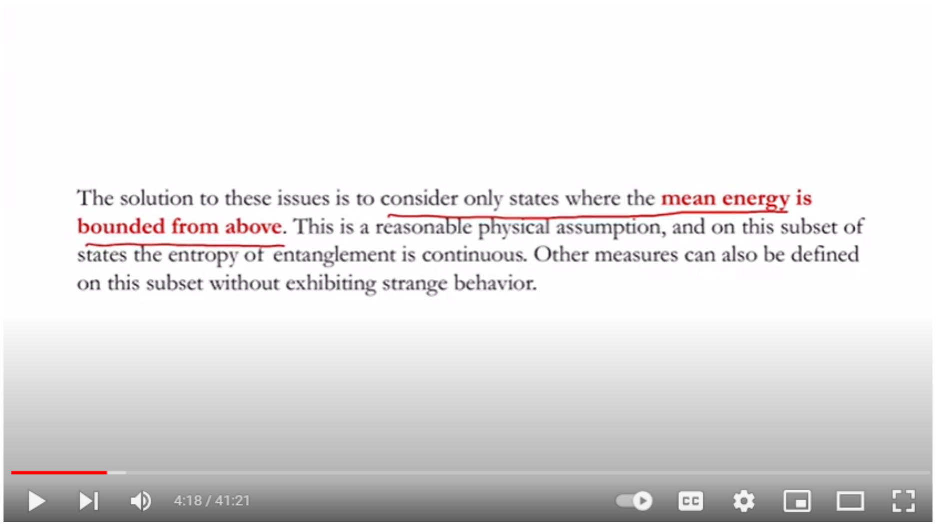

So this basically forbids us to develop quantum entanglement measures in the context of continuous variable system. But there people have suggested some solution and maybe it is some of the solution would be that okay consider only states where the mean energy is bounded from above.

And this is a reasonable physical assumption and on this subset of states the entropy of entanglement is continuous. Other measures can also be defined on this subset without exhibiting any strange behavior.

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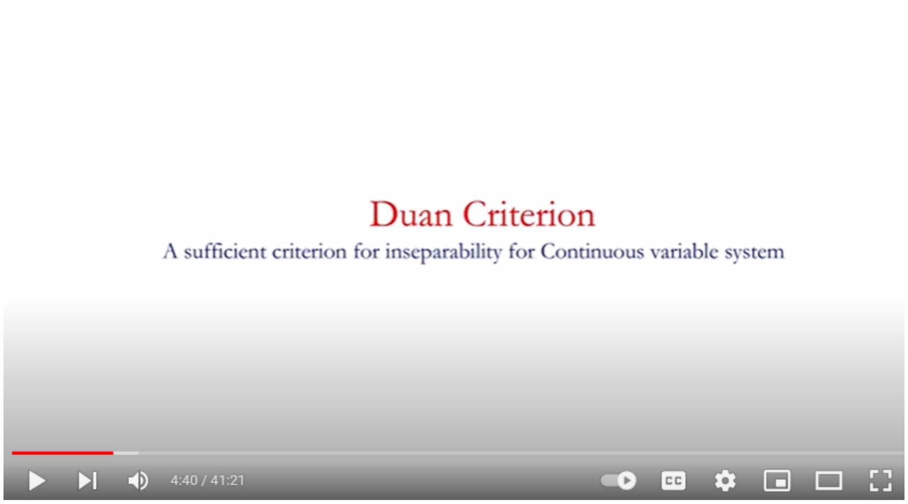

Now I will not go into much of technical details here but I will just mention one continuous variable criterion which is related to the so called EPR paradox and this criterion is called Duan criterion. And this criterion is a sufficient criterion for inseparability for continuous variable system. So let us discuss it briefly.

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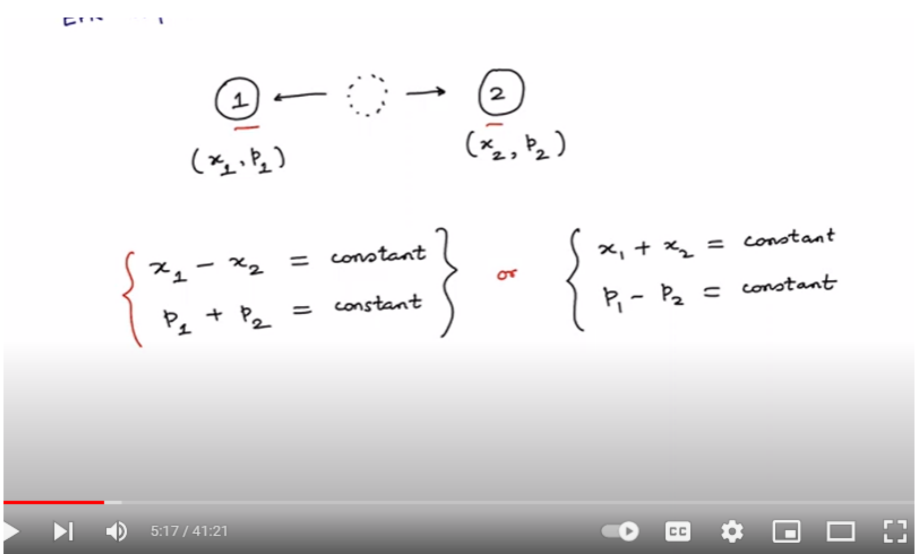

In EPR thought experiment, Einstein, Podolsky and Rosen considered a pair of particles 1 and 2 created at some point at some time moment so that conservation of momentum led to the equation x1 minus x2 is equal to constant. And p1 plus p2 is equal to constant or one may also get x1 plus x2 is equal to constant and p1 minus p2 is equal to constant.

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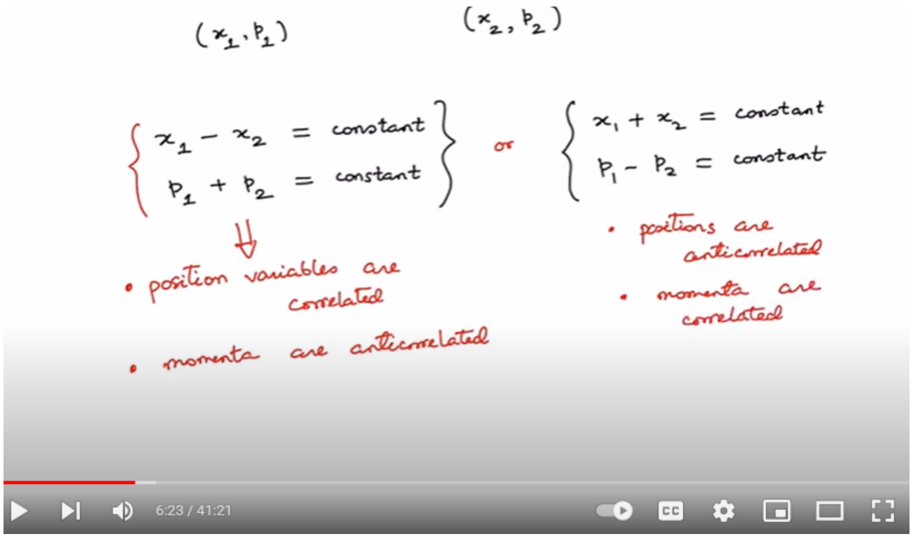

In this first case here this implies that the position variables are correlated and these relations we have discussed in an earlier class on EPR paradox. Here positions variables are correlated and momentum variables, momenta, momenta are anti-correlated. On the other hand in this case here also we may get the opposite thing where positions are anti-correlated and momenta are correlated.

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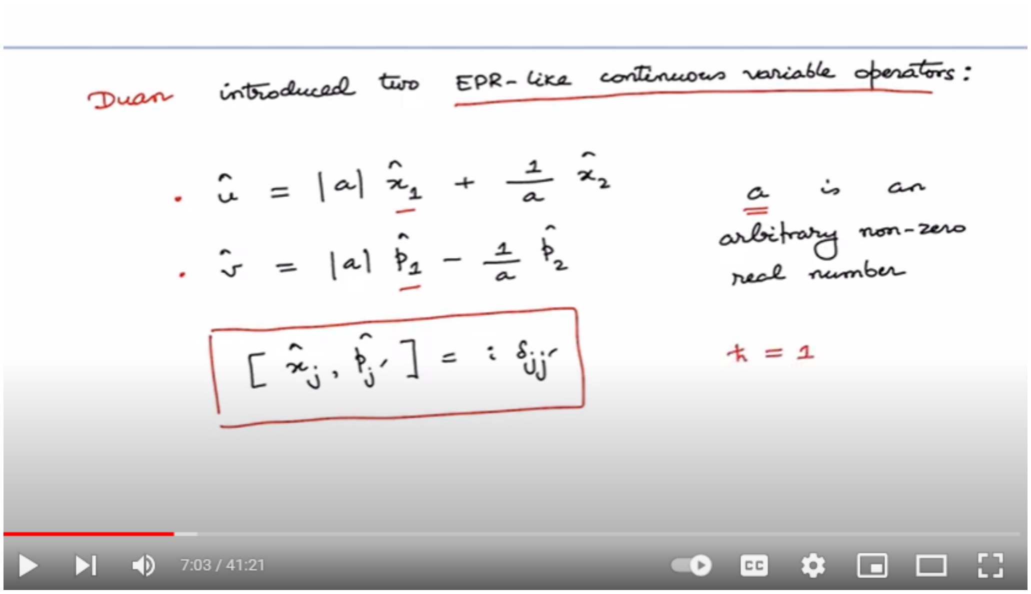

Alright. Now taking the idea from EPR experiment, Duan introduced two EPR like continuous variable operators u and v. u is equal to mod a x1 plus 1 by a x2 and v is equal to mod a p1 minus 1 by a p2 where a is an arbitrary non-zero real number and this position variable and the momentum variable has to satisfy this commutator. Here we take generally h cross is equal to 1 which is in the natural unit.

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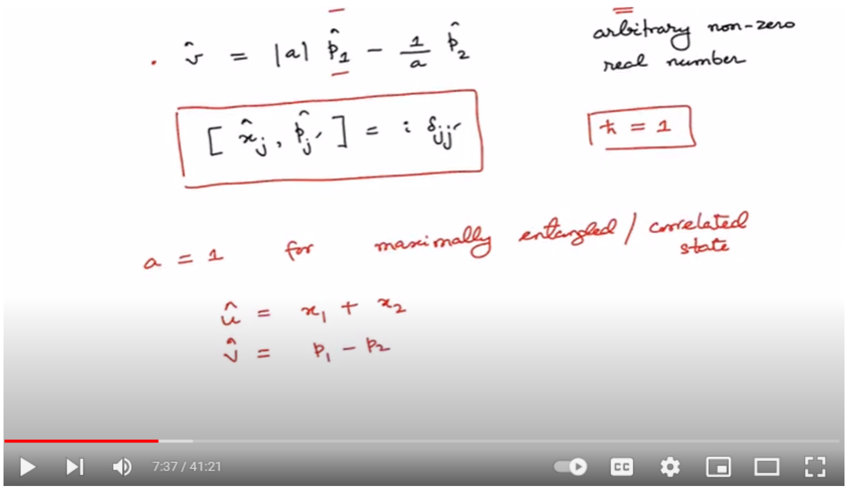

And a is equal to 1 for maximally entangled state, maximally entangled or correlated state. And in that case you will get u is equal to x1 plus x2 and v is equal to p1 minus p2.

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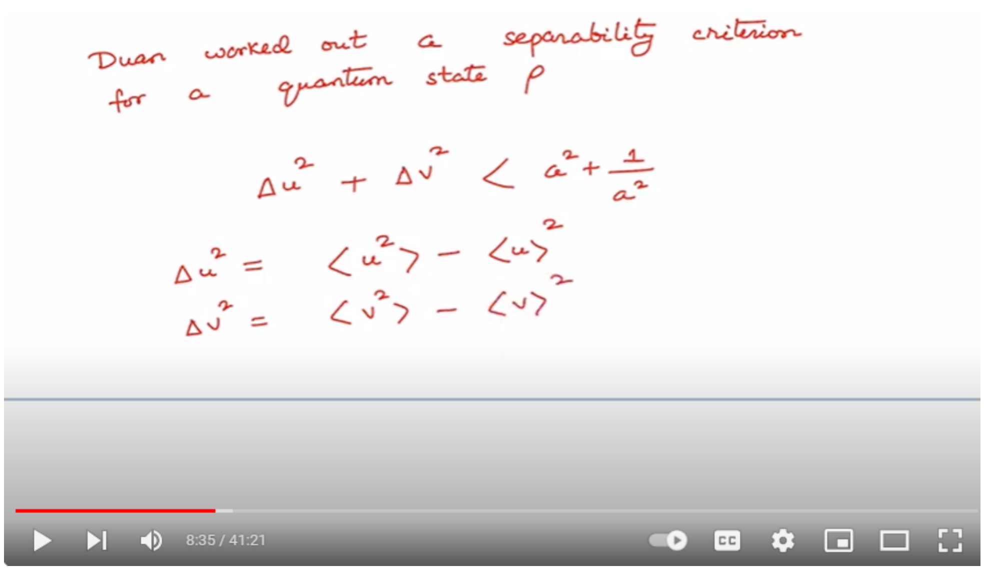

Okay. So Duan worked out a separability criterion based on these two operators for a quantum state rho representing a continuous variable system and he found that the variance in u that is delta u square and variance in v that is delta v square is less than a square plus 1 by a square. Where delta u is the variance in u and it is defined as it is the expectation value of u square calculated for the state rho minus square of the expectation value of u. And delta v square is similarly the expectation value of v square minus square of the expectation value of v.

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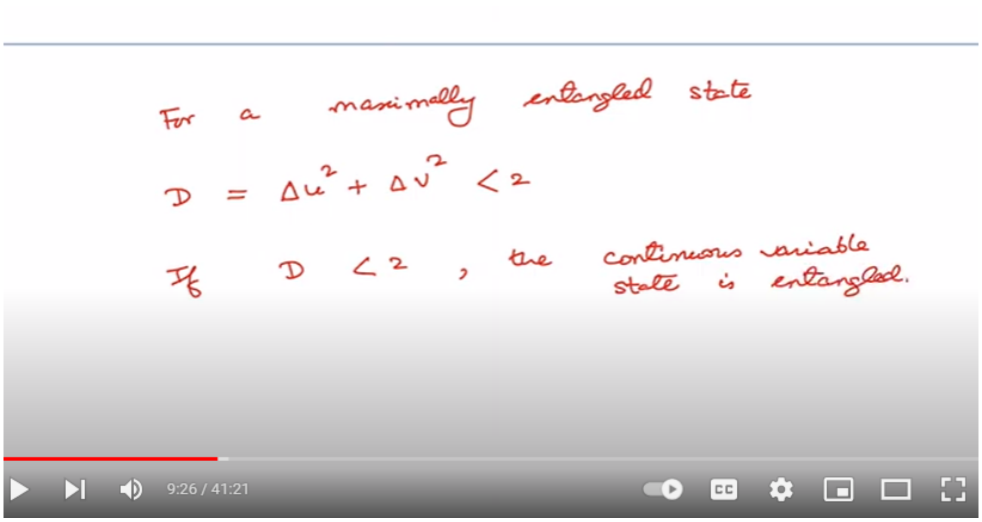

And for a maximally entangled state, this criterion the total variance delta u square plus delta v square which is in short denoted by this symbol D is less than 2. And if that means if this quantity capital D is less than 2, this implies the continuous variable state, the continuous variable state is entangled.

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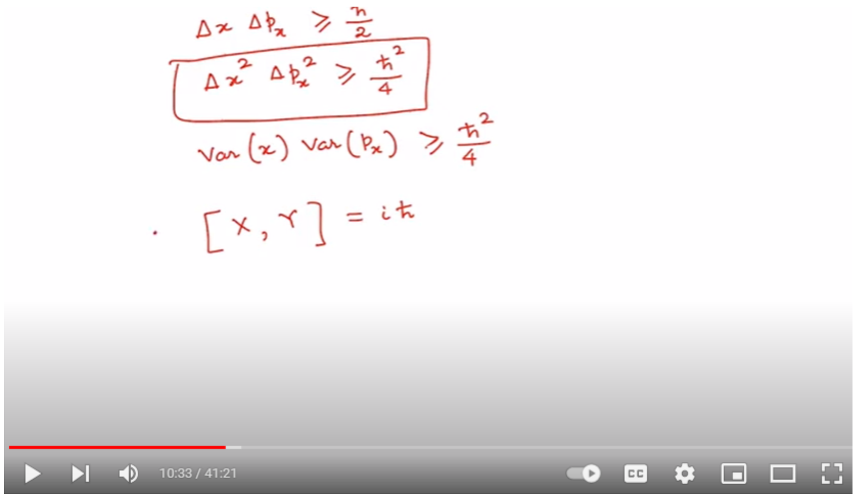

Let me now make a useful comment as a reminder to you as we are talking about variances. You know the famous Heisenberg uncertainty principle has this form delta x delta px is greater than or equal to h cross by 2 where h cross is the reduced Planck's constant. This we can write in terms of variance in this form delta x square delta px square is greater than or equal to h cross square by 4. Sometimes people write this expression in this form as well. Variance var variance of x into variance of p of x is greater than or equal to h cross by 4. In fact, you know that any variable, any set of variable x and y if it satisfy this commutation relation xy is equal to ih cross, then you can write a uncertainty relation for these two set of variables x and y.

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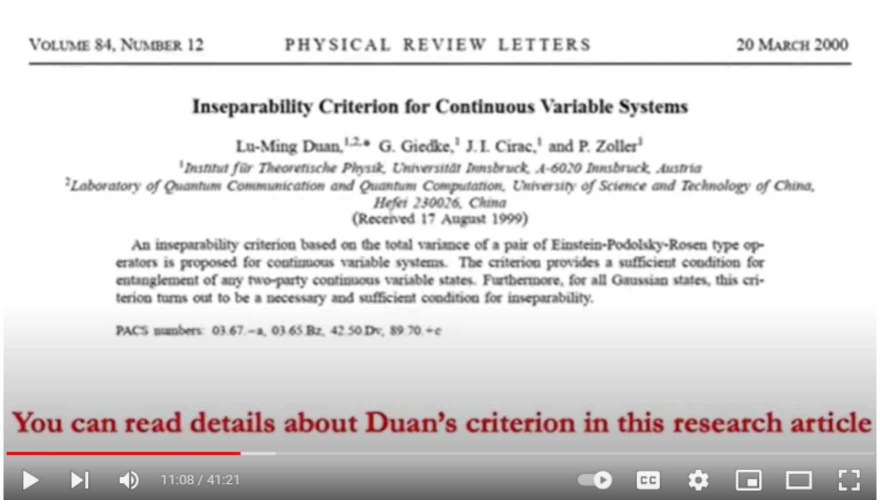

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Inseparability Criterion for Continuous Variable Systems

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An inseparability criterion based on the total variance of a pair of Einstein-Podolsky-Rosen type operators is proposed for continuous variable systems. The criterion provides a sufficient condition for entanglement of any two-party continuous variable states. Furthermore, for all Gaussian states, this criterion turns out to be a necessary and sufficient condition for inseparability.

PACS numbers: 03.67.–a, 03.65.Bz, 42.50.Dv, 89.70.+c

Here, as regards Duan criterion is concerned, I have given you a prescription only without going into any technical details. In fact, I have not even derived the Duan criterion which is too complicated and this is beyond the scope of our work also. But if time permits, in a later class I will focus exclusively on continuous variable entanglement. However, you can consider that as extra material only and it is not going to be part of your exam.

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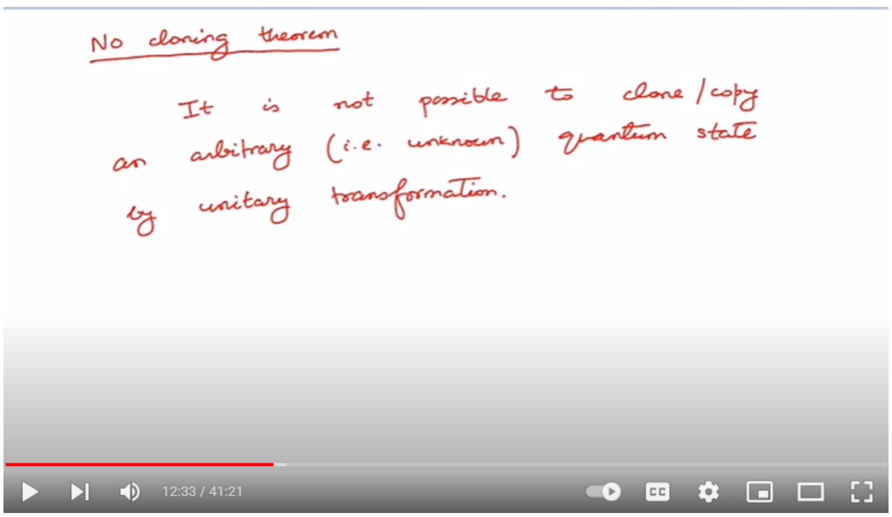

I will now discuss how quantum entanglement can be useful for the realization of so called super dense coding. In the next lecture, I will discuss quantum teleportation which is somewhat analogous to super dense coding. However, before I discuss that, we will need some concept and ideas. One such concept is the so called no cloning theorem. So let us discuss it first.

No cloning theorem is of paramount importance in quantum information science. No cloning theorem says that it is not possible to clone or copy an arbitrary quantum state by unitary transformation. So this is the no cloning theorem and by now all of you know what is called unitary transformation. Now as per no cloning theorem, it is not possible to clone or copy an unknown or arbitrary quantum state.

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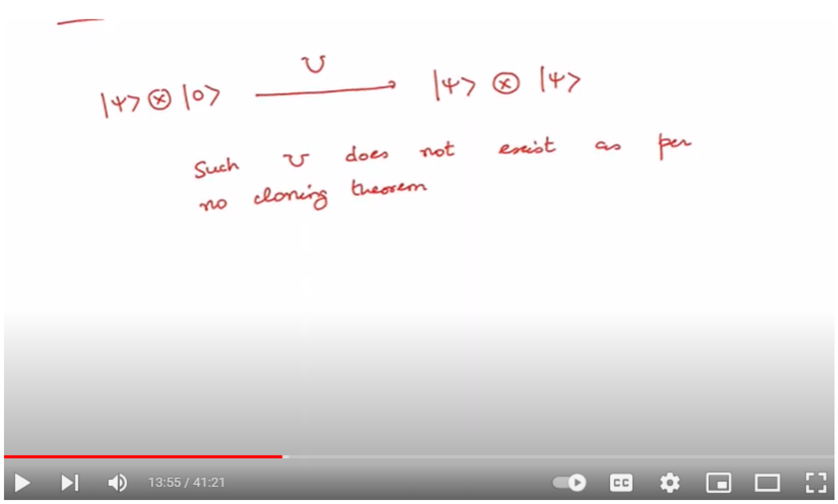

Let us prove it. In order to prove it, let me first explain what we mean by cloning. Cloning basically requires that you find an unitary transformation U such that suppose you have a state ket psi and you want to copy it to an empty state represented by this, say blank state ket 0. And doing cloning means that by this unitary transformation you can write this ket psi into this blank state. This is what we mean by cloning. No cloning theorem says that no such unitary transformation does not exist as per no cloning theorem.

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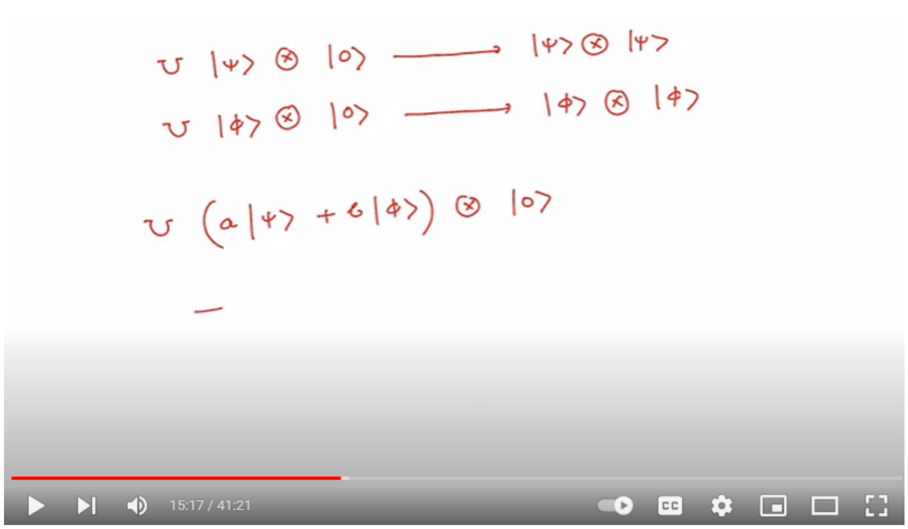

In order to prove that, let us assume the opposite thing, the contrary thing. That means let us assume that such U exists. Then what is going to happen? Suppose you have two state, two linearly independent quantum states, say you have ket psi and ket phi are two linearly independent states. And you have this unitary transformation by which you can copy this state psi to the blank state U, empty state U. So because of this you are going to get ket psi direct product ket psi.

And in this case, the other case you can copy ket phi to the empty state or the blank state and you will get ket phi tensor product ket phi. Now again as I said k, this ket psi and ket phi are linearly independent so we can have a superposition state, say a ket psi plus b ket phi. And this state also if such unitary transformation exists, I can copy it to the blank state.

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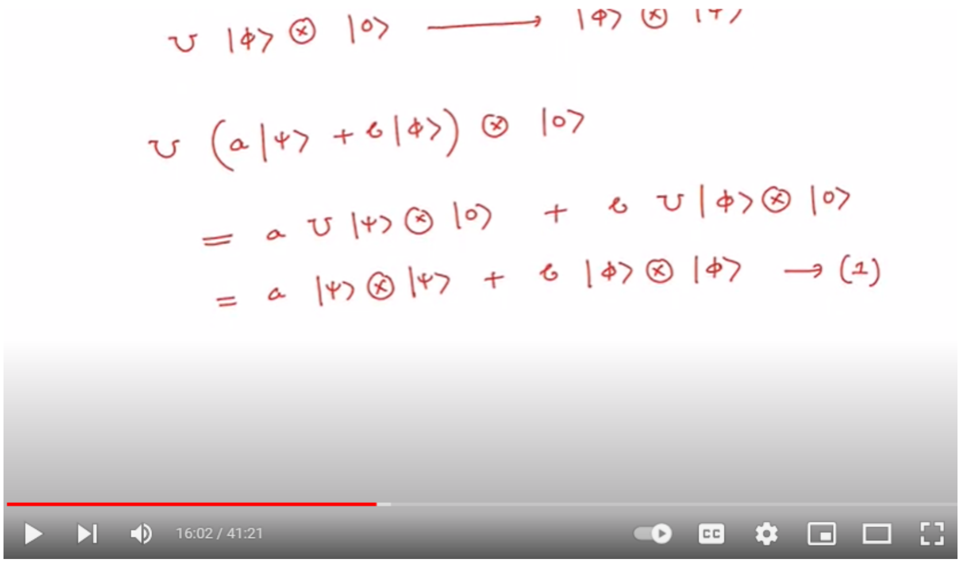

So if I do that, if we can do that then this will give us, say because of the linearity let me just write, you will get A U ket psi direct product ket 0 plus B unitary operator applied on ket phi direct product or tensor product 0. And because of this unitary operation you will get A ket psi direct product ket psi plus B ket phi direct product ket phi. Okay, so this is what you will get. Let me say this is my equation number 1.

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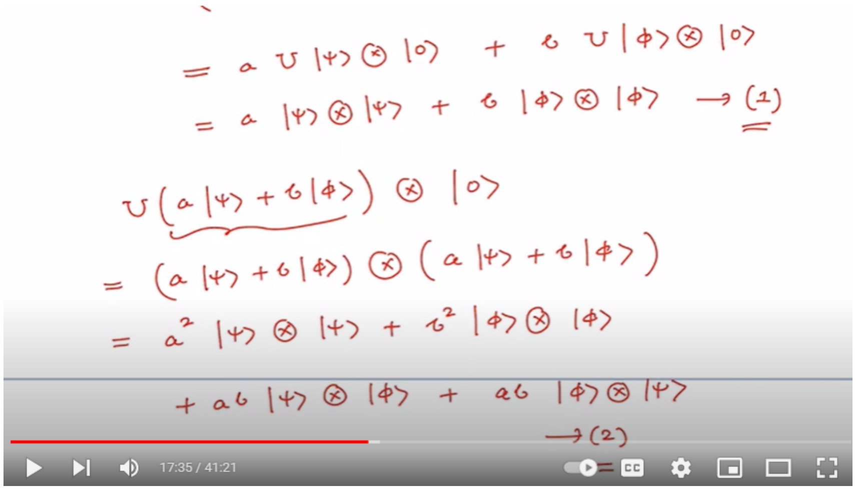

But you see if this transformation you can clone arbitrary states, it should give for any arbitrary A and B you should be able to write A say ket psi plus B ket phi. Then if I can copy this to the whole state, this whole thing I can copy to the blank state that means I will get A ket psi plus B ket phi direct product or tensor product A ket psi plus B ket phi. Right, and if I now open it up then I am going to get A square ket psi direct product ket psi plus B square ket phi direct product ket phi plus A B ket psi direct product ket phi plus A B direct product of ket phi and ket psi. So you will get this. Let us say this is my equation number two. Now if you look at equation one and equation two, what you see is that they are different unless A and B A or B is zero. So clearly the transformation u does not exist.

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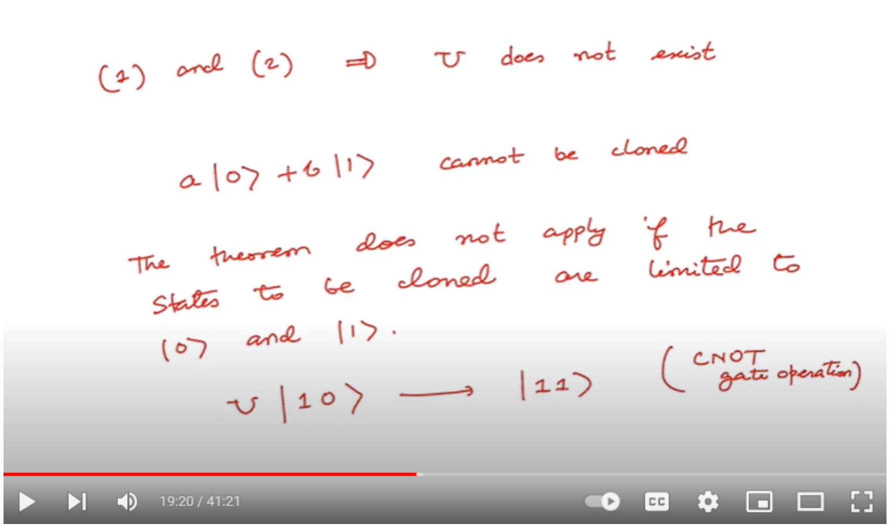

So equation one and two implies that u does not exist because A or B has to be equal to zero and then the whole thing would have no meaning at all. So no cloning theorem says that we cannot clone any arbitrary states such as say A ket zero plus B ket one where A and B are arbitrary.

These kind of states cannot be cloned. However there is a loophole and the loophole says that the theorem does not apply if the states to be cloned are limited to ket zero and ket one. Because the arbitrariness would not be there in that case. So you know that this ket zero and ket one and this is the reason why we have unitary transformation operations where we can go from ket one zero to ket one one. You may recall that this we can achieve by the so called C not ket operation.

So this is possible and in this case the no cloning theorem does not apply and this fact is actually exploited to construct quantum error correcting codes. So if the data under consideration are limited to ket zero and ket one we can copy qubit states even in a quantum computer.

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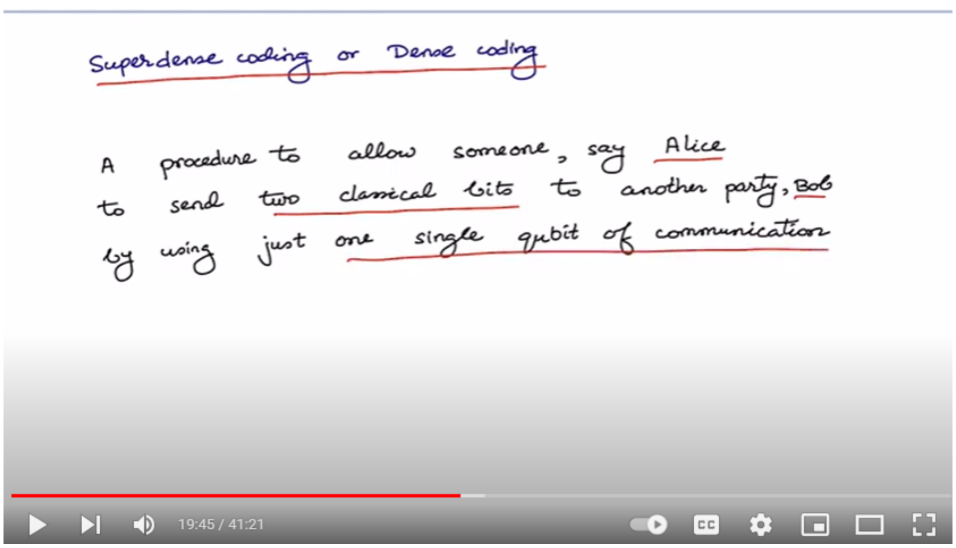

Now we will discuss our first application of quantum entanglement. We will discuss super dense coding or simply called dense coding. It is a procedure to allow someone say Alice to send two classical bits to another party say Bob by using just one single qubit of communication.

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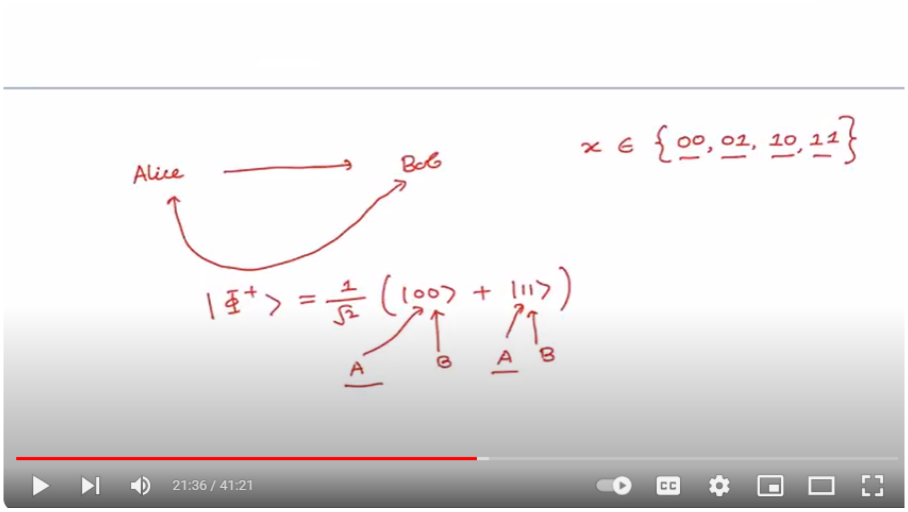

In other words Alice can transfer two classical bits to Bob by using a quantum channel. Here Alice will be the encoder and Bob will be the decoder. In this scheme Alice and Bob share an entangled state between them. Generally this entangled state is preferably a Bell state because you know that the Bell state is maximally entangled. This sharing of entangled state is at the core of super dense coding protocol.

Let me explain this protocol in some details. Alice wants to send classical bits to Bob and this classical bit may be anyone of this binary number say 00, 01,10,11 any of this four binary number Alice wants to send to Bob and Alice and Bob as per the super dense protocol they share the entangled state say a Bell state say phi plus and this phi plus as you know is one by root two ket zero zero plus ket one one. By sharing means the first qubit belongs to Alice and the second qubit belongs to Bob. And here the first qubit belongs to Alice and the second qubit belongs to Bob. Now depending on which of the classical bits whether it is zero zero zero one one zero or one one Alice wants to send that depending on that which one she wants to send Alice is going to carry out certain unitary operation on her qubits.

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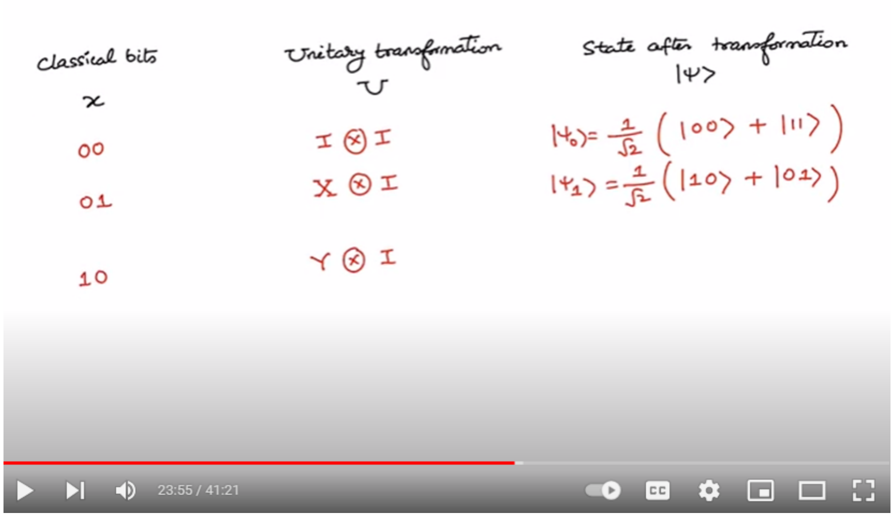

Say Alice wants to send the classical bits zero zero to Bob then Alice is going to apply the unitary transformation I to her qubits and on Bob's qubit also identity operation I is going to make.

That means applying identity operation means that nothing is done no experiment is done on either of the qubits. Anyway Bob's qubit cannot be touched by Alice and in this case if she wants to send the state zero zero classical bit zero zero to Bob then the protocol says that Alice is going to make identity operation on her qubit. And because of this operation the state after the transformation will remain as it is so because it is a Bell state is as I said is shared by both Alice and Bob and that is ket zero zero plus ket one one this is the state after the transformation. So let's say let me denote it by psi zero. So this is the state after the transformation.

Now if the classical bits that Alice wants to send this to Bob is zero one that in that case Bob is going to Alice is going to make the unitary transformation X which is basically not operation on her qubit. Bob qubit remaining as it is and then the state after this transformation would become after this unitary transformation Bob's qubit Alice qubit will become if it is zero initially then because of the not operation it will become one Bob's qubit will remain as it is. And if the qubit of Alice is one then it will become zero and Bob qubit remaining as it is. Now say classical bits one zero is to be sent by Alice to Bob then Alice is going to make the Y gate operation on her qubit Bob's qubit remaining unchanged.

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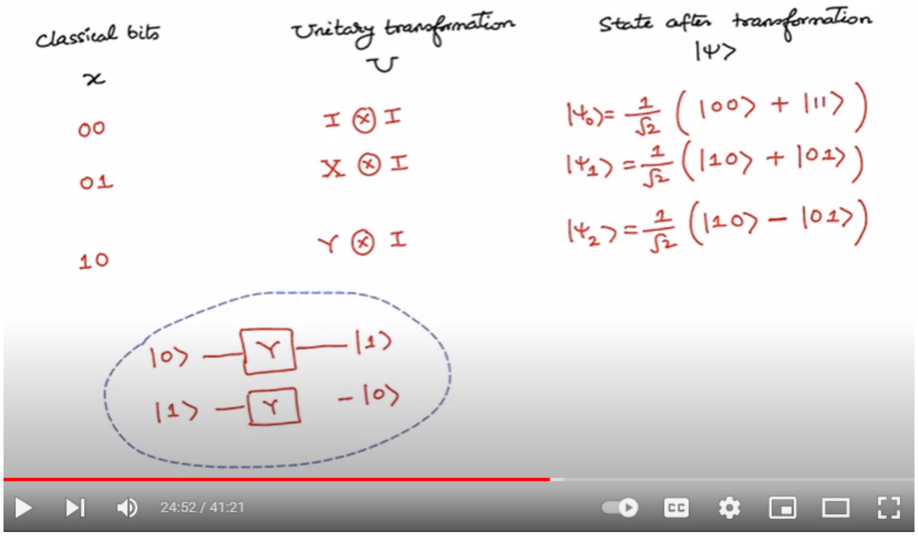

Now a Y gate is just like a NOT gate only with some difference. You know if ket zero is applied at the input then Y gate makes it flips it to one but if the input is ket one then Y gate flips it to zero but with a phase change of pi so therefore a minus sign will appear.

Now because of this Y gate operation the state after the transformation will get it as one by root two if Alice's qubit is zero then it will become one and Bob's qubit remaining unchanged and if Alice's qubit is one it will become zero with a phase change of pi so therefore minus sign will be there and Bob's qubit remaining as it is. So this is the state will get you know because of the Alice's operation.

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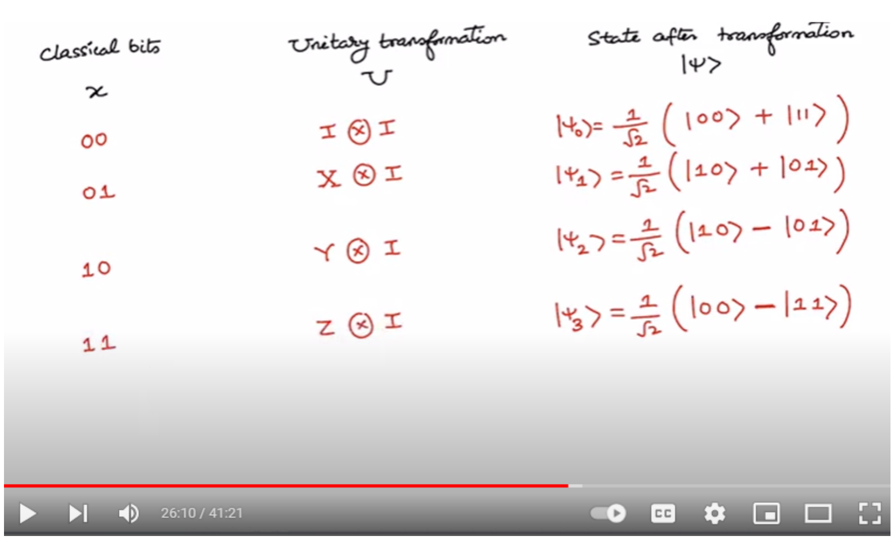

Finally let's say finally let's say Alice wants to send the classical bits one one to Bob then Alice is going to make Z transformation on her qubit and Z transformation is a simple operation where if the input is zero it will remain as it is at the output it will remain ket zero. But if the input is ket one then because of the Z operation at the output one will get a change of phase only so this is what one will obtain. So therefore the state after this transformation would become one by root two if the state is zero Alice's qubit is zero it will remain zero Bob's qubit is anyway getting unaffected and if the qubit Alice's qubit is one there would be a change of phase that will be a minus sign will appear here and Bob's qubit will remain as it is. So this is the state after the unitary transformation made by Alice one will get.

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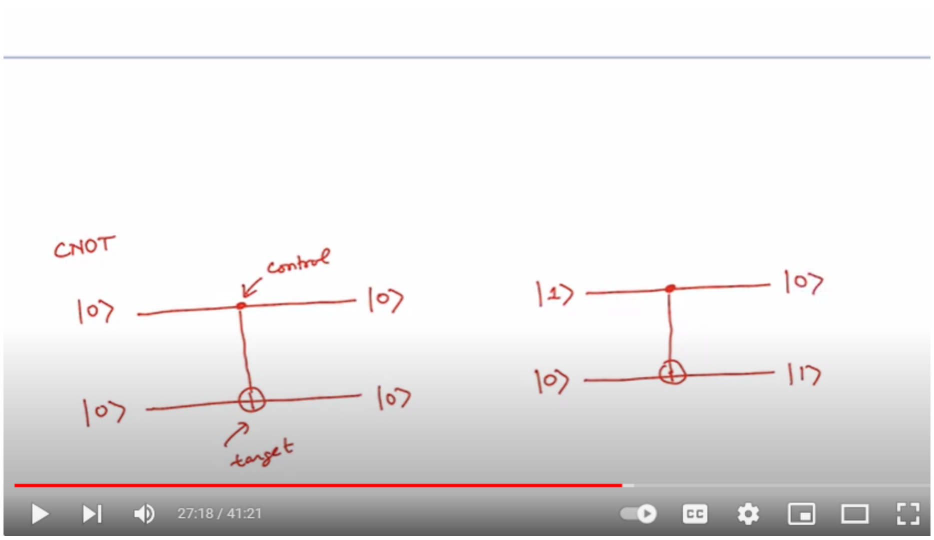

Okay now this transformed qubits after Alice does her experiment or operations this states is going to be transferred to Bob and after receiving the qubit from Alice Bob is going to make some operation in his laboratory. After Bob received the qubit from Alice he does CNOT gate operation on the entangled pair in which the first qubit which is the received qubit is the control bit and the second qubit which Bob already has is the target qubit. Let me remind you about the CNOT gate operation if the here we have a control bit the first qubit is the control and the second qubit is the target if the control is zero and say the target qubit is zero then target qubit remains as it is the output would be zero zero. But if the control qubit is one then the target qubit gets flipped so target qubit will get flipped and the output will get in this case would be just like this. So this is the essence of CNOT gate operation.

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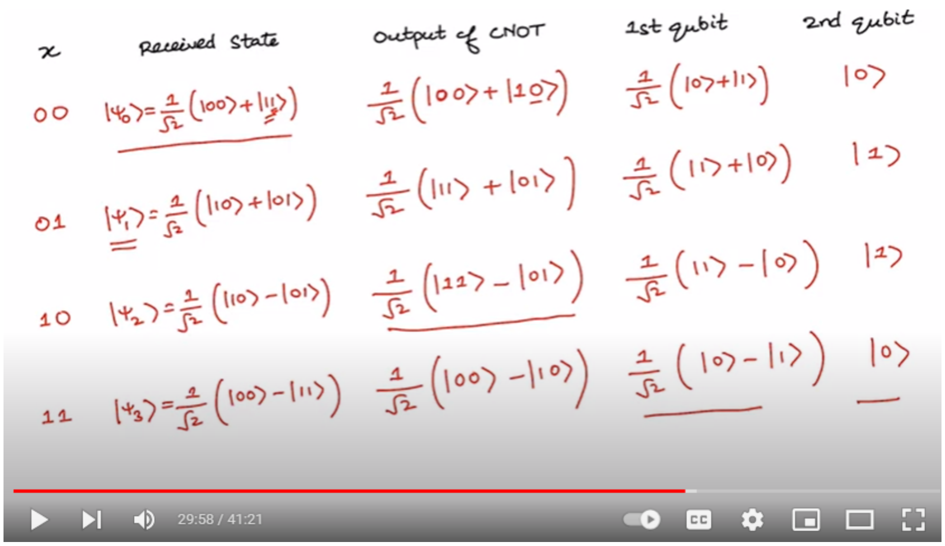

Now in Bob's laboratory suppose the received qubit is psi zero then as a result of the operation by Bob CNOT operation the output would be one by root two in Bob's laboratory. Because the Bob's Alice qubit is now going to be control qubit so Bob's qubit will remain as it is if it is zero it will zero we are just applying the CNOT gate operation so as you can see here it would become one zero because the Bob's Alice qubit is one so therefore Bob's qubit is one so it would get flipped. Similarly if it is zero one or the state received is ket psi one then the resultant output because of the CNOT operation in Bob's lab will become one one plus zero one. If it is psi two the resultant of state because of the CNOT operation would be ket one one minus zero one as you can see. And finally if it is psi three then the after the CNOT operation the resultant state would be ket zero zero minus ket one zero.

Now this actually will result in a tensor product and we can write it in terms of product state like this the first qubit would become one by root two ket zero plus ket one the second qubit here would become zero. And if the state is psi one then the resultant tensor product state would be one by root two ket one plus ket zero and here you will have ket one and the first qubit in this case if we get this state then it would one by root two ket one minus ket zero. Second qubit would be ket one and finally the first qubit in the last case would be one by root two ket zero minus ket one and here it would become ket zero.

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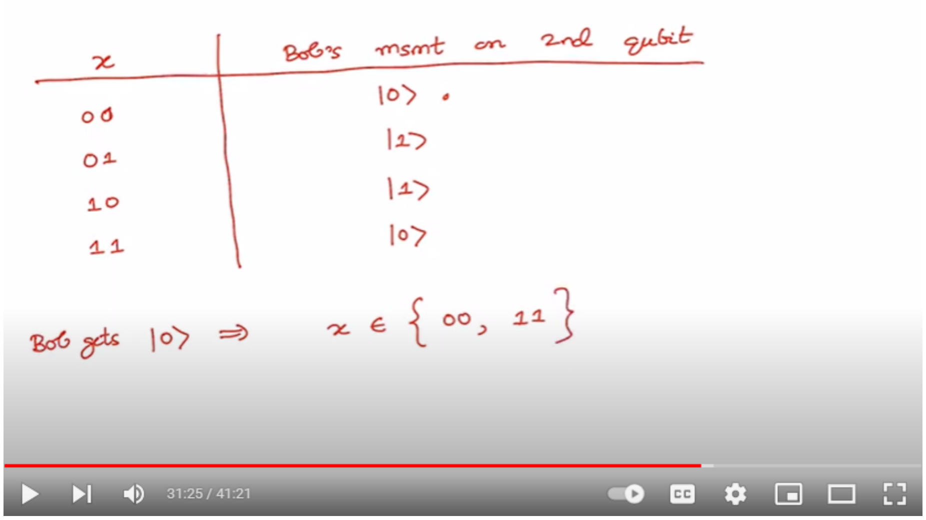

As you can see if Bob makes a measurement of second qubits he may get either zero or one and in fact let me make a table here regarding Bob's measurement of the second qubit. If the and the classical corresponding classical bits are zero zero zero one one zero one one and Bob's measurement result measurement on second qubit. So if classical bit is zero zero then Bob may get ket zero because of his measurement on the second qubit. Second qubit may be ket zero if it is zero one then the second qubit may be one and if it is one zero the second qubit measurement will result in one and if it is one one if classical bits are one one then second qubit measurement will give Bob zero. So if Bob gets zero if Bob gets zero then classical bits send by Alice if Bob gets zero Bob gets zero implies the classical bits may be zero zero or one one right.

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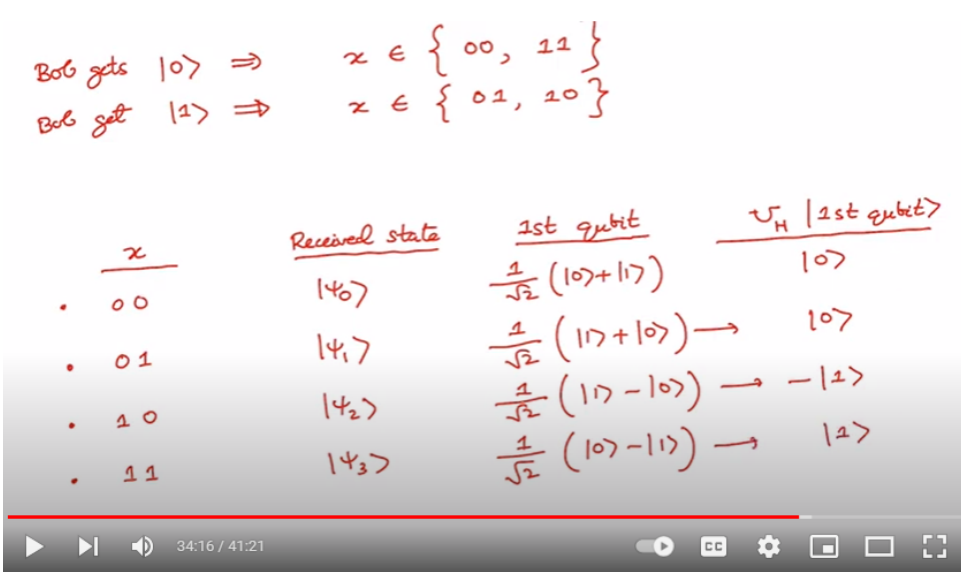

As you can see from the table and if Bob gets the second qubit measurement result of the second qubit measurement if he gets it to be one ket one then the classical bit may be zero one or one zero. So clearly with this Bob would not be able to make a definite conclusion.

However now as you as I have said that this first qubit and second qubit are independent and Bob can make measurement on the second on the first qubit also. So if Bob make a Hadamard operation on the second qubit then he is going to get the following result that I am now going to write that in again in a tabular form. Say you have X classical bits are zero zero zero one one zero one one and the received state by Bob is which already we discussed was a psi zero psi one psi two and psi three. And the first qubit first qubit is after Bob makes the C not operation on the received qubit because of the C not operation it resulted in a product state first qubit and second qubit we can write. The first qubit was for zero zero the first qubit was one by root two ket zero plus ket one and in this if it is zero one classical bit is zero one then the first qubit we got was one by root two ket one plus ket zero. And if it is one zero then the first qubit was one by root two ket one minus ket zero and lastly if it is one one then we had one by root two ket zero minus ket one as the first qubit.

Now Bob is going to make a measurement on the first qubit so if he makes the measurement of the first qubit Hadamard operation if he make then this first qubit will turn to ket zero in this case and here this qubit will turn to ket zero. And here this qubit will turn to with a sign change it will turn into one with a minus sign and then lastly this one will turn into ket one.

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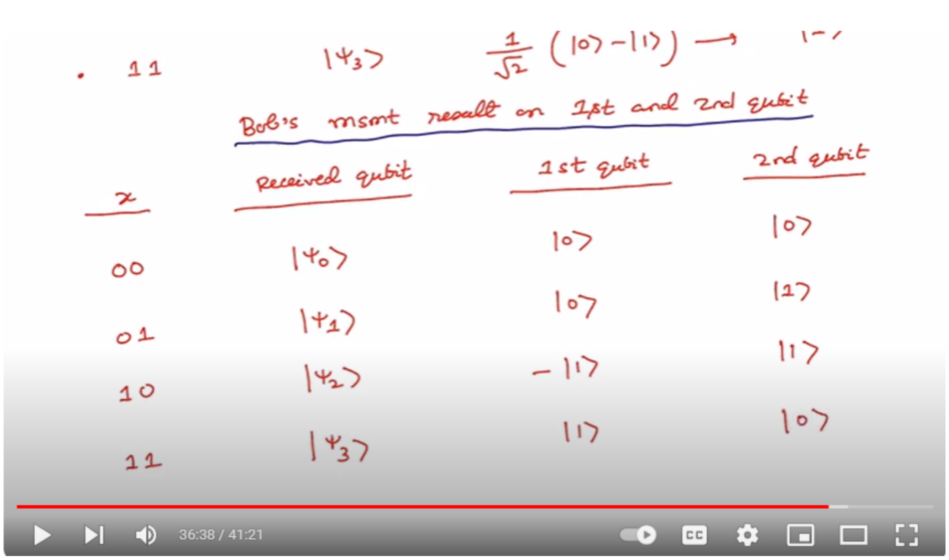

Now let me tabulate the results of Bob's measurement on the first and the second qubit then from here from there we will be able to decide the classical bits actually you will see it very quickly. So let me make the table so first of all what is the result of first qubit measurement and then the second qubit measurement by Bob.

Okay this is basically I am tabulating Bob's measurement result on first and second qubit. Okay this is Bob's measurement result. Now if the classical bit is zero zero received qubit by Bob would be psi zero if it is zero one the received qubit would be psi one if it is one zero received qubit would be psi two if it is one one. As per the super dense protocol I am saying received qubit by Bob would be psi three and then he is going to make C not operation on psi zero psi one psi two psi three as a result we will get a tensor product state and then we can write it as a first qubit tensor product second qubit. We will be able to separate and then he makes the measurement on the first qubit which is a Hadamard operation then if first qubit measurement he makes he gets zero here zero here minus one here I am summarizing both the tables here now and here it will one and if the second qubit measurement would be just you can see we have wrote it earlier.Yes here you see these are the second qubit measurement and if you put it here you will get zero one one and zero.

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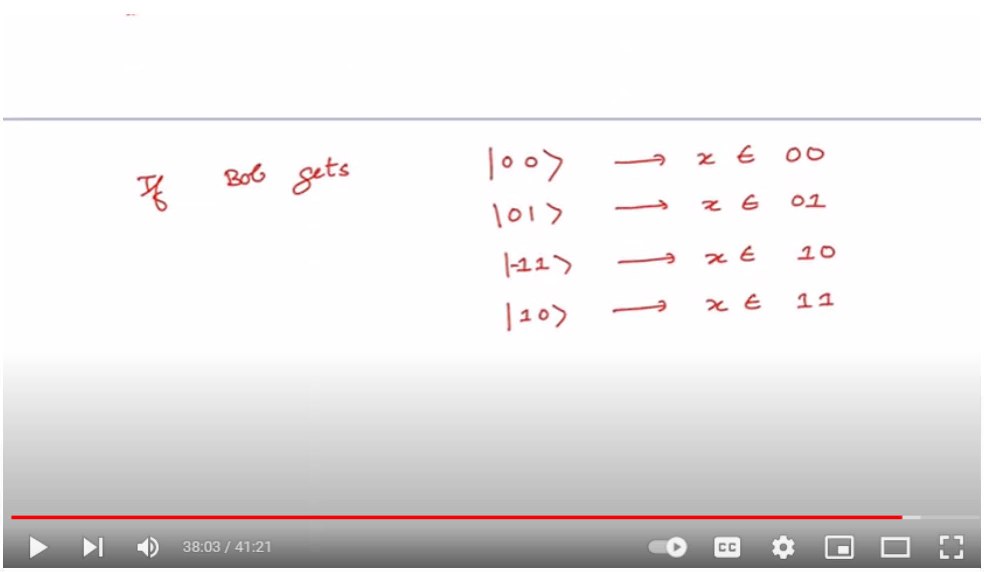

Now what you see as you can see here there is no ambiguity in arriving at the decision regarding which classical bits are sent by Alice. So if say Bob gets because of his measurement first qubit and second qubit measurement first qubit he gets zero and the second qubit he gets zero then that clearly tells that the classical bit is zero zero. If Bob gets zero one as a result of the measurement of first qubit and second qubit then classical bit is going to be simply zero one and if it is one zero then the classical bit will correspond to one zero. No here you have to be careful if the first qubit and second qubit Bob gets is one one of course with a minus sign for the first qubit then the classical bit will correspond to one zero and finally if the first qubit measurement is one and the second qubit measurement is zero then the classical bit will correspond to one zero. So if the first qubit measurement is one and the second qubit measurement is zero then the classical bit will correspond to one one.

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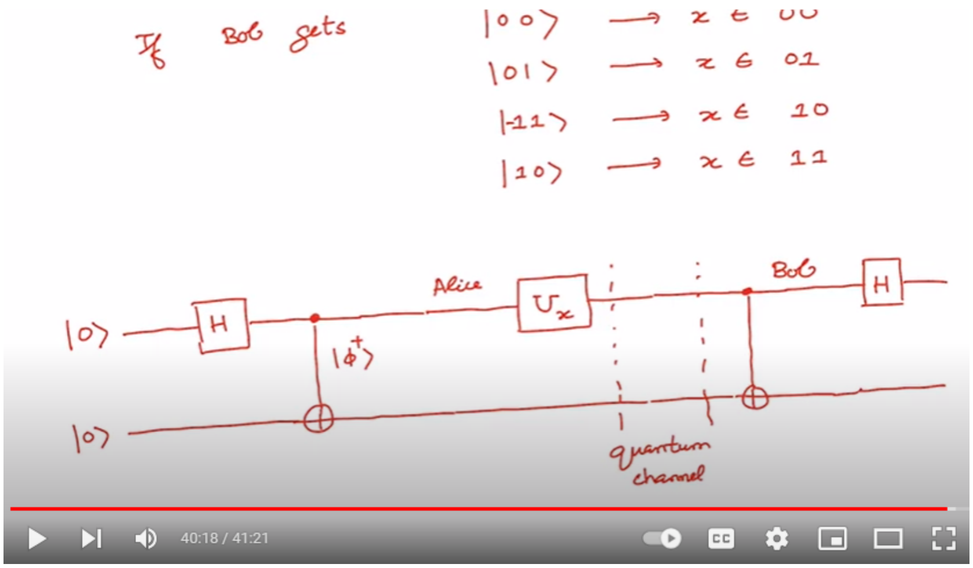

Okay and this is the super dense coding protocol in fact the whole protocol can be represented by a scheme in a schematic diagram let me show you that. First of all you have to create the entangled state Phi plus let me we already know how this entangled state can be created we have to we have to give the input as zero zero in this quantum circuit and then after the Hadamard gate it has the Hadamard gate has to be followed by a C not gate operation. Then at the output we are going to get the Bell state Phi plus and Phi plus is one by root two ket zero zero plus ket one one this is the state you will get and this would be shared by both Alice and Bob. And then what Alice does Alice does depending on what classical bit she wants to send or transmit depending on that she is going to make some unitary operations.

After doing the unitary operations the qubits will be sent to Bob and then Bob will do a C not operation on the received qubit. Now the Bob now Bob is going to do a C not operation on the received qubit and this is basically means that the Alice is going to send the information of her result by a quantum channel. Then Bob receives that qubit and he does the C not operation on the received qubit and on the first qubit after he carry out the C not operation he applies the Hadamard operation on the first qubit along with his measurement on the second qubit. And this will make Bob able to find out what is the two classical bits that Alice actually send thereby Bob will be able to decode the message of Alice.

So this is in a sense the super dense protocol is. Let me stop here for today. In this lecture we briefly discuss about the so called Duan criterion. Then we started discussing applications of quantum entanglement and in this context I first discussed the No Cloning Theory. And I discussed super dense coding as an application of quantum entanglement. In the next lecture we will continue discussing applications of quantum entanglement.

In particular I will discuss the so called quantum teleportation. So see you in the next class. Thank you. .